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The Marshall–Olkin exponential Weibull distribution

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Abstract

A new four-parameter model called the Marshall–Olkin exponential–Weibull probability distribution is being introduced in this paper, generalizing a number of known lifetime distributions. This model turns out to be quite flexible for analyzing positive data. The hazard rate functions of the new model can be increasing and bathtub shaped. Our main objectives are to obtain representations of certain associated statistical functions, to estimate the parameters of the proposed distribution and to discuss its modality. As an application, the probability density function is utilized to model two actual data sets. The new distribution is shown to provide a better fit than related distributions as measured by the Anderson–Darling and Cramér–von Mises goodness–of–fit statistics. The proposed distribution may serve as a viable alternative to other distributions available in the literature for modeling positive data arising in various fields of scientific investigation such as reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

Keywords: Marshall–Olkin exponential–Weibull distribution, goodness–of–fit statistics, moments, median, mode, unimodal distribution, quantile function, Fox–Wright ${}_p\Psi_q$ function, Goyal–Laddha generalized Hurwitz–Lerch zeta function.

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1. Introduction

The Weibull distribution is a popular life time distribution model in reliability engineering. However, this distribution does not have a bathtub or upside–down bathtub shaped hazard rate function, which is why it cannot be utilized to model the life time of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Extensions of the Weibull distribution arise in different areas of research as is often pointed out in the literature, see for instance [2] and the references therein. Various extended Weibull models have an upside–down bathtub shaped hazard rate, which is the case for the extensions discussed by [11] and [20], among others.

Adding parameters to an existing distribution enables one to obtain classes of more flexible distributions. Marshall and Olkin [9] introduced a method for adding a new parameter to an existing distribution, which results in improved flexibility to model different types of data. They consider the so–called *baseline distribution* having *cumulative distribution function* (CDF) G_b, with the associated *probability density function* (PDF) $g_b(x)$, being the Radon-Nikodým derivative of the CDF G_b with respect to the ordinary Lebesgue measure. Then, the associated Marshall–Olkin extended distribution CDF F is given by

$$F(x) = \frac{G_b(x)}{G_b(x) + \alpha \overline{G}_b(x)}\,,$$

where $\overline{G}_b = 1 - G_b$ stands for the survival function of the baseline CDF G_b. Accordingly, *via* the baseline PDF g_b the Marshall–Olkin PDF becomes

$$f(x) = \frac{\alpha\, g_b(x)}{\big(G_b(x) + \alpha \overline{G}_b(x)\big)^2}\,.$$

Recently Cordeiro *et al.* [2] introduced a type of exponential–Weibull distribution by considering the baseline CDF ¶

$$G_b(x) = \Big(1 - e^{-\lambda\, x - \beta\, x^k}\Big) \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \lambda > 0\,,\, \beta > 0\,,\, k > 0\,, \tag{1.1}$$

with the associated PDF

$$g_b(x) = \Big(\lambda + \beta\, k\, x^{k-1}\Big)\; e^{-\lambda\, x - \beta\, x^k} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,. \tag{1.2}$$

Now, we generalize the model (1.1) by Cordeiro *et al.* by applying the Marshall–Olkin technique, which results in what we are referring to as the Marshall–Olkin Exponential–Weibull (MOEW) distribution. Another implementation of the Marshall–Olkin technique was recently considered by Saboor and Pogány, see [20].

Let $\theta = (\lambda, \beta, k, \alpha)$ be a vector parameter having positive coordinates. The random variable (rv) ξ defined on a fixed probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ possesses the *Marshall–Olkin exponential–Weibull distribution* when its CDF and PDF are respectively given by

$$F(x) = \frac{1 - e^{-\left(\lambda\, x + \beta\, x^k\right)}}{1 - (1-\alpha)\, e^{-\left(\lambda\, x + \beta\, x^k\right)}} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,, \tag{1.3}$$

$$f(x) = \frac{\alpha\, \left(\lambda + \beta\, k\, x^{k-1}\right)\, e^{-\lambda\, x - \beta\, x^k}}{\Big(1 - (1-\alpha)\, e^{-\left(\lambda\, x + \beta\, x^k\right)}\Big)^2} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,, \qquad \lambda, \beta, k, \alpha > 0\,; \tag{1.4}$$

¶In this paper, $\boldsymbol{I}_A(x)$ denotes the indicator function of the set A.

and we write $\xi \sim \text{MOEW}(\theta)$ with $\theta = (\lambda, \beta, k, \alpha)$ to indicate that the rv ξ follows this distribution.

One of the main reasons for introducing the MOEW distribution is the following. Consider a sequence of random variables (X_n), $n \in \mathbb{N}$ with IID elements from $G(x)$ distribution, the rv N which possesses geometric distribution with parameter $\alpha \in [0,1]$, that is with probability mass function $\alpha(1-\alpha)^{n-1}$ for $n \in \mathbb{N}$, and $m_N = \min\{X_1, X_2, \cdots, X_N\}$. Then

$$\mathsf{P}\{m_N < x\} = 1 - \sum_{n \geq 1} \mathsf{P}\{m_N \geq x \,|\, N = n\}\, \mathsf{P}\{N = n\} = \frac{G(x)}{G(x) + \alpha\, \overline{G}(x)}.$$

Graphical illustrations of the effect of the parameter α, considered on the whole set $\mathbb{R}_+$ are included in Section 2. Representations of certain statistical functions are provided in Section 3. The parameter estimation technique described in Section 4 is utilized in Section 5 in connection with the modeling of two actual data sets originating from the engineering and biological sciences, where the new model is compared with several related distributions.

2. Graphical Presentations of the MOEW Distribution

Graphs of the PDF (1.4) and the hazard rate function (2.1) are presented in this section for certain values of the parameters.

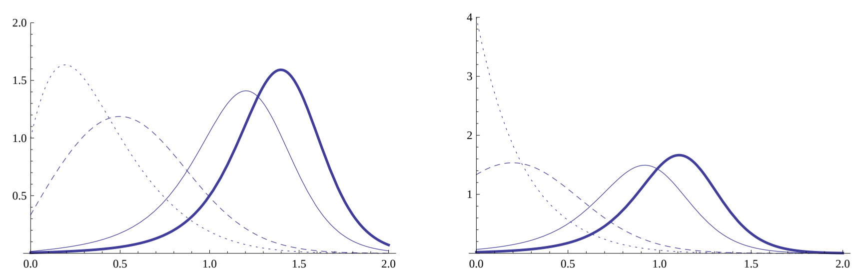

Figure 1. The MOEW PDF. Left panel: $\lambda = 0.5$, $\beta = 2.1$, $k = 2$, and $\alpha = 0.5$ (dotted line) $\alpha = 1.5$ (dashed line), $\alpha = 30$ (solid line), $\alpha = 100$ (thick line). Right panel: $\lambda = 2$, $k = 2$, $\beta = 2.1$ and $\alpha = 0.5$ (dotted line) $\alpha = 1.5$ (dashed line), $\alpha = 30$ (solid line), $\alpha = 100$ (thick line).

Figures 1 and 2 illustrate how the additional parameter α affect the MOEW(θ) density (1.4). The graphs illustrate the versatility of the MOEW distribution and indicate that the new parameter α has a noticeable effect on the skewness and kurtosis. Both Figures 1 and 2 suggest that the parameter α acts somewhat as a location parameter. The left and right panels of Figure 3 indicate that the hazard rate function

$$h(x) = \frac{\lambda + \beta\, k\, x^{k-1}}{1 - (1-\alpha)\, e^{-\left(\lambda\, x + \beta\, x^k\right)}} \cdot \boldsymbol{I}_{(0,\infty)}(x) \qquad (2.1)$$

can be increasing or bathtub shaped for certain values of the parameters.

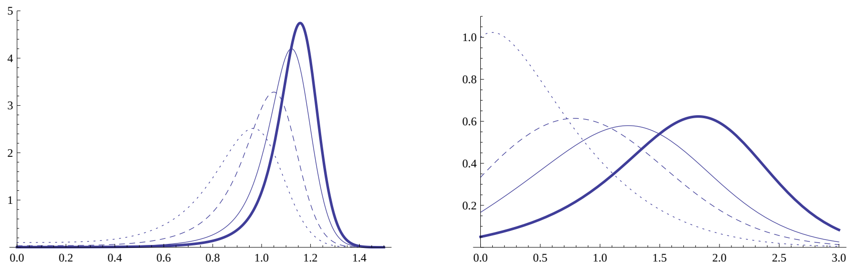

Figure 2. The MOEW PDF. Left panel: $\lambda = 0.5$, $k = 5$, $\beta = 2.1$, and $\alpha = 5$ (dotted line), $\alpha = 15$ (dashed line), $\alpha = 50$ (solid line), $\alpha = 100$ (thick line). Right panel: $\lambda = 0.5$, $\beta = 0.5$, $k = 2$, and $\alpha = 0.5$ (dotted line) $\alpha = 1.5$ (dashed line), $\alpha = 3$ (solid line), $\alpha = 10$ (thick line).

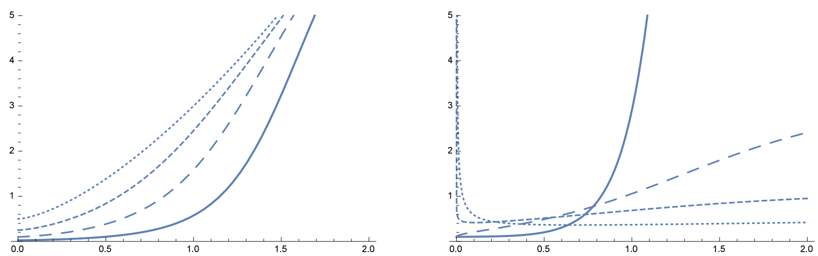

Figure 3. The MOEW hazard rate function. Left panel: $\lambda = 0.5$, $\beta = 1$, $k = 2.5$ and $\alpha = 1$ (dotted line), $\alpha = 2$ (short dashes), $\alpha = 5$ (long dashes), $\alpha = 20$ (solid line). Right panel: $\lambda = 0.5$, $\beta = 1$, $\alpha = 5$, and $k = 0.2$ (dotted line) $k = 0.8$ (small dashed line), $k = 1.5$ (long dashed line), $k = 5$ (solid line).

3. Special Cases

We point out some special cases of the MOEW$(\lambda, \beta, k, \alpha)$ distribution which are obtained by specifying some of its parameters values. For example, the MOEW$(\lambda, \beta, k, 1)$ corresponds to the *exponential–Weibull* distribution [2], the MOEW$(\lambda, \beta, 2, 1)$ is the *modified Rayleigh* distribution, the MOEW$(\lambda, \beta, 1, 1)$ turns out to be the *modified exponential* distribution and finally the MOEW$(0, \beta, k, 1)$ stands for the classical two-parameter Weibull distribution. If $k = 1$ and $k = 2$ in addition to $\alpha = 1$ and $\lambda = 0$, it coincides with the exponential and Rayleigh distributions, respectively.

4. Moments, Quantile Function, Modality Analysis and Mixture representation of the MOEW Distribution

In this section, we derive computable representations of some general order moments associated with the MOEW(θ) distribution having the PDF specified by (1.4). The Fox–Wright generalized hypergeometric ${}_1\Psi_0$, function has been used to obtain the series representations; in the case $k = 1$, the Goyal–Laddha generalized Hurwitz–Lerch Zeta function provides a closed form for the general order moments; in this case, the MOEW distribution is close to the classical Gamma distribution. The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, *MATLAB* or *Maple*. In numerical applications, infinite sum can be truncated whenever convergence is observed.

4.1. Moments. Before concentrating on the derivation of the $r^{\rm th}$ raw moment of the MOEW(θ) distribution, we introduce the Fox-Wright function ${}_p\Psi_q$, which is a generalization of the familiar generalized hypergeometric function ${}_pF_q$, with $p \in \mathbb{N}_0$ numerator parameters $a_1, \cdots, a_p \in \mathbb{C}$ and $q \in \mathbb{N}_0$ denominator parameters $b_1, \cdots, b_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$, defined by

$$ {}_p\Psi_q\left[\begin{array}{c} (a_1, A_1)\,, \;\cdots, (a_p, A_p) \\ \\ (b_1, B_1)\,, \;\cdots, (b_q, B_q) \end{array} \middle| \, z \right] = \sum_{n\geq 0} \frac{\Gamma(a_1 + A_1 n) \cdots \Gamma(a_p + A_p n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_q + B_q n)} \, \frac{z^n}{n!}, $$

where the empty products are conventionally taken to be equal 1, while

$$ A_j > 0,\, j = \overline{1,p};\; B_k > 0,\, k = \overline{1,q}; \quad \Delta = 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0\,, $$

(see, for instance [5, p. 56]). Convergence will occur for suitably bounded values of $|z|$ such that

$$ |z| < \nabla := \left(\prod_{j=1}^p A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^q B_j^{B_j} \right). $$

We now derive closed form representations of the real order moments of a r.v. $\xi \sim$ MOEW(θ). First, we expand the denominator of the PDF (1.4) into a power series in $\exp\{-\left(\lambda\, x + \beta\, x^k\right)\}$. Then, interchanging the integral and the sum, we have

$$ \begin{aligned} \mathsf{E}\, \xi^r &= \alpha \sum_{n\geq 0} \frac{(2)_n\, (1-\alpha)^n}{n!} \int_0^\infty x^r\, \left(\lambda + \beta\, k\, x^{k-1} \right)\, e^{-(n+1)\,\lambda\, x - (n+1)\,\beta\, x^k}\, {\rm d}x \\ &= \alpha\, \lambda \sum_{n\geq 0} \frac{(2)_n\, (1-\alpha)^n}{n!} \int_0^\infty x^r\, e^{-(n+1)\,\lambda\, x - (n+1)\,\beta\, x^k}\, {\rm d}x \\ &\quad + \alpha\, \beta\, k \sum_{n\geq 0} \frac{(2)_n\, (1-\alpha)^n}{n!} \int_0^\infty x^{r+k-1}\, e^{-(n+1)\,\lambda\, x - (n+1)\,\beta\, x^k}\, {\rm d}x\,, \end{aligned} $$

where the Pochhammer symbol $(a)_b := \Gamma(a+b)/\Gamma(a)$, $\min(a, a+b) > 0$, and conventionally $(0)_0 = 1$. The $r^{\rm th}$ moment is a linear combination of integrals $\mathcal{I}(\omega)$ (considered already for a similar purpose by Nadarajah and Kotz in [12, Eq. (2.1)]) where

$$ \mathcal{I}(\omega) = \int_0^\infty x^{\kappa-1}\, e^{-(\mu x + a x^\eta)}\, {\rm d}x\,, \qquad \omega = (\kappa, \mu, a, \eta) > 0\,. $$

The following representation of this integral for general parameter values was obtained by Pogány and Saxena in [16, p. 515, Corollary 1.1]:

$$ \mathcal{I}(\omega) = \begin{cases} \mu^{-\kappa}\; {}_1\Psi_0\left[\begin{array}{c} (\kappa, \eta) \\ \overline{} \end{array} \middle| -\dfrac{a}{\mu^\eta} \right] & 0 < \eta < 1 \\ \dfrac{\Gamma(\kappa)}{(\mu + a)^\kappa} & \eta = 1 \\ \dfrac{1}{\eta a^{\kappa/\eta}}\; {}_1\Psi_0\left[\begin{array}{c} \left(\dfrac{\kappa}{\eta}, \dfrac{1}{\eta}\right) \\ \overline{} \end{array} \middle| -\dfrac{\mu}{a^{1/\eta}} \right] & \eta > 1 \end{cases}. $$

Thus, for all $k \in (0,1)$, we have

$$\begin{aligned} \mathsf{E}\,\xi^r &= \alpha\,\lambda \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!}\,\mathcal{I}(r+1,(n+1)\lambda,(n+1)\beta,k) \\ &\quad + \alpha\,\beta\,k \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!}\,\mathcal{I}(r+k-1,(n+1)\lambda,(n+1)\beta,k) \\ &= \frac{\alpha}{\lambda^r} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+1}\,n!}\,{}_1\Psi_0\left[\begin{array}{c} (r+1,k) \\ \hline \end{array} \middle| -\frac{\lambda\beta^{-k}}{(n+1)^{k-1}}\right] \\ &\quad + \frac{\alpha\,\beta\,k}{\lambda^{r+k}} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+k}\,n!}\,{}_1\Psi_0\left[\begin{array}{c} (r+k,k) \\ \hline \end{array} \middle| -\frac{\lambda\beta^{-k}}{(n+1)^{k-1}}\right]. \end{aligned} \tag{4.1}$$

When $k=1$, we have

$$\mathsf{E}\,\xi^r = \frac{\alpha\,\Gamma(r+1)}{(\lambda+\beta)^r} \sum_{n\geq 0} \frac{(2)_n}{n!}\,\frac{(1-\alpha)^n}{(n+1)^{r+1}}\,. \tag{4.2}$$

Now, consider the *Goyal–Laddha generalized Hurwitz–Lerch Zeta function* [4, p. 100, Eq. (1.5)] defined by the series

$$\Phi^*_\mu(z,s,a) = \sum_{n\geq 0} \frac{(\mu)_n}{n!}\,\frac{z^n}{(n+a)^s}\,, \tag{4.3}$$

where $\mu \in \mathbb{C}$; $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z|<1$; $\Re(s-\mu)>1$ for $|z|=1$. Applying (4.3) to the moment expression (4.2) for all $\alpha \in (0,2)$, while for $\alpha \in \{0,2\}$, $r>2$, we obtain

$$\mathsf{E}\,\xi^r = \frac{\alpha\,\Gamma(r+1)}{(\lambda+\beta)^r}\,\Phi^*_2(1-\alpha,r+1,1)\,.$$

The remaining values of the parameter $k>1$ lead to the expected value

$$\begin{aligned} \mathsf{E}\,\xi^r &= \frac{\alpha\,\lambda}{k\beta^{\frac{r+1}{k}}} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r+1}{k}}\,n!}\,{}_1\Psi_0\left[\begin{array}{c} \left(\frac{r+1}{k},\frac{1}{k}\right) \\ \hline \end{array} \middle| -\frac{(n+1)^{1-\frac{1}{k}}\,\lambda}{\beta^{\frac{1}{k}}}\right] \\ &\quad + \frac{\alpha}{\beta^{\frac{r}{k}}} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r}{k}+1}\,n!}\,{}_1\Psi_0\left[\begin{array}{c} \left(\frac{r}{k}+1,\frac{1}{k}\right) \\ \hline \end{array} \middle| -\frac{\lambda}{(n+1)^{k-1}\beta^k}\right]. \end{aligned}$$

Thus, the following result:

4.1. Theorem. *Let the rv* $\xi \sim \mathrm{MOEW}(\theta), \theta = (\lambda, \beta, k, \alpha) > 0$*. Then, for all* $r > -1$*, we have*

$$
(4.4) \qquad \mathsf{E}\,\xi^r = \begin{cases}
\dfrac{\alpha}{\lambda^r} \displaystyle\sum_{n\geq 0} \dfrac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+1}\,n!}\, {}_1\Psi_0\left[\begin{matrix} (r+1,k) \\ \text{---} \end{matrix} \,\middle|\, \dfrac{-\lambda\beta^{-k}}{(n+1)^{k-1}} \right] \\
\quad + \dfrac{\alpha\,\beta\,k}{\lambda^{r+k}} \displaystyle\sum_{n\geq 0} \dfrac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+k}\,n!}\, {}_1\Psi_0\left[\begin{matrix} (r+k,k) \\ \text{---} \end{matrix} \,\middle|\, \dfrac{-\lambda\beta^{-k}}{(n+1)^{k-1}} \right] & 0 < k < 1 \\
\dfrac{\alpha\,\Gamma(r+1)}{(\lambda+\beta)^r}\, \Phi_2^*(1-\alpha, r+1, 1) & k = 1 \\
\dfrac{\alpha\,\lambda}{k\beta^{\frac{r+1}{k}}} \displaystyle\sum_{n\geq 0} \dfrac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r+1}{k}}\,n!}\, {}_1\Psi_0\left[\begin{matrix} \left(\frac{r+1}{k}, \frac{1}{k}\right) \\ \text{---} \end{matrix} \,\middle|\, \dfrac{-\lambda\beta^{-\frac{1}{k}}}{(n+1)^{\frac{1}{k}-1}} \right] \\
\quad + \dfrac{\alpha}{\beta^{\frac{r}{k}}} \displaystyle\sum_{n\geq 0} \dfrac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r}{k}+1}\,n!}\, {}_1\Psi_0\left[\begin{matrix} \left(\frac{r}{k}+1, \frac{1}{k}\right) \\ \text{---} \end{matrix} \,\middle|\, \dfrac{-\lambda\beta^{-\frac{1}{k}}}{(n+1)^{\frac{1}{k}-1}} \right] & k > 1
\end{cases},
$$

where in the case $k = 1$*, the additional conditions* $\alpha \in (0, 2)$*, or when* $\alpha \in \{0, 2\}$*,* $r > 2$*, have to be satisfied.*

Proof. It only remains to verify the convergence conditions of the Fox–Wright series which depend only on the parameter k. Note that, when $k \in (0, 1)$, $\Delta = 1 - k > 0$, so that both series in (4.1) converge. So does the Goyal–Laddha function when $k = 1$. Finally, when $k > 1$, the value $\Delta = 1 - \frac{1}{k} > 0$ ensures that the moment $\mathsf{E}\,\xi^r$ is finite for any $r > -1$. □

4.2. Remark. For certain integer and rational values of the parameter k, we can make use of a representation of the Fox–Wright ${}_1\Psi_0$ in terms of generalized hypergeometric ${}_pF_q$ functions, which is discussed in detail in [10]. By their [10, Eq. (3.3)], for all positive rational $A = \frac{m}{M}$, one has

$$
{}_1\Psi_0\left[\begin{matrix} (a, \frac{m}{M}) \\ \text{---} \end{matrix} \,\middle|\, z \right] = \Gamma(a) + \sum_{j=1}^{M} \frac{\Gamma(a + \frac{m}{M}j)\, z^j}{j!} \times {}_{m+1}F_M\left[\begin{matrix} 1, \frac{j}{M} + \frac{a}{m}, \cdots, \frac{j}{M} + \frac{a+m-1}{m} \\ \text{---} \end{matrix} \,\middle|\, \frac{m^m\, z^M}{M^M} \right],
$$

where ${}_pF_q$ stands for the generalized hypergeometric function which is a built–in *Mathematica* function specified by

```
HypergeometricPFQ[{a_1, ..., a_p},{b_1, ..., b_q},z].
```

The same authors also transform Fox–Wright Ψ functions into Meijer G–functions for rational arguments. Referring to [10, Eq. (5.1)], one has

$$
{}_1\Psi_0\left[\begin{matrix} (a, \frac{m}{M}) \\ \text{---} \end{matrix} \,\middle|\, z \right] = \frac{2\sqrt{M}\, m^a}{\Gamma(a)\,\sqrt{m}\,\pi^{\frac{M+m-1}{2}}} \times G^{M,m}_{m,M}\left(\frac{m^m\,(-z)^M}{M^M} \,\middle|\, \begin{matrix} 1 - \frac{a}{m}, \cdots, 1 - \frac{a+m-1}{m} \\ 0, \frac{1}{M}, \cdots, \frac{M-1}{M} \end{matrix} \right).
$$

The G–function in *Mathematica* code reads

```
MeijerG[{{a_1,...,a_n},{a_{n+1},...,a_p}},{{b_1,...,b_m},{b_{m+1},...,b_q}},z].
```

See, for example, the monographs [8, Ch. V] and [5] for an introduction to the G–function. ■

The factorial moments of order $N \in \mathbb{N}$ for a r.v. ξ are

$$\Phi_N = \mathsf{E}(\xi(\xi-1)(\xi-2)\cdots(\xi-N+1)) = \frac{\mathrm{d}^N\left(\mathsf{E}\,t^\xi\right)}{\mathrm{d}t^N}\bigg|_{t=1}.$$

By virtue of the Viète–Girard formulae for expanding $\xi(\xi-1)(\xi-2)\cdots(\xi-N+1)$, we obtain

$$\Phi_N = \sum_{r=1}^N (-1)^{N-r}\left\{\sum_{1\le \ell_1<\cdots<\ell_r\le N-1} \ell_1\cdots\ell_r\right\}\mathsf{E}\xi^r,$$

where the second sum represents elementary symmetric polynomials:

$$e_r = e_r(\ell_1,\cdots,\ell_r) = \sum_{1\le \ell_1<\cdots<\ell_r\le N-1} \ell_1\cdots\ell_r, \qquad r=\overline{0,N-1}.$$

This in conjunction with the positive integer r^{th} order moment expression given in formula (4.4) provides an exact series representation for the fractional order moments.

4.2. Quantile Function. The next statistical function being considered is the *quantile function* $\mathscr{Q}_\xi$ for the rv $\xi \sim \mathrm{MOEW}(\theta)$. The rv ξ possesses the CDF $F(x)$ given by (1.3) and its quantile function is

$$\mathscr{Q}_\xi(p) = \inf\{x\in\mathbb{R}\colon p\le F(x)\}, \qquad p\in(0,1);$$

it consists of the generalized inverse of the CDF for a fixed probability p. A closed form is given in the next theorem for the MOEW distribution.

4.3. Theorem. *Let $\xi\sim\mathrm{MOEW}(\theta)$, $\theta=(\lambda,\beta,k,\alpha)$ with parameter space $\theta\in\mathbb{R}_+^4$. For all $p\in(0,1)$, the quantile function of ξ is*

$$\mathscr{Q}_\xi(p) = \ln\Big(\frac{1-(1-\alpha)p}{1-p}\Big)^{\frac{1}{\lambda}}\left\{1+\sum_{n\ge1}\binom{kn}{n-1}\frac{w^n}{n!}\right\}, \tag{4.5}$$

where

$$w = \left(-\frac{1}{\lambda}\right)^k\left[\ln\frac{1-p}{1-(1-\alpha)p}\right]^{k\beta}.$$

Moreover, for $k>1$ we have

$$\mathscr{Q}_\xi(p) = \ln\Big(\frac{1-(1-\alpha)p}{1-p}\Big)^{\frac{1}{\lambda}}\cdot\left\{1+w\cdot{}_1\Psi_2\Big[\begin{matrix}(k+1,k)\\(k+1,k-1),\,(2,1)\end{matrix}\Big|\,w\Big]\right\}. \tag{4.6}$$

Proof. The quantile function is the solution of $F(x)=p$ in x. Thus, for $p\in(0,1)$ fixed, one has

$$\beta x^k + \lambda x + \ln\frac{1-p}{1-(1-\alpha)p} = 0,$$

which is equivalent to

$$1-t+wt^k=0; \qquad t=-\frac{\lambda}{c}; \qquad c=\ln\frac{1-p}{1-(1-\alpha)p}. \tag{4.7}$$

Applying the Bürmann–Lagrange series expansion [17, p. 153, p. 348, **211.**] for the three-term equation (4.7), we obtain

$$t = 1+\sum_{n\ge1}\binom{kn}{n-1}\frac{w^n}{n!},$$

which leads to the solution (4.5).

Further, assuming $k>1$, transforming and writing the generalized binomial coefficient in (4.7) in terms of gamma function, that is,

$$\binom{a}{\ell} = \frac{\Gamma(a+1)}{\Gamma(a-\ell+1)\,\ell!}, \qquad \ell \in \mathbb{N},$$

we have

$$\begin{aligned} t &= 1 + \sum_{n\geq 1} \binom{kn}{n-1} \frac{w^n}{n!} = 1 + \sum_{n\geq 0} \binom{kn+k}{n} \frac{w^{n+1}}{(n+1)!} \\ &= 1 + w \sum_{n\geq 0} \frac{\Gamma(k+1+kn)}{\Gamma(k+1+(k-1)n)\,\Gamma(2+n)}\,\frac{w^n}{n!}\,. \end{aligned}$$

Since $\Delta = 1+k-1+2-k = 2$ and all coefficients of the running indices are positive, we recognize the sum as the appropriate converging Fox–Wright generalized ${}_1\Psi_2$ function as stated in (4.6). $\square$

The distribution of ξ being absolutely continuous, the corresponding *median* turns out to be $\mathfrak{m}_\xi = \mathscr{Q}_\xi(\frac{1}{2})$. Therefore we have

4.4. Corollary. *Under the assumptions made in the* Theorem 3.3 *we have*

$$\mathfrak{m}_\xi = \frac{1}{\lambda}\,\ln(1+\alpha)\cdot\left(1 + \sum_{n\geq 1}\binom{kn}{n-1}\frac{w^n}{n!}\right)\,, \tag{4.8}$$

where

$$w = \frac{(-1)^{k(\beta+1)}}{\lambda^k}\,\big[\,\ln(1+\alpha)\,\big]^{k\beta}\,.$$

Accordingly, for $k>1$ we have

$$\mathfrak{m}_\xi = \frac{1}{\lambda}\,\ln(1+\alpha)\cdot\left\{1 + w\cdot {}_1\Psi_2\Big[\begin{array}{c}(k+1,k)\\ (k+1,k-1),\,(2,1)\end{array}\Big|\,w\Big]\right\}\,. \tag{4.9}$$

Finally, we point out that Theorem 4.1 yields the characteristic function $\phi_\xi(t) = \mathsf{E}\,e^{\mathrm{i}t\xi}$ *via* the well–known Maclaurin series expansion $\phi_\xi(t) = \sum_{n\geq 0}(\mathrm{i}t)^n\,\mathsf{E}\,\xi^n/n!$. Further, the moment generating function $M_\xi(t) = \phi_\xi(-\mathrm{i}t)$, while the hazard rate function $h(x)$ and the survival function $\overline{F}(x) = 1-F(x)$ can be expressed in obvious ways in terms of the PDF and the CDF of the rv $\xi \sim \mathrm{MOEW}(\theta)$.

4.3. Modality Analysis. To close this section, we carry out a modality analysis for the $\mathrm{MOEW}(\theta)$ distribution.

Let us recall that in the case of continuous distributions having PDF f, the argument value x_0 belonging to its support $\mathrm{supp}(f) := \{x\colon f(x)>0\}$ for which $f(x_0) = \max$, is called the *mode* (*peak*)[‖]. The PDF can attain local maximum at several values from $\mathrm{supp}(f)$; the distributions with a single mode are *unimodal.* The following theorem gives certain sufficient conditions for the unimodality of a $\mathrm{MOEW}(\theta)$ distribution for different cases.

4.5. Theorem. *Let $\xi \sim \mathrm{MOEW}(\theta)$, $\theta = (\lambda,\beta,k,\alpha)$ where $(\lambda,\beta,k) \in \mathbb{R}_+^3$, $\alpha \in (0,1]$. Then*

(i) $k \in (0,1]$. *No mode.*

[‖]Let us mention that there are other definitions of the modality in terms of the related CDF or the characteristic function or its Laplace–Stieltjes transform [22]

(ii) $k \in (1,2)$. *The rv* $\xi \sim \mathrm{MOEW}(\theta)$ *is unimodal with* $x_0 \in (0,1)$*, when*

$$\beta k[(\beta-1)k + 2\lambda + 1] + \lambda^2 > 0\,.$$

(iii) $k = 2$. *No mode exists when* $\lambda \geq \sqrt{2\beta}$. *For* $\lambda < \sqrt{2\beta}$ *the distribution is unimodal with the peak at* $x_0 \in (0, x^*)$*, where*

$$x^* = \frac{\sqrt{2\beta} - \lambda}{2\beta}\,.$$

(iv) $k \in (2,4)$. *The rv* $\xi \sim \mathrm{MOEW}(\theta)$ *is unimodal with* $x_0 \in (0,1)$*, when*

$$(\beta k + \lambda)^2 < (4-k)\beta\,.$$

Proof. For MOEW distribution $\mathrm{supp}(f) = \mathbb{R}_+$. As for the peak value of the PDF (1.4), we consider its logarithmic derivative

$$\begin{aligned}\frac{\partial \ln f(x)}{\partial x} &= \frac{\beta k(k-1)x^{k-2}}{\lambda + \beta k x^{k-1}} - \left(\lambda + \beta k x^{k-1}\right)\\ &\quad - \frac{2(1-\alpha)(\lambda + \beta k x^{k-1})}{1-(1-\alpha)e^{-\lambda x - \beta k x^k}}\, e^{-\lambda x - \beta k x^k}\,. \end{aligned} \tag{4.10}$$

The case **(i)**, when $k \in (0,1)$ is obvious, since

$$f'(x) = f(x)\,\frac{\partial \ln f(x)}{\partial x} < 0, \qquad x > 0\,,$$

that is, $f(x)$ monotonically decreases from $f(0^+) = +\infty$ to zero. The case $k = 1$ is actually generated by the exponential baseline distribution with parameter $\lambda + \beta$, see (1.1). In all those cases no mode exists.

As for the case **(ii)**, when $k \in (1,2)$, we consider the first two terms on the right-hand-side expression

$$\begin{aligned} h_k(x) &= \frac{\beta k(k-1)x^{k-2}}{\lambda + \beta k x^{k-1}} - \left(\lambda + \beta k x^{k-1}\right)\\ &= -\frac{\beta^2 k^2 x^{2k-2} + 2\lambda\beta k x^{k-1} - \beta k(k-1)x^{k-2} + \lambda^2}{\lambda + \beta k x^{k-1}} =: \frac{-q_k(x)}{\lambda + \beta k x^{k-1}}\,,\end{aligned}$$

say. For $\alpha \in (0,1]$ the third term in (4.10) is negative for all $x > 0$. Since $q_k(0^+) = -\infty$, but $q_k(1) = \beta k[(\beta-1)k + 2\lambda + 1] + \lambda^2 > 0$ and

$$q_k'(x) = \beta k(k-1)x^{k-3}(2\beta k x^k + 2\lambda x + 2 - k) > 0$$

exactly one sign change occurs inside $(0,1)$, so $x_0 \in (0,1)$.

Consider now

$$q_2(x) = 4\beta^2 x^2 + 4\lambda\beta x - 2\beta + \lambda^2 = 0\,.$$

The roots of $q_2(x) = 0$ are

$$x_1 = -\frac{\sqrt{2\beta} + \lambda}{2\beta} < 0\,, \qquad x^* = \frac{\sqrt{2\beta} - \lambda}{2\beta}\,.$$

The solution $x^* > 0$ for $\sqrt{2\beta} - \lambda > 0$, which confirms the assertion **(iii)**.

Finally, for $k > 2$, $q_k(0^+) = \lambda^2 > 0$ and $q_k(1) = (\beta k + \lambda)^2 - \beta k(k-1)$ should be negative. However,

$$(\beta k + \lambda)^2 - \beta k(k-1) < (\beta k + \lambda)^2 - (4-k)\beta\,,$$

which can take negative values for $k \in (2,4)$. For $k \geq 4$, the last estimate becomes redundant. □

4.6. Remark. Obviously, the modality analysis in the cases $\alpha \in (0,1], k > 4$ and $\alpha > 1$ requires another approach to be solved, since in the latter case the third right-hand-side addend in (4.10) becomes

$$\frac{2(\alpha-1)(\lambda+\beta k x^{k-1})}{1+(\alpha-1)e^{-\lambda x-\beta k x^k}}\, e^{-\lambda x-\beta k x^k} > 0\,. \qquad \blacksquare$$

We now show that the density (1.4) can be expressed as a mixture of EW densities. Using the identity

$$(1-z)^{-\tau} = \sum_{n=0}^{\infty} \frac{(\tau)_n}{n!}\, z^n\,, \qquad |z|<1\,,\ \tau>0\,,$$

one has the following mixture representation for the density function (1.4):

$$f(x) = \alpha \sum_{n\ge 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)!}\, g_{n+1}(x),$$

where $g_{n+1}(x)$ denotes the PDF of the EW model with parameters $\lambda^\star = (n+1)\lambda$, $\beta^\star = (n+1)\beta$ and k. Thus, the MOEW density function is a mixture of EW densities.

5. Parameter Estimation

This section provides a system of equations that can be utilized to determine the maximum likelihood estimates of the parameters of the MOEW distribution. Additionally, two goodness-of-fit measures are proposed to compare the density estimates.

5.1. Maximum Likelihood Estimation. In order to estimate the parameters of the proposed MOEW density function as defined in Equation (6), the loglikelihood of the sample is maximized with respect to the parameters. Given the data $\boldsymbol{x} = (x_1, \cdots, x_n)$, the loglikelihood function is

$$\begin{aligned}\ell(\theta) &= n\,\log\alpha + \sum_{i=1}^n \log\left(\lambda+\beta\,k\,x_i^{k-1}\right) - \sum_{i=1}^n \left(\lambda\,x_i+\beta\,x_i^k\right)\\ &\quad - \sum_{i=1}^n \log\left(\left(1-(1-\alpha)\,e^{-\left(\lambda\,x_i+\beta\,x_i^k\right)}\right)^2\right)\,,\end{aligned}$$

where $f(x)$ is as given in (1.4). The associated nonlinear loglikelihood system $\frac{\partial\ell(\theta)}{\partial\theta} = 0$ for MLE's is

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial\lambda} &= -\sum_{i=1}^n x_i - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}(1-\alpha)x_i}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)} + \sum_{i=1}^n \frac{1}{\lambda+k\beta x_i^{-1+k}} = 0\\ \frac{\partial\ell(\theta)}{\partial\beta} &= -\sum_{i=1}^n x_i^k - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}(1-\alpha)x_i^k}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)} + \sum_{i=1}^n \frac{k x_i^{-1+k}}{\lambda+k\beta x_i^{-1+k}} = 0\\ \frac{\partial\ell(\theta)}{\partial k} &= -\beta\sum_{i=1}^n x_i^k\,\log x_i - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}(1-\alpha)\beta x_i^k\,\log x_i}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)}\\ &\quad + \sum_{i=1}^n \frac{\beta x_i^{-1+k} + k\beta x_i^{-1+k}\,\log x_i}{\lambda+k\beta x_i^{-1+k}} = 0\\ \frac{\partial\ell(\theta)}{\partial\alpha} &= \frac{n}{\alpha} - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)} = 0\,.\end{aligned}$$

Solving these equations simultaneously yields the maximum likelihood estimates (MLEs) of the four parameters. Numerical iterative techniques are then necessary to estimate the model parameters. It is possible to determine the global maximum of the log-likelihood by taking different initial values for the parameters. However, we observed that the MLEs for this model are not very sensitive to the initial estimates. For interval estimation on the model parameters, we require the Fisher information matrix; however in this article we leave this routine calculation to the interested reader.

5.2. Goodness-of-Fit Statistics. The Anderson-Darling and the Cramér-von Mises statistics are widely utilized to determine how closely a specific distribution whose associated cumulative distribution function fits the empirical distribution associated with a given data set. These statistics are

$$A_0^* = -\left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\,\log\left(z_j\,(1 - z_{n-j+1})\right)\right\}$$

$$W_0^* = \left(\frac{1}{2n} + 1\right)\left\{\sum_{j=1}^{n}\left(z_j - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\},$$

respectively, where $z_j = F(y_j)$, the y_j values being the *ordered* observations. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness–of–fit statistics were tabulated in [13].

6. Applications

Now, we will make use of the MOEW, beta transmuted Weibull (BTW) [14], Kumaraswamy modified Weibull (KMW) [3], extended Weibull (ExtW) [15], exponential–Weibull (EW) [2], gamma–Weibull (GW) [18] **, generalized gamma (GG) [21], two parameter Weibull (Weibull) and two parameter gamma (Gamma) distributions to model two well–known real data sets, namely the 'Carbon fibres' [13] and the 'Cancer patients' [6] data sets. The parameters of the MOEW distribution can be estimated from the log-likelihood of the samples in conjunction with the *NMaximize* command in the symbolic computational package *Mathematica*. More specifically, the models being considered are:

- The classical gamma distribution with PDF

$$f(x) = \frac{x^{\xi-1}\,e^{-x/\phi}}{\phi^{\xi}\,\Gamma(\xi)} \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \phi,\,\xi > 0\,.$$

- The classical Weibull distribution with PDF

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad k,\lambda > 0\,.$$

- The generalized gamma (GG) distribution [21] with PDF

$$f(x) = \frac{k\,\lambda^{-\xi}\,x^{\xi-1}\,e^{-\lambda^{-k}\,x^k}}{\Gamma(\xi/k)} \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad k,\,\lambda > 0\,.$$

- The gamma–Weibull (GW) distribution [18] with PDF

$$f(x) = \frac{k\,\lambda^{-k-\xi}\,x^{\xi+k-1}e^{-\lambda^{-k}\,x^k}}{\Gamma(1+\xi/k)} \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \xi+k,\,\lambda > 0\,.$$

**It is worth mentioning that following another approach, that is, renormalizing the product of the gamma and the Weibull distribution's PDF, Leipnik and Pearce [7] introduced a five–parameter gamma–Weibull distribution; for further results on this type of investigations consult also [12] and [16]. In turn, the independently introduced, different type of PDF proposed by Provost *et al.* [18] is actually a specific case of Leipnik–Pearce type gamma-Weibull distribution. Fortunately, the both turn out to be good candidates for various applications.

- The gamma exponentiated exponential (GEE) distribution [19] with PDF

$$f(x) = \frac{\lambda\,\alpha^{\delta}}{\Gamma(\delta)}\, e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\alpha-1} \left(-\log\left(1 - e^{-\lambda x}\right)\right)^{\delta-1} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,,$$

where $\lambda,\, \alpha,\, \delta > 0\,.$

- The exponential–Weibull (EW) distribution [1] with PDF

$$f(x) = \left(\lambda + \beta\, k\, x^{k-1}\right) e^{-\lambda\, x - \beta\, x^{k}} \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \lambda,\, \beta,\, k > 0\,.$$

- The extended Weibull (ExtW) distribution [15] with PDF

$$f(x) = a\,(c + b\,x)\,x^{-2+b}\, e^{-c/x - a x^{b} e^{-c/x}} \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad a,\, b,\, c \geq 0\,.$$

- The Kumaraswamy modified Weibull (KMW) distribution [3] with PDF

$$f(x) = a\,b\,\alpha\,x^{\gamma-1}(\gamma + \lambda\,x)\exp\left(\lambda\,x - \alpha\,x^{\gamma}\,e^{\lambda\,x}\right)\left(1 - \exp\left(-\alpha\,x^{\gamma}\,e^{\lambda\,x}\right)\right)^{a-1}$$
$$\cdot\,\left(1 - \left(1 - \exp\left(-\alpha\,x^{\gamma}\,e^{\lambda\,x}\right)\right)^{a}\right)^{b-1} \cdot \boldsymbol{I}_{(0,\infty)}(x)$$

where $a,\, b,\, \alpha, \gamma > 0\,, \lambda \geq 0$.

- The beta transmuted Weibull (BTW) distribution [14] with PDF

$$f(x) = \frac{\alpha\beta x^{\beta-1}}{\mathrm{B}(a,b)}\, e^{-\alpha\,x^{\beta}}(1 - \lambda + 2\,\lambda\,e^{-\alpha\,x^{\beta}})(1 - e^{-\alpha\,x^{\beta}})^{a-1}(1 + \lambda e^{-\alpha x^{\beta}})^{a-1}$$
$$\cdot\,(1 - (1 - e^{-\alpha\,x^{\beta}})(1 + \lambda\,e^{-\alpha\,x^{\beta}}))^{b-1} \cdot \boldsymbol{I}_{(0,\infty)}(x)$$

where $a,\, b,\, \alpha,\, \beta > 0,\; |\lambda| \leq 1$.

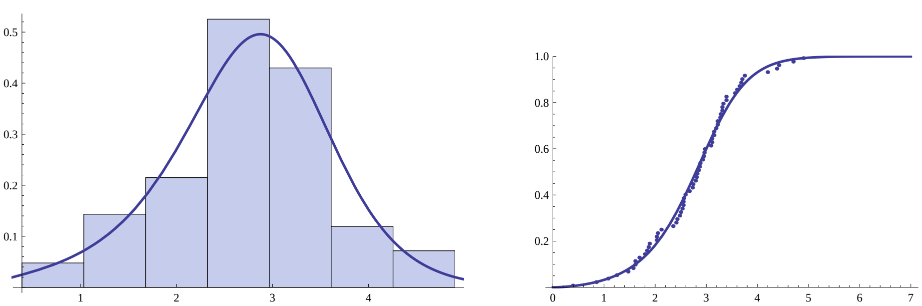

Figure 4. The Carbon fibres data fitted using the maximum likelihood approach; Left panel: The MOEW PDF estimate superimposed on the histogram for Carbon fibres data. Right panel: The MOEW CDF estimate and empirical CDF.

6.1. The Carbon Fibres Data Set. We shall consider the uncensored real data set on the breaking stress of carbon fibres (in Gba) as reported in [13]. The data are ($n = 66$):

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

Table 1. Estimates of the Parameters and Goodness-of-Fit Statistics for the Carbon Fibres Data

Distributions	Estimates					A_0^*	W_0^*
Gamma(ξ, ϕ)	7.48803	0.36853				1.32674	0.24815
Weibull(k, λ)	3.44120	47.0505				0.49168	0.08430
GG(k, λ, ξ)	4.07350	3.34592	3.09225			0.48757	0.08111
GW(k, ξ, λ)	3.44120	$1.6 * 10^{-7}$	3.06226			0.49168	0.08430
GEE$(\lambda, \alpha, \delta)$	0.26555	10.0365	7.23658			1.43415	0.26682
EW(k, λ, β)	3.73666	0.01710	0.01402			0.40365	0.06479
ExtW(a, b, c)	16.1979	$1 * 10^{-7}$	8.05671			2.26745	0.41615
KMW$(\alpha, \gamma, \lambda, a, b)$	0.14981	1.79940	0.49987	0.64975	0.17111	1.29338	0.21322
BTW$(\alpha, \beta, \lambda, a, b)$	0.00395	3.49999	0.99982	0.95052	2.39533	0.51603	0.09143
MOEW$(\lambda, \beta, k, \alpha)$	1.62267	$1 * 10^{-6}$	0.61610	25.3808		**0.2565**	**0.0374**

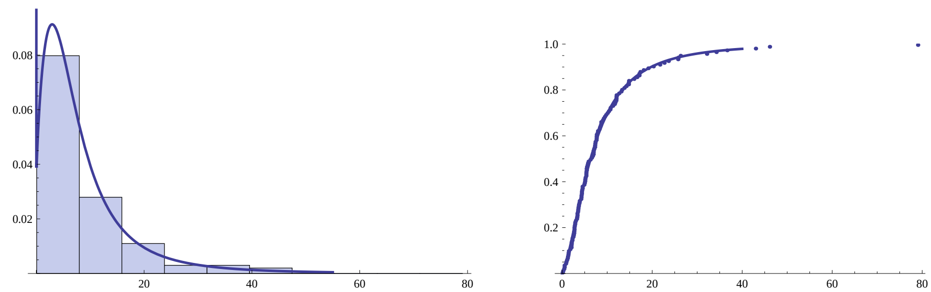

Figure 5. The Cancer Patients data fitted using the maximum likelihood approach; Left panel: The MOEW PDF estimate superimposed on the histogram for Cancer patients data. Right panel: The MOEW CDF estimate and empirical CDF.

6.2. The Cancer Patients Data Set. The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in [6]. The data are

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The PDF and CDF estimates of the MOEW distribution are plotted in Figures 4 and 5 for the Carbon fibres and Cancer patients data, respectively. The estimates of the parameters and the values of the Anderson-Darling and Cramér-von Mises goodness–of–fit statistics are given in Tables 1 and 2. It is seen that the proposed MOEW model provides the best fit for the both data sets.

To compare MOEW model with its sub-model EW, the likelihood-ratio (LR) test is applied to both data sets. The LR in this case is $L^* = L_0(k, \lambda, \beta)/L_a(k, \lambda, \beta)$, where L_0 and L_a are the likelihood values for the EW and MOEW distributions, respectively. The LR statistic $-2 \log L^*$ follows a chi-square distribution (asymptotically) with 1 degrees

Table 2. Estimates of the Parameters and Goodness-of-Fit Statistics for the Cancer Patients Data

Distributions	Estimates					A_0^*	W_0^*
Gamma(ξ, ϕ)	1.17251	7.98766				0.77625	0.13606
Weibull(k, λ)	1.04783	10.6510				0.96345	0.15430
GG(k, λ, ξ)	0.52010	0.59510	1.94927			0.30087	0.04526
GW(k, ξ, λ)	0.52001	1.42917	0.59510			0.30087	0.04526
GEE$(\lambda, \alpha, \delta)$	0.12117	1.21795	1.00156			0.71819	0.12840
EW(k, λ, β)	1.04780	$1 * 10^{-7}$	0.09389			0.96345	0.15430
ExtW(a, b, c)	1.96210	$1 * 10^{-21}$	3.74383			13.3317	2.49818
KMW$(\alpha, \gamma, \lambda, a, b)$	0.63962	0.38186	0.02960	0.37500	0.32284	18.8864	3.68568
BTW$(\alpha, \beta, \lambda, a, b)$	0.21333	0.99990	0.97623	1.52665	0.32699	0.16057	0.02367
MOEW$(\lambda, \beta, k, \alpha)$	0.12080	0.01234	10.9988	$1 * 10^6$		**0.09052**	**0.0141**

of freedom. For the first data set $-2 \log L^* = 1.613$ with a p-value of 0.2041 whereas for the second data set $-2 \log L^* = 9.344$ with a p-value of 0.0022. Both values of the LR statistics suggest that in both cases the MOEW model performs significantly better when compared with its sub-model EW.

7. Discussion

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. As a result, significant progress has been made towards generalizing some well–known lifetime models, which have been successfully applied to problems arising in several areas of research. In particular, several authors proposed new distributions that are based on the traditional Weibull model. In this paper, we introduce a four–parameter distribution which is obtained by applying the Marshall–Olkin technique to the exponential Weibull model. We studied some of its mathematical and statistical properties. We also provided computable representations of the moments of order $r > -1$, the factorial moments and the quantile function. Also the unimodality analysis was performed for suitable sub-domains of the parameter space of the MOEW(θ) distribution.

The proposed distribution was utilized to model two data sets; it was shown to provide a better fit than several other related models, including some with more parameters. The distributional results developed in this article should find numerous applications in reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

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