Int.J. Thermodynamics,
Vol.6, (No.2), pp.69-78, June-2003

ISSN 1301-9724

Optimal Endoreversible Heat Engines with Polytropic Branches

Josef Maximilian BURZLER, Karl Heinz HOFFMANN*
Institut für Physik, Technische Universität Chemnitz,
09107 Chemnitz, Germany
Tel: +49 371 531 3212, Fax:+49 371 531 3233
E-mail: hoffmann@physik.tu-chemnitz.de

Sergey Anatolievich AMELKIN, Anatoliy Mikhailovich TSIRLIN
Program Systems Institute of Russian Academy of Sciences,
Botik, Pereslavl-Zalesskiy, 152140 Russian Federation.
Tel: +7 085 35 98 246, Fax: +7 085 35 20 593

Abstract

Endoreversible engine cycles with two adiabatic and two heat transfer branches are investigated and optimized for maximum work output. The heat transfer branches are described as general polytropic processes which include common standard branches, like isotherms, isobars and isometrics, as special cases. The study considers the finite heat capacity of the working fluid and the finite-time character of the heat transfer processes, determines the optimal allocation of branch times, and derives analytic expressions for the maximized work output. The efficiency at maximum work is found to coincide with the Curzon-Ahlborn efficiency for endoreversible Carnot engines and does not depend on design parameters of the engine if the degree of the polytropic processes is equal in both heat transfer branches.

Key words: endoreversible, finite-time thermodynamics, optimization, polytropic, heat engine

1. Introduction

After the advent of modern thermodynamics at the beginning of the 19th century, when Sadi Carnot discovered his famous efficiency expression for the conversion of heat into work, classical equilibrium thermodynamics has established fundamental bounds for the performance of reversible systems. As it became apparent that these reversible bounds are rarely approached in practice, finite-time thermodynamics was developed to account for the irreversibilities caused by the finite times and rates at which real systems operate (see for example Andresen et al. 1984, Bejan 1988, Sieniutycz and Salamon 1990, De Vos 1992, Berry et al. 1999).

A common concept in finite-time thermodynamics is the assumption of *endoreversibility* (Rubin 1979, Ondrechen et al. 1983). The assumption of endorversibility helps to reduce the complexity of thermodynamic models by composing them of reversible subsystems and confining the irreversibilities to the interactions, typically heat transfer, between these subsystems. A comprehensive review on the topic may be found in Hoffmann et al. (1997). Finite-time thermodynamics of endoreversible systems has been applied to heat engines and has yielded far more realistic results than classical equilibrium thermodynamics. The majority of studies have analyzed endoreversible cycles with two isothermal heat transfer branches. Rozonoer and Tsirlin (1983) have proved that this kind of cycles provide the maximal efficiency at any given work output.

However, there are engine cycles which consist of non-isothermal branches. To describe the state change of the working fluid during such a heat transfer branch one typically uses polytropic processes, since common standard branches, such as isotherms, isometrics and isobars, are just special cases of the more general polytropic process. A polytropic process is characterised by a working fluid obeying the relation $pV^n = \text{const}$ for a given polytropic degree n.

[*] Author to whom all correspondence should be addressed.

Such polytropic processes are standard in textbooks on classical thermodynamics (Spalding 1973).

Landsberg and Leff (1989) have investigated engine cycles with two adiabatic branches and two generalised heat transfer branches, which can be interpreted as polytropic branches. They considered a quasi-static, reversible engine model where heat is transfered between the temperature reservoir and the working fluid even if the temperature differences are infinitesimally small. Such a description apparently neglects the finite rates of real heat transfer processes. Other authors have studied cycles for the case of given variable temperatures of the heat reservoirs (Gordon and Huleihil 1991) or finite heat capacities of the heat reservoirs: Ondrechen et al. (1983), Kuznetsov et al. (1985), Lee and Kim (1991), Yan and Chen (1997).

In our paper we examine the case where the working fluid undergoes polytropic heat transfer processes while the temperatures of the two heat reservoirs remain constant. Particularly we study engine cycles which consist of two polytropic and two adiabatic branches. Note that due to the endo-reversibility adiabatic branches are always isentropic. The polytropic degree along the two polytropic branches can of course be different, depending on the physical engine which is modelled. For such cycles the power/work optimal temperatures as well as the optimal time allocation between the branches are determined.

Interestingly it turns out that from a technical point a part of our analysis is similar to a problem which has previously been treated in a paper of Pathria, Nulton and Salamon. They extended their existing theory (Nulton et al. 1993) of finite-time heat engines to cases with non-isothermal heat transfer branches (Pathria et al. 1996) considering a sytem with a finite constant heat capacity. The similarity is due to the fact that along a polytropic branch the effective heat capacity remains finite and constant. There are, however, important differences between our present paper and their study. We focus on the polytropic character of the heat transfer processes which means that the effective constant heat capacity is not a system property, but a process property. Thus it can be different on the two polytropic branches. Our analysis starts from first principles, and additionally shows how to improve the performance of an engine by adjusting both, the branch times and allocation of heat conductances, in an optimal manner. We finally discuss the influence of the polytropic degree and heat capacities on the performance of the engine.

2. Heat transfer in polytropic processes

Consider a thermodynamic system as depicted in *Figure 1* where heat is transfered between a heat reservoir at constant temperature T_0 and a working fluid at variable temperature T. In case of a Newton-type heat-transfer law, the rate of heat flowing into the working fluid is given by a linear expression

$$q(T_0, T) = K(T_0 - T) \tag{1}$$

where K is the thermal conductance. The working fluid is assumed to have a constant heat capacity and to undergo a polytropic process, where pV^n is constant, and exponent n is a given constant during the heat transfer. The polytropic degree n can in principle take any value between $-\infty$ and $+\infty$. Common standard processes are retrieved with n=1 for isothermal, n=0 for isobaric, $n=\pm\infty$ for isometric and $n=\gamma$ for isentropic processes. The property $\gamma=C_p/C_V$ is the ratio of the heat capacities C_p and C_V of the working fluid at constant pressure and constant volume, respectively. In practice, a desired polytropic process can be achieved by controlling some of state variables, for instance the volume or the pressure.

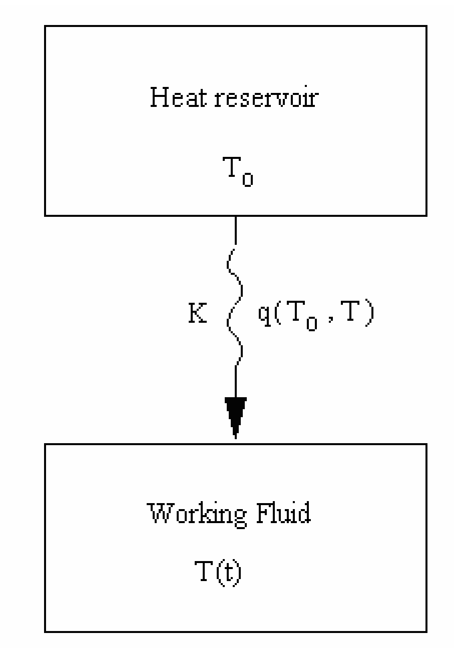

Figure 1. The heat flow q from a heat reservoir at constant temperature T_o to a working fluid at temperature T(t) is governed by a linear heat transfer law $q=K(T_o- T(t))$

If the working fluid is an ideal gas, the equations of state can be used to derive a relation between the heat flow q and the temperature change dT of the working fluid. This calculation is a standard textbook example (e.g. Granet 1965, Feidt 1996) and leads to the expression

$$\frac{dT(t)}{dt} = \frac{1}{C_n} q(T_0, T(t)) \quad (2)$$

where the polytropic heat capacity is defined as

$$C_n = C_V \frac{n-\gamma}{n-1} \quad (3)$$

for a polytropic process of degree n. C_n is independent of the fluid temperature and only depends on the type of process and the heat capacities of the fluid. Substituting the heat transfer law (1) into equation (2) and solving the resulting differential equation, we find the fluid temperature

$$T(t) = T_0 - [T_0 - T_1] \exp\left(-\frac{tK}{C_n}\right) \quad (4)$$

as a function of time t for a given initial fluid temperature $T_1 := T(0)$. The fluid temperature T(t) is monotonically approaching the temperature T_0 of the heat reservoir but never reaches T_0. This implies that the heat flow q does not change sign during the polytropic process but is directed either from the reservoir to the working fluid or in the opposite direction.

The total amount of heat Q exchanged during a polytropic process of duration τ is obtained by substituting equation (4) into the heat transfer law (1) and integrating over the time t from 0 to τ:

$$Q(\tau) = C_n \left[1 - \exp\left(-\frac{\tau K}{C_n}\right)\right] [T_0 - T_1] \quad (5)$$

Using equation (4) it is easily verified that the above expression coincides with the formula $Q = C_n(T_2 - T_1)$ for the heat transfered in a reversible polytropic process where Q only depends on the initial temperature T_1 and final temperature $T_2 := T(\tau)$ of the fluid (Granet 1965).

An important process variable is the change of the fluid entropy during the polytropic process. After substituting equation (4) into the expression for the rate of increase of the fluid's entropy,

$$s(t) = \frac{q(T_0, T(t))}{T(t)} = K\left(\frac{T_0}{T(t)} - 1\right) \quad (6)$$

the total entropy change $\Delta S(\tau)$ of the working fluid during the process is calculated by integrating equation (6):

$$\Delta S(\tau) = C_n \ln\left[\frac{1}{T_1}\left(T_0 - (T_0 - T_1)\exp\left(-\frac{\tau K}{C_n}\right)\right)\right] \quad (7)$$

The same way as for exchanged heat allows one to show that the above formula is equivalent to the expression $\Delta S = C_n \ln(T_2/T_1)$ for the entropy change of an ideal gas in a reversible polytropic process (Granet 1965).

The special case of an isothermal process is obtained in the limit of $C_n \to \infty$ where the equations for the heat (5) and entropy change (7) take the form

$$Q_{isotherm}(\tau) = \tau K (T_0 - T_1), \qquad \Delta S_{isotherm}(\tau) = \frac{\tau K}{T_1}(T_0 - T_1). \quad (8)$$

Isentropicprocesses, where $C_n \to 0$, are not considered here since no heat is transfered during such a process. Moreover, adiabats are often assumed to take place instantaneously.

3. Work Characteristic of the Engine

The thermodynamic cycle of the endoreversible engine investigated in the following is made up of two polytropic and two isentropic branches, like the example depicted in *Figure 2*. The goal of this section is to derive the performance characteristic of the engine, i.e. its work output as a function of some independent control parameters.

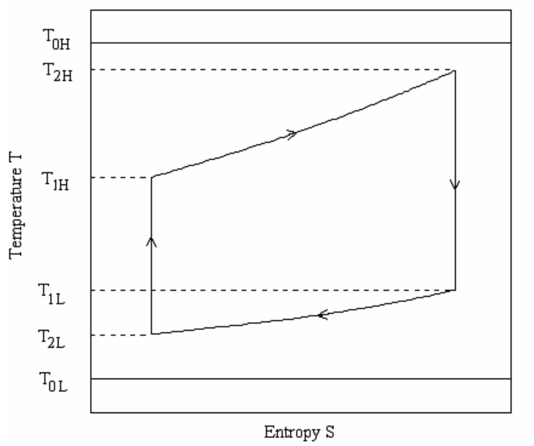

Figure 2. TS-diagram of an engine cycle with two polytropic and two adiabatic branches. The working fluid receives heat from a heat reservoir at constant temperature T_{0H} during the upper polytrope and changes its temperature from T_{1H} to T_{2H}. On the lower polytrope the working fluid releases heat to a reservoir at constant temperature T_{0L} and changes its temperature from T_{1L} to T_{2L}

3.1. Assumptions and basic relations

The engine operates between heat reservoirs at constant high and low temperatures T_{0H} and T_{0L}, respectively. The suffixes H and L indicate if a quantity belongs to the upper or the lower polytropic heat transfer process. During the heat transfer processes the entire working fluid is in thermal contact with exactly one of the two heat reservoirs and the heats Q_H and Q_L are transferred to the working fluid. The respective durations of the heat transfer processes are τ_H and τ_L. A further assumption is that the only energy transfer occurs during the polytropes and no energy is lost by heat leaks. This is in fact the assumption of endoreversibility which allows to use simple energy conservation for the work W delivered by the engine during one cycle:

$$W = Q_H + Q_L \tag{9}$$

Note that the heat Q_L is negative since heat flows out of the working fluid. Section 2 provides expressions for the total amounts of heat transfered (see equation (5)):

$$\begin{aligned} Q_H &= C_H(1-e_H)(T_{0H}-T_{1H}), \\ Q_L &= C_L(1-e_L)(T_{0L}-T_{1L}), \end{aligned} \tag{10}$$

and for the final temperatures (see equation (4)):

$$\begin{aligned} T_{2H} &= T_{0H} - (T_{0H}-T_{1H})e_H, \\ T_{2L} &= T_{0L} - (T_{0L}-T_{1L})e_L, \end{aligned} \tag{11}$$

of the working fluid in each of the polytropes. Here and in the following we are using

$$e_H = \exp\left(-\frac{\tau_H K_H}{C_H}\right),\ e_L = \exp\left(-\frac{\tau_L K_L}{C_L}\right) \tag{12}$$

as convenient abbreviations for the exponential functions to shorten the notation. Note that C_H and C_L are the *polytropic* heat capacities within the respective branches.

The heat transfer processes between the reservoirs and the working fluid are irreversible while the working fluid itself undergoes reversible processes. The total entropy change ΔS_{cycle} of the working fluid during a cycle is zero. Since the working fluid's entropy remains constant during the isentrops, we only need to consider the entropy changes ΔS_H and ΔS_L during the polytropes and thus have

$$\Delta S_H + \Delta S_L = \Delta S_{cycle} = 0 \tag{13}$$

Substituting the entropy expression (7) for the appropriate branches into equation (13) and taking the exponential yields

$$\left[\frac{T_{0H}-e_H(T_{0H}-T_{1H})}{T_{1H}}\right]^{C_H}\left[\frac{T_{0L}-e_L(T_{0L}-T_{1L})}{T_{1L}}\right]^{C_L} = 1 \tag{14}$$

3.2. The case of equal heat capacities

Equation (14) can not easily be treated analytically if C_H and C_L have different values since its transformations generally produces terms with rational exponents which render closed analytical solutions impossible. In order to continue the calculations we assume identical polytropic heat capacities

$$C := C_H \overset{!}{=} C_L \tag{15}$$

for both polytropes. Then equation (14) simplifies to

$$\frac{[T_{0H}-e_H(T_{0H}-T_{1H})][T_{0L}-e_L(T_{0L}-T_{1L})]}{T_{1H}T_{1L}} = 1 \tag{16}$$

An important property for the characterisation of a heat engine is its efficiency

$$\eta = \frac{W}{Q_H} \tag{17}$$

for which physically sensible values are between zero and the Carnot efficiency $\eta_C = 1 - T_{0L}/T_{0H}$.

Altogether we have 5 independent equations, (9), (10), (16) and (17) to eliminate four variables – two heats and two initial temperatures – and derive the desired work characteristic. Solving equations (10) for the initial temperature gives

$$T_{1H} = T_{0H} - \frac{Q_H}{C(1-e_H)} \tag{18}$$

and an analogous equation for T_{1L}. By substituting these expressions into equation (16), T_{1H} and T_{1L} are eliminated, and we obtain

$$\frac{\left[T_{0H} - \frac{e_H Q_H}{C(1-e_H)}\right]\left[T_{0L} - \frac{e_L Q_L}{C(1-e_L)}\right]}{\left[T_{0H} - \frac{Q_H}{C(1-e_H)}\right]\left[T_{0L} - \frac{Q_L}{C(1-e_L)}\right]} = 1 \tag{19}$$

which is further simplified to

$$T_{0L}Q_H + T_{0H}Q_L - \frac{Q_H Q_L(1-e_H e_L)}{C(1-e_H)(1-e_L)} = 0 \tag{20}$$

Combining equations (9) and (17) yields expressions for the heats,

$$Q_H = \frac{W}{\eta},\quad Q_L = \frac{W(\eta-1)}{\eta} \tag{21}$$

which are subsequently inserted into equation (20) to obtain

$$\begin{aligned} T_{0L}\frac{W}{\eta} + T_{0H}\frac{W(\eta-1)}{\eta} &- \\ -\frac{W^2(\eta-1)(1-e_H e_L)}{\eta^2 C(1-e_H)(1-e_L)} &= 0 \end{aligned} \tag{22}$$

The first solution of this quadratic equation is a trivial one, W=0, while the second solution is the desired work characteristic

$$W(\eta, e_H, e_L) = \\ = C\frac{\eta[(1-\eta)T_{0H} - T_{0L}]}{1-\eta}\frac{(1-e_H)(1-e_L)}{1-e_H e_L} \quad (23)$$

This fundamental relation is a function of three independent control parameters which are sufficient to describe all possible modes of operation of the investigated endoreversible engine. Our choice of the control parameters was guided by practical considerations. The efficiency η of the engine is certainly important. The design of the heat engine can be characterised by the heat conductances K_H and K_L of the heat exchangers, the branch times τ_H and τ_L, and the polytropic heat capacity C of the working fluid. These design parameters are present in the control parameters e_H and e_L as defined in equations (12).

4. Optimization for Maximal Work Output.

The optimality conditions for maximum work output can be derived by taking the respective derivatives of the work characteristic (23). As a consequence of an appropriate choice of independent control parameters, the work characteristic (23) nicely separates into a term with the efficiency η and a term with the design parameters e_L and e_H. A similar factorisation of the work characteristic has also been found for the optimal endoreversible Carnot heat engine (Gordon 1990)). The factorisation immediately implies that the efficiency of the optimized heat engine is independent of the design parameters in e_L and e_H.

4.1. Efficiency at maximum work output

The work W in equation (23) is zero for η=0 and $\eta=1-T_{0L}/T_{0H}$ and positive for intermediate values of η. The optimality condition $\partial W/\partial\eta=0$ for the efficiency η yields a quadratic equation

$$\frac{T_{0H}}{T_{0L}}(1-\eta)^2 - 1 = 0 \quad (24)$$

which has two solutions of which one solution is unphysical because of the restriction $\eta\leq\eta_C$. The second solution

$$\eta^* = 1 - \sqrt{\frac{T_{0L}}{T_{0H}}} \quad (25)$$

is equal to the well-known Chambadal-Novikov-Curzon-Ahlborn efficiency which originally was obtained for endoreversible Carnot engines by Chambadal (1958), Novikov (1957), and Curzon and Ahlborn (1975). It is quite remarkable that the efficiency η^* at maximum work output is a function of the reservoir temperatures only and does not depend on the details of the finite-rate heat transfer, i.e. the branch times and the conductances. Most important, η^* does not depend on the polytropic heat capacity. Neither the heat capacity of the working fluid nor the degree of the polytropic heat transfer processes have any influence on the efficiency of the work optimized, endoreversible heat engine, if the polytropic heat capacities are equal in both branches. This does not mean that the maximum of the work is the same for different cycles, but it means that if both processes of heat input and output are the same (cycles with two isotherms or two isochors etc.) then the efficiency corre-sponding to these points of maximum work are the same. It is remarkable that the efficiency expression (25) has also been found in a study of quasi-static engine cycles by Landsberg and Leff (1989) where the finite time character of heat transfer was neglected.

4.2. Optimal allocation of branch times

This section analyses the influence of the second factor in equation (23) on the work characteristics of the engine. The goal is to find optimal branch times τ^*_H and τ^*_L which yield maximum work output of the engine in case of a given total duration

$$\tau_{tot} = \tau_H + \tau_L \quad (26)$$

of the two heat transfer branches. Then a necessary optimality condition of the problem is $\partial W / \partial\tau_H = 0$ which leads to

$$(1-e_H e_L)[e'_H(1-e_L)+(1-e_H)e'_L] - \\ -(1-e_H)(1-e_L)(e'_H e_L - e_H e'_L) = 0 \quad (27)$$

where $e'_H = \partial e_H/\partial\tau_H$ and $e'_L = \partial e_L/\partial\tau_H$. The derivatives are calculated from the definitions of e'_H and e'_L as (12):

$$e'_H = -e_H\frac{K_H}{C}, \quad e'_L = -e_L\frac{K_L}{C} \quad (28)$$

and substituted into (27) which is further simplified to

$$\frac{(1-e_H)^2}{K_H e_H} = \frac{(1-e_L)^2}{K_L e_L} \quad (29)$$

The denominators and squared terms on both sides of this equation are positive. Taking the square root results in

$$\frac{\sinh\left(\frac{\tau_H K_H}{2C}\right)}{\sqrt{K_H}} = \frac{\sinh\left(\frac{(\tau_{tot}-\tau_H)K_L}{2C}\right)}{\sqrt{K_L}} \quad (30)$$

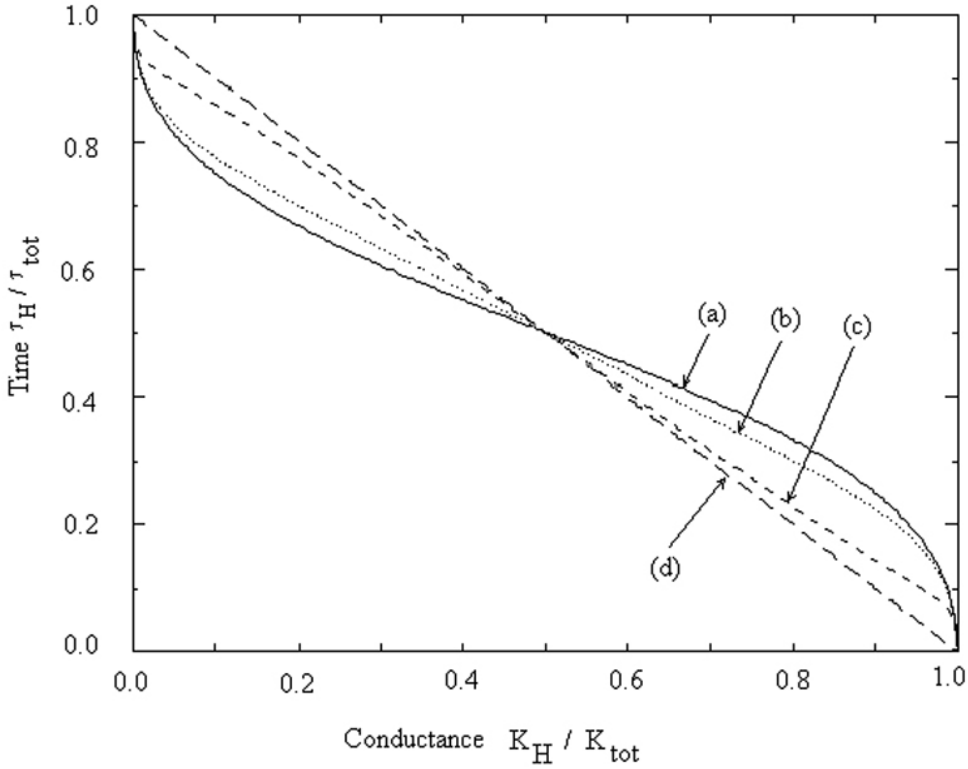

Figure 3. Optimal fraction of the branch time τ_H/τ_{tot} spent on an upper polytropic branch versus the fraction of the conductances K_H/K_{tot} on the upper polytrope plotted for different values of the (scaled) polytropic heat capacities $C/(\tau_{tot}K_{tot})$: (a) ∞ (isothermal limit), (b) 0.1, (c) 0.02 and (d) 0 (adiabatic limit)

Similar relations were obtained in a different context by Feidt (1987, 1988). Unfortunately, the optimality condition (30) is a transcendent equation and generally has to be solved numerically.

The heat conductances depend on areas of heat transfer surface: $K_i = U_i A_i$ ($i \in \{H, L\}$). Let us assume that $U_H = U_L = U$ and the total heat transfer area is restricted: $A_H + A_L = A_{tot}$. Then the maximal possible heat conductance at both sides of the engine is $K_{tot} = U A_{tot}$.

Figure 3 depicts the optimal branch time τ_H/τ_{tot} versus the heat conductance K_H/K_{tot}. The curves have been obtained by numerical solving (30) for different values of the polytropic heat capacity C. The results confirm the intuitive idea that a small heat conductance has to be compensated by a large branch time and vice versa. The curves are strictly monotonic and become linear in the unphysical limit of isentropic heat transfer branches as can bee seen from equation (30) by taking the limit $C \to 0$.

4.3. Maximal power versus polytropic heat capacity

The maximum work output at a given polytropic heat capacity C can be obtained by substituting the optimal values η^* from (25) and τ^*_H from equation (30) and $\tau^*_L = \tau_{tot} - \tau^*_H$ into the expression (23) as

$$W^* = C\left(\sqrt{T_{0H}} - \sqrt{T_{0L}}\right)^2 \cdot \frac{\left(1 - e_H\left(\tau_H^*\right)\right)\left(1 - e_L\left(\tau_L^*\right)\right)}{1 - e_H\left(\tau_H^*\right)e_L\left(\tau_L^*\right)} \tag{31}$$

Let us take a closer look at the case of equal thermal conductances $K_H = K_L = K_{tot}/2$. Then it is clear from equation (30) that $\tau^*_H = \tau_{tot}/2$ thus that the maximum work output (31) simplifies to

$$W^*_{(K_H = K_L)} = C\left(\sqrt{T_{0H}} - \sqrt{T_{0L}}\right)^2 \frac{1 - \exp\left(-\frac{\tau_{tot}K_{tot}}{4C}\right)}{1 + \exp\left(-\frac{\tau_{tot}K_{tot}}{4C}\right)} \tag{32}$$

The limit of isothermal heat transfer is derived for $C \to \infty$ thus that the exponential function in the above equation can be replaced by a power series expansion, $e^* = 1 - \tau_{tot}K_{tot}/(4\,C) + O(1/C)^2$ and simplifies to

$$W^*_{\left(\substack{K_H = K_L \\ \text{isothermal}}\right)} = \frac{\tau_{tot}K_{tot}}{8}\left(\sqrt{T_{0H}} - \sqrt{T_{0L}}\right)^2 \tag{33}$$

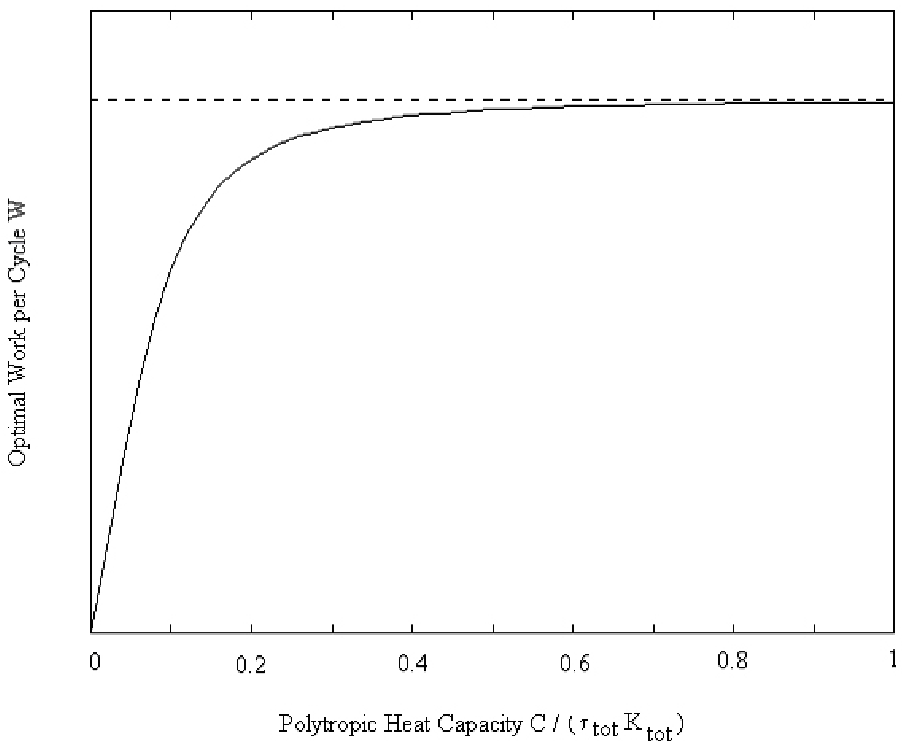

Figure 4. The optimal work output W^* is a monotonic increasing function of the (scaled) polytropic heat capacity $C/(\tau_{tot}K_{tot})$. The dashed line corresponds to the limit of isothermal branches where $C \to \infty$

Figure 4 shows a plot of the optimal work W* versus a scaled polytropic heat capacity for the case of equal thermal conductances. The graph is independent of the actual values of τ_{tot} and K_{tot}. The optimal work W* is a monotonically increasing function of the polytropic heat capacity C. The work output W* vanishes in the limit of adiabatic branches ($C \to 0$) where the heat transfer ceases. Maximum work output is achieved for $C \to \infty$. This findings are consistent with results obtained elsewhere by Rubin (1979), Gutkowicz-Krusin et al. (1978), Salamon et al. (1980), Pathria et al. (1996) where isothermal heat transfer branches were found to yield an optimal performance of endoreversible heat engines. Curves for unequal conductances qualitatively look the same as in *Figure 4* and are obtained by numerically solving equation (30) and inserting the resulting τ_H^* into the expression (31) for the optimised work output.

5. Cycles with Arbitrary Polytropic Heat Capacities

Engine cycles with arbitrary, non-equal polytropic heat capacities C_H and C_L are not easily treated analytically. The above analytical calculations have been restricted to cases where C_H=C_L because of the form of the entropy balance equation (14). This equation contains terms with rational and non-rational powers of the polytropic heat capacities. Even though analytical solutions might not be feasible one still can optimize the power output of the heat engine using a numerical scheme.

5.1. Numerical optimisation scheme

The numerical optimization scheme is based on equation (9) for the work output, which is the optimisation objective, equation (10) for the heat flows, and the entropy balance equation (13). The initial parameters of the optimisation problem are the temperatures of the heat reservoirs T_{0H} and T_{0L}, the sum of branch times τ_{tot}, the heat conductances K_H, K_L and the polytropic heat capacities C_H and C_L. The control parameters are the temperatures at the begin of the upper polytropic branch T_{1H} and the upper branch time τ_H. The objective of the optimisation is to adjust the controls such that both the physical constraints (namely entropy and energy balance) are fulfilled and the work output of the engine is maximised.

A downhill simplex method for multi-dimensions (Press et al. (1992), pp. 408-412) and a root finding bisection method (Press et al. (1992), pp. 350-354) are employed to solve the optimisation problem.

5.2. Results of numerical solution

Numerical optimisations of the heat engine are presented for the case of K_H=K_L=K_{tot}/2 and temperatures of the upper and lower heat sources of T_{0H} = 1200 K and T_{0L} = 300 K, respectively,

Figure 5 depicts the optimised work output versus the polytropic heat capacities. The work output of the optimized cycles is a monotonically increasing function of the (positive) value of both polytropic heat capacities. The scaled work output reaches its maximum value of $W^* \approx 37.5\ \tau_{tot}\ K_{tot}$ in the limit of two isothermal branches where C_H, $C_L \to \infty$. It remains at a high level for C_H, $C_L > 0.1\tau_{tot}K_{tot}$ and rapidly decreases if one of the polytropic heat capacities is below $0.1\tau_{tot}K_{tot}$. The more isothermal the heat transfer branches the better is the optimised work output.

A plot of the efficiency versus the polytropic heat capacities is displayed in *Figure 6*. Note that compared to *Figure 5* the directions of the $C_H/(\tau_{tot}\ K_{tot})$ and $C_L/(\tau_{tot}\ K_{tot})$ axis have been reversed to give a better display. *Figure 6* shows that the efficiency η^* of the maximum work cycle is equal to the Curzon-Ahlborn efficiency $\eta_{CA} = 1 - \sqrt{T_{0L}/T_{0H}}$ only if both polytropic heat capacities are equal. For all other cases the efficiency deviates from the Curzon-Ahlborn efficiency. This finding is quite remarkable since the Curzon-Ahlborn efficiency has been reoccuring in numerous publications and studies of many different systems, especially in cases of linear heat transport laws. Our example shows that the Curzon-Ahlborn efficiency can not be used as an universal figure for heat engines optimised with respect to maximum work output although it still can serve as a rough estimate of the efficiency for such engines. The plot in *Figure 5* further suggest that one should try to allocate the more isothermal-like heat transfer branch to the cold reservoir in order to improve the efficiency of an engine.

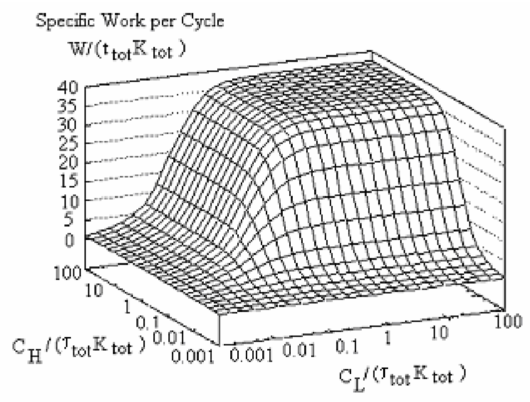

Figure 5. Optimal scaled work output $W/(\tau_{tot}K_{tot})$ versus the scaled polytropic heat capacities $C_H/(\tau_{tot}K_{tot})$ and $C_L/(\tau_{tot}K_{tot})$ for the case of equal heat conductances $K_H=K_L=K_{tot}/2$ and T_{0H} =1200 and T_L=300

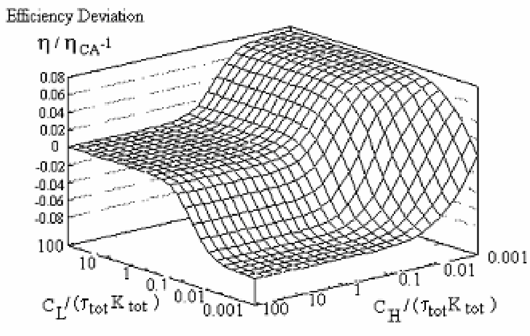

Figure 6. Deviation of the efficiency η from the Curzon-Ahlborn efficiency η_{CA}=1-$(T_{0L}/T_{0H})^{1/2}$ versus the scaled polytropic heat capacities. All other parameters of the work optimized engine are the same as in Figure 5

6. Conclusions

The presented finite-time analysis of endoreversible engines with polytropic heat transfer branches includes technologically important cycles like the Brayton cycle (two isobars), the Otto cycle (two isometrics) or the Diesel cycle (one isobar, on isometric) as special cases. The analysis shows how the times of the heat transfer processes need to be allocated to maximise the work output of the engine and confirms that the work output of the heat engine is improved for large values of the polytropic heat capacities.

If the polytropic heat capacities are equal in both branches, analytic expressions for the maximal work output are found and the efficiency of the optimised engines is equal to the Curzon-Ahlborn efficiency η_{CA} and is independent of material parameters and details of the heat transfer processes.

The optimisation in case of unequal polytropic heat capacities is performed using a numerical scheme. The resulting efficiency generally diviates from the Curzon-Ahlborn efficiency. The efficiency of the work optimized engine is increased beyond the Curzon-Ahlborn efficiency if the polytropic heat capacity in the lower heat transfer branch is larger than in the upper branch.

The findings of this study provide basic principles for the optimal design of generalised class of heat engines and may serve as a basis for models at higher levels of sophistication.

Acknowledgements

We would like to thank several institutions for their financial support which allowed us to complete this work. We acknowledge funding by

INTAS, Grant No.93-12-81. J. M. Burzler additionally thanks the Studienstiftung Cusanuswerk for a scholarship. The authors thank Bjarne Andresen for valuable remarks on the manuscript and Sven Schubert for helpful discussions.

Nomenclature

A	Heat transfer surface
C	Heat capacity of the fluid during the polytropic process
C_p	Heat capacity at constant pressure
C_V	Heat capacity at constant volume
e	Dimensionless parameter
K	Thermal conductance
n	Degree of the polytropic process
p	Pressure of the fluid
q	Heat flow
Q	Total amount of heat transferred during the process
s	The rate of increase of the fluid's entropy
t	Time
T(t)	Temperature of the fluid as function of time
T_0	Temperature of a source
T_1	Temperature of the fluid at the beginning of the process
T_2	Temperature of the fluid at the end of the process
V	Volume of the fluid
W	Work output
γ	Ratio of heat capacities
ΔS	Total change of the fluid's entropy during the process
η	Efficiency
τ	Duration of the process

Subscripts

H	In contact with the hot source
L	In contct with the cold source
tot	Total

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