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Optimal Thermostatting

A. M. TSIRLIN, D. A. ANDREEV and V. A. MOGUTOV
Program System Institute, Russian Academy of Science,
Botic, Perejaslavl-Zalesky, Russia 152140
E-mail: tsirlin@sarc.botik.ru

V. KAZAKOV*
School of Finance and Economics, Faculty of Business,
University of Technology, Sydney, PO Box 123,
Broadway NSW 2007, Australia.
E-mail: kaz@arch.usyd.edu.au

Abstract

In this paper the problem of energy-optimal heating/cooling a building is considered. Here the given subset of rooms in a building must have given temperatures. It is proven that if heat is supplied from a single heat source then it is optimal to supply it only to the rooms with given temperatures. If individual heat sources (separate air-conditioners/heat pumps in each room) are used then it is more efficient to supply/remove heat to the target rooms and also to intermediate rooms with non-fixed temperatures.

Key words: temperature control, minimum energy consumption, minimum exergy consumption, optimum control

1. Introduction

It is known that if an open thermodynamic system is in a steady state then such a distribution of thermodynamic potentials is established in it that entropy production is minimal. One of the problems considered by thermodynamics of open systems is estimation of the amount of energy required to establish a given distribution of potentials that differs from the equilibrium one. In this paper we consider one particular case of this problem, which is relatively simple but has important applications. Here a given discrete distribution of temperatures has to be established. This problem arises when minimal energy required for the thermostatting of a building needs to be estimated. Such estimate and the corresponding optimal distribution of the energy fluxes in the building allow us to calculate potential energy savings by establishing fixed temperatures only in a part of the building.

This paper uses the methods of finite-time thermodynamics, developed during the last two decades, (see for example Rozonoer and Tsirlin 1983, Bejan 1996).

Consider a building and assume that it is necessary to establish fixed temperatures in some of its rooms only (we shall call them target rooms). The temperatures in other (intermediate or passive) rooms are allowed to attain any value. The rooms that are target rooms and their temperatures may vary, depending on the season and on the time of day. We consider two versions of the problem of minimal energy consumption for heating/cooling of such a building.

Problem (A). A single source heating/cooling system is used for the whole building (one air-conditioner/heat pump or a direct supply of heat via electric, gas, water or air heating). Energy consumption here is a unique function of the sum of heat fluxes to all rooms. Therefore minimization of the energy consumption is equivalent to minimization of this combined heat flux.

Problem (B). Each room has a separate air-conditioner/heat pump. That is, each room has an individual heat source with a separate temperature.

Unlike Problem A, minimization of energy consumption (exergy losses) in Problem B is not

* Author to whom all correspondence should be addressed.

equivalent to minimization of combined heat fluxes.

We will show that for any law of heat transfer the optimal heating/cooling in Problem (A) is achieved by transferring heat to target rooms only.

We will also show that the most energy efficient way to thermostat the building in Problem B is by also supplying/removing some of the heat into intermediate rooms.

A similar problem arises in cryogenics, where the objective is to establish a pre-set low temperature in a chamber using heat pumps. It is known that for some laws of heat transfer, it is more efficient here to use so-called active insulation. It includes an “onion ring” of chambers embedding each other, where some part of the heat is removed from the central thermostatted chamber and some parts from each intermediate chamber. The temperatures in intermediate chambers are set lower than the temperature of the environment but higher than the temperature of the thermostatted chamber. The active insulation problem was first considered in Martinovskii (1979), Sertorio (1991) and then generalized in Tsirlin et al. (1998). In Tsirlin et al. (1998) it was shown for which laws of heat transfer active insulation leads to energy savings.

2. Problem Formulation

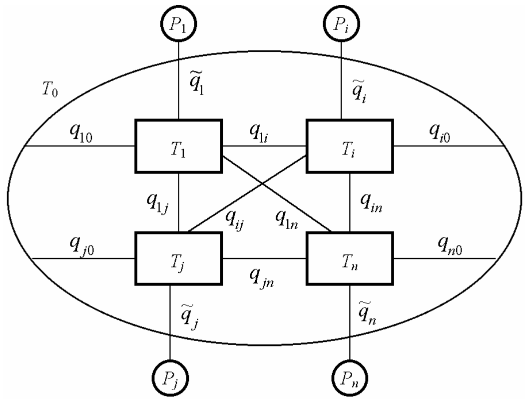

Figure 1. General structure of a building

Consider the building whose structure is shown in *Figure 1*, where the following notations are used:

T_i – is the temperature of the i-th room (i=0,1,…,n) [K];

$\alpha_{ij}(T_i, T_j)$ – is the heat transfer coefficient between i-th and j-th rooms, which can depend on the temperatures in these rooms ($\alpha_{ji} = \alpha_{ij} \geq 0$), [W/K];

$q_{ij} = \alpha_{ij}(T_i, T_j)(T_j - T_i)$ – is the heat flux from the j-th room to the i-th room, [W];

$q_{i0} = \alpha_{i0}(T_i, T_0)(T_0 - T_i)$ – is the heat flux to the i-th room from the environment with the temperature T_0, [W];

$\tilde{q}_i$ – is the heat flux, supplied (removed) to/from i-th room, [W]. We assume that the sign of this flux is positive if the heat is supplied to the i-th room. P_i is the power that runs the air-conditioner/heat pump in the i-th room.

Problem formulation: Assume that the temperatures of m rooms $T_1,\ldots T_m$ (m<n) and the temperature of the environment T_0 are fixed. It is required to find such heat fluxes $\tilde{q}_i$ (i=1,…,n) so that the total amount of heat supplied (for Problem A) or the combined power used to drive heat pumps and refrigerators (for Problem B) is minimal.

3. Thermostatting Using a Single Heat Source (Optimal Distribution of Energy)

Let us write down formally the problem of minimization of total heat supplied. This problem arises when the heating system is designed for a building in a set of rooms where the temperatures are required to be fixed as well as the temperature of the environment T_0 changes during different seasons and/or during different times of the day.

The optimality criterion here is:

$$I_A = \sum_{i=1}^{n} \tilde{q}_i \to \min \tag{1}$$

subject to the heat balance:

$$\sum_{j=1}^{n} q_{ij}(T_i, T_j) + \tilde{q}_i = 0, \qquad i = 1, \ldots, n, \tag{2}$$

constraints on the heat fluxes:

$$\tilde{q}_i \geq 0, \qquad i = 1, \ldots, n, \tag{3}$$

and constraints imposed on the temperatures of the thermostatted rooms:

$$T_i = T_i^0 > T_0, \qquad i = 0, \ldots, m. \tag{4}$$

This problem can be simplified by eliminating condition (2) and rewriting the objective function as:

$$I_A = \sum_{i=0}^{n} \sum_{j=0}^{n} q_{ij}(T_i, T_j) \to \max \tag{5}$$

subject to constraints:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) \leq 0, \qquad i = 1, \ldots, n. \tag{6}$$

The unknown variables in this problem are the temperatures of the intermediate room T_i (i=m+1,…,n).

Let us write down the Lagrange function of the problem (5), (6):

$$L = \sum_{i=0}^{n} (1+\lambda_i) \sum_{j=0}^{n} q_{ij}(T_i, T_j) \qquad (7)$$

Its optimality conditions follow the Kuhn-Tucker theorem:

$$\frac{\partial L}{\partial T_i} = (1+\lambda_i) \sum_{j=0}^{n} \frac{\partial q_{ij}(T_i, T_j)}{\partial T_i} = 0, \qquad (8)$$
$$i = m+1, \ldots n,$$

$$\lambda_i \le 0, \qquad \sum_{i=0}^{n} \lambda_i \sum_{j=0}^{n} \frac{\partial q_{ij}(T_i, T_j)}{\partial T_i} = 0. \qquad (9)$$

From Slater's complementary slackness conditions (9) it follows that if $\lambda_i = 0$, then:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) < 0$$

and if $\lambda_i \le 0$ then:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) = 0$$

It is clear that any increase of temperature T_i of any of the rooms leads to the decrease of the heat flow that enters it. Therefore for all intermediate rooms:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) < 0, \qquad i=m+1,\ldots n$$

From conditions (8) it follows that for these rooms $(1+\lambda_i) = 0$, that is, $\lambda_i = -1$ (i=m+1,…n). From Slater's complementary slackness conditions (9), it follows that on the optimal solution:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) = 0, \qquad i = m+1, \ldots n.$$

In other words, if the solution is optimal, then all the heat flows that enter intermediate rooms must equal zero.

The optimal values of heat fluxes $\tilde{q}_i$ (i=1,…,m) are uniquely determined by the heat balance equation (2) which takes the following form:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) + \tilde{q}_i = 0, \qquad i = 1, \ldots, m \qquad (10)$$

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) = 0, \qquad i = m+1, \ldots, n \qquad (11)$$

$$T_i = T_i^0, \qquad i = 0, \ldots, m. \qquad (12)$$

The conditions (10)-(12) allow us to find the fluxes $\tilde{q}_i$ ($i = 1, \ldots, m$) and (n-m) temperatures in intermediate rooms.

For Newton's (linear) law of heat transfer, the heat transfer coefficients α_{ij} are constant and the problem (4)-(6) becomes the linear programming problem, and the conditions (10)-(11) turn out to be the set of (n-m) linear equations. The solution of this set of equations completely determines the optimal values of fluxes $\tilde{q}_i$. If one of the fluxes $\tilde{q}_i$ turns out to be negative then no optimal solution exists for the original heating problem (A). The optimal solution with the given set of temperatures in the target rooms can be guaranteed only if an air-conditioner/heat pump is used for heating.

4. Example 1

Consider the building shown in *Figure 1*. The corresponding computational schematic structure is shown in *Figure 2*. The temperature of the environment T_0 and the room temperatures T_1 and T_2 are given and equal to 20°C, 18°C and 20°C, correspondingly. Heat transfer coefficients between the rooms and the environment are shown in TABLE I. It is required to determine the amount of supplied heat $\tilde{q}_1$ and $\tilde{q}_2$ and the temperatures in the non-thermostatted rooms T_3, T_4, T_5, T_6.

TABLE I. THE HEAT TRANSFER COEFFICIENTS, α_{ij} [W/K]

i,j	0	1	2	3	4	5	6
0		16.8	84	16.8	0	33.6	50.4
1	16.8		0	0	33.6	33.6	33.6
2	84	0		33.6	33.6	0	33.6
3	16.8	0	33.6		33.6	33.6	0
4	0	33.6	33.6	33.6		0	33.6
5	33.6	33.6	0	33.6	0		0
6	50.4	33.6	33.6	0	33.6	0	

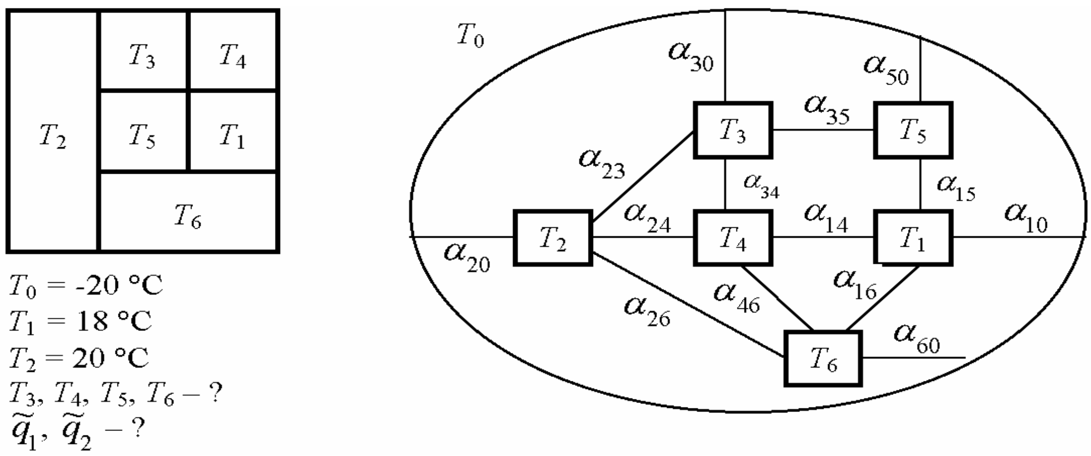

Figure 2. A fragment of building plan (a). Computational schema of heat transfer in this fragment (b).

The equations (10)-(12) yield the set of heat balance equations

$$q_{10}(T_1,T_0)+q_{14}(T_1,T_4)+q_{15}(T_1,T_5)$$
$$+q_{16}(T_1,T_6)+\tilde{q}_1=0,$$
$$q_{20}(T_2,T_0)+q_{23}(T_2,T_3)+q_{24}(T_2,T_4)$$
$$+q_{26}(T_2,T_6)+\tilde{q}_2=0,$$
$$q_{30}(T_1,T_0)+q_{32}(T_3,T_2)+q_{34}(T_3,T_4)$$
$$+q_{26}(T_2,T_6)=0,$$
$$q_{41}(T_4,T_1)+q_{42}(T_4,T_2)+q_{43}(T_4,T_3)$$
$$+q_{46}(T_4,T_6)=0,$$
$$q_{50}(T_5,T_0)+q_{51}(T_5,T_1)+q_{53}(T_5,T_3)=0,$$
$$q_{60}(T_6,T_0)+q_{61}(T_6,T_1)+q_{62}(T_6,T_2)$$
$$+q_{64}(T_6,T_4)=0.$$

Substitution of the given temperatures T_0, T_1 and T_2 yields the following results:

$$\tilde{q}_1=1832\text{ W};\ \tilde{q}_2=4579\text{ W};$$
$$T_3=6.8\text{ °C};\ T_4=12.3\text{ °C};\ T_5=1.6\text{ °C};\ T_6=4.5\text{ °C}.$$

4.1. Minimization of exergy losses for heating using individual room pumps/air conditioners/ heat pumps

The problem of minimization of the combined energy used by air-conditioners/heat pumps takes the following form:

$$I_B=\sum_{i=1}^{n}P_i\rightarrow\min \tag{13}$$

subject to conditions (2) and (4). We denote the efficiencies of heat pumps (coefficient of performance) as $r_i=\dfrac{\tilde{q}_i}{P_i}$. These efficiencies depend on the design of the pump (the heat transfer coefficients in the heater and refrigerator k_o and k_i), the form of the cycle, the temperatures on the hot and cold side of the cycles T_o and T_i and on the power used P_i. The reversible estimate of the heat efficiency of the heat engine does not depend on P_i.

$$r_i=\frac{T_i}{T_i-T_0} \tag{14}$$

Here and later on we measure temperatures in the Kelvin scale.

The more accurate lower estimate for the efficiency of a heat pump and a refrigerator cycle, which takes into account the irreversibility of heat transfer was obtained in Rozonoer and Tsirlin (1983) and Berry et al. (2000). For Newton's law of heat transfer with the heat transfer coefficient k_o for the heat removal from the environment and k_i for the heat supply into the room, this estimate for a heat pump has the following form (Berry et al. 2000):

$$r_i(T_0,T_i,P_i)=1+\frac{1}{2P_i}\cdot\left(\sqrt{P_i^2+\frac{k(T_i+T_0)}{2}P_i+\frac{k^2(T_i-T_0)^2}{16}}-P_i-\frac{k(T_i-T_0)}{4}\right) \tag{15}$$

here $\bar{k}_i=\dfrac{4k_ik_0}{(\sqrt{k_i}+\sqrt{k_0})^2}$ is the equivalent heat transfer coefficient.

For refrigerator $T_i<T_0$ and its efficiency $\breve{r}_i$ is expressed in terms of r_i defined in (16) as $\breve{r}_i=r_i(T_i,T_0,P_i)-1$. The equality (17) follows from the known relation between the efficiency of the refrigerating cycle and the efficiency of heat pump (Berry et al. 2000). In particular, for a reversible cycle:

$$\breve{r}_i^{\,0}=\frac{T_i}{T_i-T_0}=r_i^{\,0}-1$$

Let us rewrite condition (2) in the following form:

$$\sum_{j=0}^{n} q_{ij}(T_i, T_j) + P_i r_i(T_0, T_i, P_i) = 0, \quad i = 1,\dots,n \tag{16}$$

In the problem (13), (16), (4), the unknown variables are powers $P_i \geq 0$ ($i = 1,\dots,n$) and the temperatures of the intermediate rooms T_i ($i = m+1,\dots,n$).

If $\sum_{j=0}^{n} q_{ij} < 0,$

then the air-conditioner for the i-th room operates as a heat pump and its r_i has the form (15).

If $\sum_{j=0}^{n} q_{ij} > 0,$

then it operates as a refrigerator, with $T_i < T_0$. The efficiencies r_i in conditions (16) and all equations that follow them should be replaced with refrigerators' efficiencies:

$$\breve{r}_i = -r_i(T_i, T_0, P_i) - 1 \tag{17}$$

Note that the temperatures T_0 and T_i in equation (17) changed places.

The Lagrange function of the problem (13), (14), (4) has the form:

$$L = \sum_{i=1}^{1} \left\{ P_i \left[1 + \lambda_i r_i(T_0, T_i, P_i)\right] + \lambda_i \sum_{j=0}^{n} q_{ij}(T_i, T_j) \right\}$$

which yields the following optimality conditions:

$$\frac{\partial L}{\partial P_i} = 0 \rightarrow r_i(T_0, T_i, P_i) + P_i \frac{\partial r_i}{\partial P_i} = -\frac{1}{\lambda_i}, \quad i = 1,\dots,n \tag{18}$$

$$\frac{\partial L}{\partial T_\nu} = 0 \rightarrow P_\nu \frac{\partial r_\nu}{\partial T_\nu} + \sum_{j=0}^{n} \frac{\partial q_{\nu j}}{\partial T_\nu} + \sum_{\substack{i=1,\\ i\neq\nu}}^{n} \frac{\partial q_{i\nu}}{\partial T_\nu} = 0, \quad \nu = m+1,\dots,n \tag{19}$$

These conditions jointly with the conditions (16) and expressions (15) and (17) determine the unknown variables.

If a reversible efficiency estimate is used, then the problem is simplified and the system (16), (18), (19) leads to the following equations:

$$P_i = -\left(1 - \frac{T_0}{T_i}\right) \sum_{j=0}^{n} q_{ij}(T_i, T_j), \quad i = 1,\dots,n \tag{20}$$

$$\lambda_i = -1 + \frac{T_0}{T_i}, \qquad i = 1,\dots,n \tag{21}$$

$$\lambda_\nu \sum_{\nu=0}^{n} \frac{\partial q_{\nu j}}{\partial T_\nu} + \sum_{\substack{i=1,\\ i\neq\nu}}^{n} \frac{\partial q_{i\nu}}{\partial T_\nu} - P_\nu \lambda_\nu \frac{T_0}{(T_\nu - T_0)^2} = 0, \quad \nu = m+1,\dots,n \tag{22}$$

Thus the temperatures of the intermediate rooms are:

$$\frac{T_\nu}{T_0 - T_\nu} \sum_{j=0}^{n} \frac{\partial q_{\nu j}}{\partial T_{\nu i}} + \sum_{\substack{i=1,\\ i\neq\nu}}^{n} \frac{T_i - T_0}{T_i} \frac{\partial q_{i\nu}}{\partial T_\nu} + \frac{T_0}{{T_\nu}^2} \sum_{j=0}^{n} \frac{\partial q_{\nu i}}{\partial T_i} = 0 \tag{23}$$
$$\nu = m+1,\dots,n$$

This system of equations allows us to find all the temperatures because all the temperatures for $i \leq m$ are fixed (see (12)). After finding the temperatures the powers can be found from the conditions (20) for all $i = 1,\dots,n$.

5. Example 2

Consider the building shown in *Figure 3*. The temperatures are T_0=253 K and T_1=293 K and the heat transfer coefficients are:

$$K_0 = K_1 = K_2 = 3000 \frac{W}{K} \text{ and}$$

$$\alpha_{10} = \alpha_{20} = 94.08 \frac{W}{K} \text{ and}$$

$$\alpha_{12} = \alpha_{21} = 180 \frac{W}{K}.$$

It is required to find the temperature T_2 in the second room and the powers of heat pumps/refrigerators.

The problem of minimal energy used to drive heat pumps has the following form:

$$I = P_1 + P_2 \rightarrow \min$$

subject to heat balance

$$q_{10}(T_1, T_0) + q_{12}(T_1, T_2) + P_1 r_1(T_0, T_1, P_1) = 0$$

$$q_{20}(T_2, T_0) + q_{21}(T_2, T_1) + P_2 r_2(T_0, T_2, P_2) = 0$$

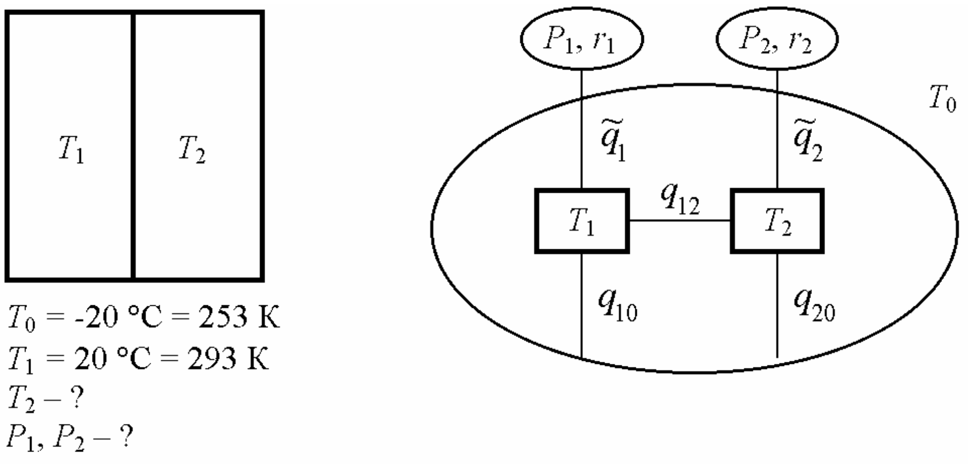

Figure 3. The plan and the computational structure of the building used in Example 2

Now power can be expressed in terms of T_2 as

$$P_1(T_2) = 1.6 \cdot \frac{848560349 - 4468950T_2 + 5625T_2^2}{76737 - 50T_2}$$

$$P_2(T_2) = 0.48 \cdot \frac{1769323311113 - 1313982242T_2 + 2436457T_2^2}{4267T_2 - 318926}$$

Thus, the optimality criterion I depends on T_2 only and attends its minimum at T_2=282 K.

Substitution of the obtained temperature T_2 into the expressions for the powers yields P_1= 910.36 W and P_2=79.32 W.

6. Conclusion

In this paper we demonstrated that if the building is heated from a single-temperature heat source (single air-conditioner/heat pump, electrical heating, heating using hot water/air, natural gas heating), then for any law of heat transfer it is most energy efficient to supply heat only into the set of rooms where the temperatures are fixed. The temperatures in the intermediate rooms are allowed to attain any value freely and are determined by the conditions of heat transfer.

If separate air-conditions/heat pumps are used for heating/cooling, then it is most efficient to use some power to establish optimal temperatures in the intermediate non-target set of rooms.

The obtained formulas allow us to find these temperatures and to estimate the lower bound on the total energy consumption for thermostatting of the building.

Acknowledgments

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