Hacettepe Journal of Mathematics and Statistics
Volume 43 (4) (2014), 625 – 634

Fuzzy Γ-ideals in Γ-AG-groupoids

Tariq Shah*, Inayatur-Rehman† and Asghar Khan‡

Abstract

In this paper we study fuzzy Γ-ideals and prime, semiprime fuzzy Γ-ideals of a Γ-AG-groupoid S. We prove that, if S is a Γ-AG-groupoid with left identity, then every fuzzy Γ-ideal of S is idempotent if and only if every fuzzy Γ-ideal of S is semiprime. We also show that, if S is a Γ-AG-groupoid with left identity e, then every fuzzy Γ-ideal of S is prime if and only if every fuzzy Γ-ideal of S is idempotent and the set of fuzzy Γ-ideals of S is totally ordered by inclusion.

Received 11/04/2011 : Accepted 09/04/2013

Keywords: Γ-AG-groupoids; Fuzzy sets; Fuzzy Γ-ideals; Prime fuzzy Γ-ideals; Band; Irreducible fuzzy Γ-ideals.

1. Introduction

In [17], Zadeh introduced the notion of a fuzzy subset f of a set S as a function from S into unit interval $[0, 1]$. The notion of a fuzzy ideal in Γ-rings was first introduced by Jun and Lee [5]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Dutta and Chanda [4], studied the structures of fuzzy ideals of a Γ-ring and characterized Γ-field and Noetherian Γ-ring. Jun [6] defined fuzzy prime ideal of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal.

Kazim and naseerudin [8], have introduced the concept of LA-semigroups (also known as AG-groupoids). An LA-semigroup S is a groupoid which satisfies the left invertive law $(ab)c = (cb)a$ for all $a, b, c \in S$. It is a midway structure between a commutative semigroup and a groupoid. It is a non-associative structure which has wide applications in the theory of flocks and in automata theory.

In 1981, the notion of Γ-semigroups was introduced by Sen [13, 14]. Let M and Γ be any nonempty sets. M is said to be a Γ-semigroup if there exists a mapping $M \times \Gamma \times M \longrightarrow M$, such that M satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$

*Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan
E-mail: stariqshah@gmail.com
†Department of Mathematics, COMSATS Institute of Information Technology Abbottabad, Pakistan
Corresponding author. E-mail: sinayat@ciit.net.pk
‡Department of Mathematics, Abdul Wali Khan University Mardan KPK, Pakistan
E-mail: azhar4set@yahoo.com

and $\alpha, \beta \in \Gamma$. Whereas the Γ-semigroups are a generalization of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups.

Recently, Shah and Rehman [15], have introduced the notion of Γ-AG-groupoids and discussed some properties of Γ-ideals and Γ-bi-ideals in Γ-AG-groupoids. Moreover in [16], they have discussed M-systems in Γ-AG-groupoids. Though Γ-AG-groupoid is a non-associative and non-commutative structure, but due to its peculiar characteristics, it possesses properties which we usually encounter in associative algebraic structures.

In this paper we define fuzzy Γ-ideals in a Γ-AG-groupoid and study some of its properties. We also define prime and semiprime fuzzy Γ-ideals in Γ-AG-groupoids and study those Γ-AG-groupoids in which each fuzzy Γ-ideal is (prime) semiprime.

2. Definitions and Preliminary Lemmas

In this section we recall certain definitions and results from [15], which are needed for our discussion.

• Let S and Γ be nonempty sets. We call S to be a Γ-AG-groupoid if there exists a mapping $S \times \Gamma \times S \longrightarrow S$, written as (a, γ, c) and denoted by $a\gamma c$ such that S satisfies the identity $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

• In a Γ-AG-groupoid S, $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$ for all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

• An element e of a Γ-AG-groupoid S is called left identity if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

• A non-empty subset A of a Γ-AG-groupoid S is called a sub Γ-AG-groupoid of S if $a\gamma b \in A$ for all $a, b \in S$ and $\gamma \in \Gamma$.

• A Γ-AG-groupoid S whose all elements are idempotent, that is $a\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$ is called a band.

In a Γ-AG-band the following are true

$$\begin{aligned} \circ\ (a\alpha b)\beta a &= a\alpha(b\beta a) \text{ for all } a, b \in S \text{ and } \alpha, \beta \in \Gamma. \\ \circ\ (a\alpha b)\beta c &= (a\alpha c)\beta(b\gamma c) \text{ for all } a, b, c \in S \text{ and } \alpha, \beta, \gamma \in \Gamma. \\ \circ\ (a\alpha b)\beta b &= b\alpha a \text{ for all } a, b \in S \text{ and } \alpha, \beta \in \Gamma. \end{aligned}$$

If S is Γ-AG-groupoid and $A, B \subseteq S$, then we denote $A\Gamma B := \{a\gamma b | a \in A, \gamma \in \Gamma, b \in B\}$.

A non-empty subset I of a Γ-AG-groupoid S is called a left (right) Γ-ideal of S if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). A non-empty subset I of a Γ-AG-groupoid S is called a Γ-ideal if it is both a left and a right Γ-ideal of S.

If S is a Γ-AG-groupoid with left identity e, then every right Γ-ideal of S is a left Γ-ideal of S [15, Proposition 1].

3. Fuzzy Concepts in Γ-AG-groupoids

3.1. Fuzzy Subset of Γ-AG-groupoids. A function f from a non empty set X to the unit interval $[0, 1]$ is called a fuzzy subset of S. For fuzzy subsets f, g of X, $f \subseteq g$ means that $f(x) \leq g(x)$ for all $x \in X$.

Let f be a fuzzy subset and $t \in (0, 1]$. Then the set

$$\mathbb{U}(f; t) := \{x \in X | f(x) \geq t\},$$

is called the level set of f.

If S is a Γ-AG groupoid and f, g are any fuzzy subsets of S. We define the product $f\Gamma g$ of f and g as follows:

$$(f\Gamma g)(x) := \begin{cases} \bigvee\limits_{x=y\gamma z} \min\{f(y), g(z)\} & \text{if } \exists\ x, y \in S \text{ and } \gamma \in \Gamma \text{ such that } x = y\gamma z, \\ 0 & \text{if } x \neq y\gamma z. \end{cases}$$

3.2. Definition. Let S be a Γ-AG- groupoid and $\emptyset \neq A \subseteq S$. Then the characteristic function χ_A of A is defined by:

$$\chi_A : S \longrightarrow [0,1], \longmapsto \chi_A(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases}$$

3.3. Definition. Let S be a Γ-AG-groupoid and f a fuzzy subset of S. Then f is called a fuzzy sub Γ-AG-groupoid of S if $f(x\gamma y) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$ and $\gamma \in \Gamma$.

3.4. Definition. Let S be a Γ-AG-groupoid and f a fuzzy subset of S. Then f is called a fuzzy left (right) Γ-ideal of S if $f(x\gamma y) \geq f(y)(f(x\gamma y) \geq f(x))$ for all $x, y \in S$ and $\gamma \in \Gamma$.

If f is both a fuzzy left Γ-ideal and a fuzzy right Γ-ideal of S. Then f is called a two-sided fuzzy Γ-ideal of S.

Note that if f is a fuzzy right Γ-ideal of a Γ-AG-groupoid S with left identity e. Then

$$f(x) = f(e\alpha x) \geq f(e) \text{ for all } x \in S \text{ and } \alpha \in \Gamma.$$

In the following, the proofs of Propositions 3.5, 3.6, 3.7 and 3.8 are obvious and are omitted.

3.5. Proposition. Let S be a Γ-AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is a sub Γ-AG-groupoid if and only if the characteristic function χ_A of A is a fuzzy sub Γ-AG-groupoid.

3.6. Proposition. Let S be a Γ-AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is a left (right) Γ-ideal of S if and only if the characteristic function χ_A of A is a fuzzy left (right) Γ-ideal of S.

3.7. Proposition. Let S be a Γ-AG-groupoid f a fuzzy subset of S. Then f is a fuzzy sub Γ-AG-groupoid if and only if $\mathbb{U}(f;t)(\neq \emptyset)$ is a sub Γ-AG groupoid for all $t \in (0,1]$.

3.8. Proposition. Let S be a Γ-AG-groupoid f a fuzzy subset of S. Then f if a fuzzy left (right) Γ-ideal of S if and only if $\mathbb{U}(f;t)(\neq \emptyset)$ is a left (right) Γ-ideal of S for all $t \in (0,1]$.

3.9. Example. Let $S = \{1,2,3,4,5\}$ and define a binary operation "$\cdot$" in S as follows:

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	5	3	4
4	1	1	4	5	3
5	1	1	3	4	5

Then $(S,\cdot)$ is an AG-groupoid. Now let $\Gamma = \{1\}$ and define the mapping $S \times \Gamma \times S \longrightarrow S$ by $a1b = ab$ for all $a, b \in S$. Then S is a Γ-AG-groupoid (see [15]) and $\{1\}, \{1,2\}, \{1,3,4,5\}$ and S are ideal of S.

Define $f : S \longrightarrow [0,1]$ by $f(1) = 0.9,\ f(2) = 0.8,\ f(3) = 0.5,\ f(4) = 0.5,\ f(5) = 0.5.$

$$\mathbb{U}(f;t) := \begin{cases} S & \text{if } t \in (0, 0.5] \\ \{1,2\} & \text{if } t \in (0.5, 0.8] \\ \{1\} & \text{if } t \in (0.8, 0.9] \\ \emptyset & \text{if } t \in (0.9, 1] \end{cases}$$

Then by Proposition 3.8, f is a fuzzy ideal of S.

3.10. Lemma. If S is a Γ-AG-groupoid with left identity e. Then every fuzzy right Γ-ideal of S is a fuzzy left Γ-ideal of S.

Proof. : Let S be a Γ-AG-groupoid with left identity e and f a fuzzy right Γ-ideal of S. Let $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} f(a\alpha b) &= f((e\gamma a)\alpha b) \text{ because } e \text{ is left identity} \\ &= f((b\gamma a)\alpha e) \text{ by left invertive law} \\ &\geq f(b\gamma a) \text{ because } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\geq f(b) \text{ because } f \text{ is a fuzzy right } \Gamma\text{-ideal.} \end{aligned}$$

Thus $f(a\alpha b) \geq f(b)$ for all $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Hence f is a fuzzy left Γ-ideal.

Note that the intersection and union of any family of fuzzy Γ-ideals of a Γ-AG-groupoid S is a fuzzy Γ-ideal of S. □

3.11. Proposition. If S is a Γ-AG-groupoid and f, g, h are fuzzy subsets of S. Then $(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f$.

Proof. Let $x \in S$ and $\alpha, \beta \in \Gamma$, then

$$\begin{aligned} ((f\Gamma g)\Gamma h)(x) &= \bigvee_{x = y\alpha z} \{(f\Gamma g)(y) \wedge h(z)\} \\ &= \bigvee_{x = y\alpha z} \left\{ \begin{array}{ll} \left(\bigvee_{y = a\beta b} (f(a) \wedge g(b)) \right) \wedge h(z) & \\ 0 & \text{if } y \neq a\beta b \end{array} \right\} \\ &= \bigvee_{x = (a\beta b)\alpha z} \{(f(a) \wedge g(b)) \wedge h(z)\} \end{aligned}$$

Since $x = (a\beta b)\alpha z = (z\beta b)\alpha a$, so $((f\Gamma g)\Gamma h)(x) = \bigvee_{x = (z\beta b)\alpha a} [(h(z) \wedge g(b)) \wedge f(a)]$. Also $h(z) \wedge g(b) \leq \bigvee_{z\alpha b = c\alpha d} \{h(c) \wedge g(d)\}$), so

$$\begin{aligned} ((f\Gamma g)\Gamma h)(x) &\leq \bigvee_{x = (z\beta b)\alpha a} \left[\left\{ \bigvee_{z\alpha b = c\alpha d} \{h(c) \wedge g(d)\} \right\} \wedge f(a) \right] \\ &= \bigvee_{x = (z\beta b)\alpha a} \{(h\Gamma g)(z\beta b) \wedge f(a)\} \\ &\leq \bigvee_{x = s\gamma t} \{(h\Gamma g)(s) \wedge f(t)\} \\ &= ((h\Gamma g)\Gamma f)(x). \end{aligned}$$

Hence $(f\Gamma g)\Gamma h \leq (h\Gamma g)\Gamma f$. Similarly, $(h\Gamma g)\Gamma f \leq (f\Gamma g)\Gamma h$. Thus $(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f$.

3.12. Remark. The above Proposition shows that if S is a Γ-AG-groupoid and $F(S)$ is the collection of all fuzzy subsets of S, then $(F(S), \Gamma^{'})$ is a Γ-AG-groupoid.

3.13. Lemma. Let S be a Γ-AG-groupoid with left identity e and f, g, h be fuzzy subsets of S, then

$$f\Gamma(g\Gamma h) = g\Gamma(f\Gamma h).$$

Proof. The proof follows from Proposition 3.11 and from a generalized form of Lemma 4 in [12].

3.14. Lemma. Let S be a Γ-AG-groupoid with left identity e and f a fuzzy right Γ-ideal of S, then $f\Gamma f$ is a fuzzy Γ-ideal of S.

Proof: Since f is a fuzzy right Γ-ideal of S, by Lemma 3.10, f is a fuzzy left Γ-ideal of S.

Let $a, b \in S$ and $\alpha \in \Gamma$. If $(f\Gamma f)(a) = 0$, then $(f\Gamma f)(a\alpha b) \geq (f\Gamma f)(a)$. Otherwise

$$\begin{aligned}
(f\Gamma f)(a\alpha b) &= \bigvee_{a=y\gamma z} \{f(y) \wedge f(z)\} \\
(\text{If } a &= y\gamma z, \text{ then } a\alpha b = (y\gamma z)\alpha b = (b\gamma z)\alpha y \text{ by left invertive law}) \\
\text{So, } (f\Gamma f)(a\alpha b) &= \bigvee_{a=y\gamma z} \{f(y) \wedge f(z)\} \\
&\leq \bigvee_{a=y\gamma z} \{f(b\gamma z) \wedge f(y)\}, \text{ since } f \text{ is a fuzzy left } \Gamma\text{-ideal} \\
&\leq \bigvee_{a\alpha b=c\gamma d} \{f(c) \wedge f(d)\} = (f\Gamma f)(a\alpha b).
\end{aligned}$$

Thus $(f\Gamma f)(a\alpha b) \geq (f\Gamma f)(a)$. Hence $f\Gamma f$ is a fuzzy right Γ-ideal of S and by Lemma 3.10, a fuzzy Γ-ideal of S.

3.15. Lemma. If S is a Γ-AG-groupoid and f, g are fuzzy Γ-ideals of S, then $f\Gamma g \subseteq f \cap g$.

proof: Let f and g be fuzzy Γ-ideals of S and $x \in S$. If $(f\Gamma g)(x) = 0$, then $(f\Gamma g)(x) \leq (f \cap g)(x)$, otherwise

$$\begin{aligned}
(f\Gamma g)(x) &= \bigvee_{x=y\alpha z} (f(y) \wedge g(z)) \\
&\leq \bigvee_{x=y\alpha z} (f(y) \wedge g(z))(f(y\alpha z) \wedge g(y\alpha z)), \text{ since } f \text{ and } g \text{ are fuzzy } \Gamma\text{-ideals of } S \\
&= \bigvee_{x=y\alpha z} (f(x) \wedge g(x)) \\
&= (f \cap g)(x).
\end{aligned}$$

Thus $f\Gamma g \subseteq f \cap g$.

3.16. Remark. If S is a Γ-AG-groupoid with left identity e and f and g are fuzzy right Γ-ideals of S, then $f\Gamma g \subseteq f \cap g$.

3.17. Remark. If S is a Γ-AG-groupoid and f a fuzzy Γ-ideal of S, then $f\Gamma f \subseteq f$.

3.18. Lemma. If S is a Γ-AG-groupoid with left identity e and f, g are fuzzy Γ-ideals of S, then $f\Gamma g$ is a fuzzy Γ-ideal of S.

Proof. Let f, g be fuzzy Γ-ideals of S and $a, b \in S$, and $\alpha, \beta, \gamma \in \Gamma$. If $(f\Gamma g)(a) = 0$, then $(f\Gamma g)(a) \leq (f\Gamma g)(a\alpha b)$, otherwise

$$\begin{aligned}
(f\Gamma g)(a) &= \bigvee_{a=c\gamma d} (f(c) \wedge g(d)) \\
(\text{since } a &= c\gamma d, \text{ so } a\beta b = (c\gamma d)\beta b = (c\gamma d)\beta(e\alpha b) = (c\gamma e)\beta(d\alpha b) \text{ by medial law}) \\
(f\Gamma g)(a) &\leq \bigvee_{a=c\gamma d} (f(c\gamma e) \wedge g(d\alpha b)), \text{ since } f \text{ and } g \text{ are fuzzy } \Gamma\text{-ideals} \\
&\leq \bigvee_{a\beta b=x\gamma y} (f(x) \wedge g(y)) \\
&= (f\Gamma g)(a\beta b).
\end{aligned}$$

Thus $(f\Gamma g)(a\beta b) \geq (f\Gamma g)(a)$.

Therefore $f\Gamma g$ is a fuzzy right Γ-ideal of S and by Lemma 3.10, $f\Gamma g$ is a fuzzy Γ-ideal of S.

4. Fuzzy points in Γ-AG-groupoids

Let S be a Γ-AG-groupoid and $x \in S$. Then for $a \in S$ and $t \in (0, 1]$, we define

$$a_t : S \longrightarrow [0,1], x \longmapsto a_t(x) := \begin{cases} t & \text{if } x = a \\ 0 \text{ otherwise.} \end{cases}$$

Then a_t is a fuzzy subset of S and is called a fuzzy point with support t and value t.

By $a_t \in f$, we mean $f(a) \geq t$.

4.1. Theorem. Let S be a Γ-AG-groupoid with left identity e. If f is a fuzzy left Γ-ideal of S, then $a_t\Gamma f$ is a fuzzy left Γ-ideal of S, where a_t is a fuzzy point of S.

Proof. Suppose that f is a fuzzy left Γ-ideal of S. Let $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. If $(a_t\Gamma f)(x) = 0$, then $(a_t\Gamma f)(x\gamma y) \geq (a_t\Gamma f)(x)$, otherwise $(a_t\Gamma f)(y) = \bigvee_{y=p\alpha q} \{a_t(p) \wedge f(q)\}$. Since $y = p\alpha q$, so $x\gamma y = x\gamma(p\alpha q) = (e\beta x)\gamma(p\alpha q) = (e\beta p)\gamma(x\alpha q) = p\gamma(x\alpha q)$ by medial law. Thus $(a_t\Gamma f)(y) = \bigvee_{y=p\alpha q} \{a_t(p) \wedge f(q)\} \leq \bigvee_{y=p\alpha q} \{a_t(p) \wedge f(x\alpha q)\} \leq \bigvee_{x\gamma y=c\alpha d} \{a_t(c) \wedge f(d)\} = (a_t\Gamma f)(x\gamma y)$. Thus $(a_t\Gamma f)(x\gamma y) \geq (a_t\Gamma f)(y)$. Consequently, $a_t\Gamma f$ is a fuzzy left Γ-ideal of S.

4.2. Definition. Let S be a Γ-AG-groupoid and $a \in S$. A fuzzy left (two-sided) Γ-ideal f of S is called the fuzzy left (two-sided) Γ-ideal of S, generated by a_t for $t \in (0, 1]$ if f is the smallest fuzzy left (two-sided) Γ-ideal of S containing a_t.

4.3. Theorem. Let S be a Γ-AG-groupoid with left identity e and a_t a fuzzy point of S. Then the fuzzy left Γ-ideal of S, generated by a_t is l_{a_t}, defined by:

$$l_{a_t} : S \longrightarrow [0,1],\ x \longmapsto l_{a_t}(x) := \begin{cases} t \text{ if } x \in S\Gamma a \\ 0 \text{ otherwise.} \end{cases}.$$

Proof. Let $x, y \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Case 1. If $y \notin S\Gamma a$, then $l_{a_t}(y) = 0 \leq l_{a_t}(x\gamma y)$.

Case 2. If $y \in S\Gamma a$, then $y = s\alpha a$ for some $s \in S$ and $\alpha \in \Gamma$. Hence

$$\begin{aligned} x\gamma y &= x\gamma(s\alpha a) = (e\beta x)\gamma(s\alpha a) \\ &= ((s\alpha a)\beta x)\gamma e \\ &= ((s\alpha a)\beta(e\delta x))\gamma e \\ &= ((s\alpha e)\beta(a\delta x))\gamma e \\ &= (e\beta(a\delta x))\gamma(s\alpha e) \\ &= (a\delta x)\gamma(s\alpha e) \\ &= ((s\alpha e)\delta x)\gamma a \in Sa. \end{aligned}$$

Hence $l_{a_t}(y) = t = l_{a_t}(x\gamma y)$. Thus in any case $l_{a_t}(x\gamma y) \geq l_{a_t}(x)$. Consequently, l_{a_t} is a fuzzy left Γ-ideal of S.

Also by the definition of a_t, we have $a_t \leq l_{a_t}$.

Now let f be a fuzzy left Γ-ideal of S containing a_t.

Case 1. If $x \in S\Gamma a$, then $x = s\alpha a$ for some $s \in S$ and $\alpha \in \Gamma$ and so $l_{a_t}(x) = t$.

Also

$$\begin{aligned} t &= a_t(a) \leq f(a) \\ &\Longrightarrow t \leq f(a) \leq f(s\alpha a) = f(x) \\ &\Longrightarrow f(x) \geq t = l_{a_t}(x). \end{aligned}$$

Case 2. If $x \notin S\Gamma a$, then $l_{a_t}(x) = 0 \leq f(x)$.

Thus $l_{a_t} \subseteq f$ in any case. This shows that l_{a_t} is the smallest fuzzy left Γ-ideal of S containing a_t.

4.4. Theorem. Let S be a Γ-AG-groupoid with left identity e and a_t a fuzzy point of S. Then the fuzzy right Γ-ideal R_{a_t} of S, generated by a_t is defined by:

$$R_{a_t} : S \longrightarrow [0,1], x \longmapsto R_{a_t}(x) := \begin{cases} t \text{ if } x \in a\Gamma S \cup S\Gamma a, \\ 0 \qquad \text{otherwise.} \end{cases}.$$

Proof. The proof follows from Theorem 4.3.

A Γ-AG-groupoid S is called regular if for every $a \in S$, there exists x in S and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$, or equivalently, $a \in (a\Gamma S)\Gamma a$.

For a regular Γ-AG-groupoid it is easy to see that $S\Gamma S = S$.

4.5. Proposition. Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is a fuzzy left Γ-ideal of S.

Proof. Let f be a fuzzy right Γ-ideal of S and $a, b \in S$ and $\gamma \in \Gamma$. Since S is regular, there exist $x \in S$, and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Then

$$\begin{aligned} f(a\gamma b) &= f(((a\alpha x)\beta a)\gamma b) \\ &= f((b\beta a)\gamma(a\alpha x)) \\ &\geq f(b\beta a), \text{ since } f \text{ is a fuzzy right } \Gamma\text{-ideal of } S \\ &\geq f(b). \end{aligned}$$

Thus $f(a\gamma b) \geq f(b)$. Therefore f is a fuzzy left Γ-ideal of S.

4.6. Corollary. In a regular Γ-AG-groupoid S, every fuzzy right Γ-ideal of S is a fuzzy Γ-ideal of S.

4.7. Lemma. If f and g are fuzzy right Γ-ideals of a regular Γ-AG-groupoid S, then $f\Gamma g = f \cap g$.

Proof. Since S is regular, by Proposition 4.5, every fuzzy right Γ-ideal of S is a fuzzy Γ-ideal of S. Since S is regular, so for every $a \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Thus

$$\begin{aligned}(f \cap g)(a) &= f(a) \wedge g(a) \\ &\leq f(a\alpha x) \wedge g(a), \text{ since } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\leq \bigvee_{a = y\gamma z} (f(a) \wedge g(b)) \\ &= (f\Gamma g)(a).\end{aligned}$$

Thus $f \cap g \subseteq f\Gamma g$. On the other hand by Lemma 3.15, we have $f\Gamma g \subseteq f \cap g$. Therefore $f\Gamma g = f \cap g$.

4.8. Corollary. Let f be a fuzzy right Γ-ideal of a regular Γ-AG-groupoid S, then $f = f\Gamma f$.

4.9. Definition. A fuzzy Γ-ideal f of a Γ-AG-groupoid S is called *prime (semiprime)* if:

$g\Gamma h \subseteq f$ $(g\Gamma g \subseteq f)$ implies $g \subseteq f$ or $h \subseteq f$ $(g \subseteq f)$ for every fuzzy Γ-ideal g and h of S.

Note that every prime fuzzy Γ-ideal of a Γ-AG-groupoid is semiprime.

4.10. Definition. A fuzzy Γ-ideal f of a Γ-AG-groupoid S is called *irreducible* if:

$g \cap h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$ for every fuzzy Γ-ideal g and h of S.

4.11. Proposition. If f is a prime fuzzy Γ-ideal of a Γ-AG-groupoid$S$, then it is both semiprime and irreducible.

Proof. Suppose that f is a prime fuzzy Γ-ideal of S. Then clearly, f is semiprime. Let g and h be fuzzy Γ-ideals of S such that $g \cap h \subseteq f$. As

$$\begin{aligned}g\Gamma h &\subseteq g \cap h \subseteq f \Longrightarrow g\Gamma h \subseteq f \\ &\Longrightarrow g \subseteq f \text{ or } h \subseteq f, \text{ since } f \text{ is prime.}\end{aligned}$$

Hence f is irreducible fuzzy Γ-ideal of S.

4.12. Definition. A Γ-AG-groupoid S is called fully fuzzy prime if every fuzzy Γ-ideal of S is prime fuzzy Γ-ideal of S.

4.13. Theorem. A regular Γ-AG-groupoid S is fully fuzzy prime if and only if the set of all fuzzy Γ-ideals $FI(S)$ of S is totally ordered under inclusion.

Proof. Suppose that S is fully fuzzy prime. Let f, g be fuzzy Γ-ideals of S. Then by Lemma 3.15, $f\Gamma g \subseteq f \cap g$. As $f \cap g$ is a prime fuzzy Γ-ideal of S, hence either $f \subseteq f \cap g$ or $g \subseteq f \cap g$. This implies that either $f \subseteq g$ or $g \subseteq f$. Hence $FI(S)$ is totally ordered under inclusion.

Conversely, assume that $FI(S)$ is totally ordered under inclusion. Let f, g, h be fuzzy Γ-ideals of S such that $g\Gamma h \subseteq f$. As $FI(S)$ is totally ordered under inclusion, so either $g \subseteq h$ or $h \subseteq g$. Suppose that $g \subseteq h$. Then $g = g\Gamma g \subseteq g\Gamma h \subseteq f$. It follows that $g \subseteq f$. Thus f is prime fuzzy Γ-ideal of S and hence S is fully fuzzy prime.

4.14. Theorem. Every fuzzy Γ-ideal f in a regular Γ-AG-groupoid S is prime if and only if f is irreducible.

Proof. Let f be a prime fuzzy Γ-ideal of S. Let g, h be fuzzy Γ-ideals of S such that $g \cap h \subseteq f$. By Lemma 3.15,

$$\begin{aligned} g\Gamma h &\subseteq g \cap h \subseteq f \\ &\Longrightarrow g \subseteq f \text{ or } h \subseteq f, \text{ since } f \text{ is prime.} \end{aligned}$$

Hence f is irreducible.

Conversely, assume that f is irreducible. Let g, h be fuzzy Γ-ideals of S such that $g\Gamma h \subseteq f$. Since S is regular, by Lemma 4.7, $g\Gamma h = g \cap h$. Thus $g \cap h \subseteq f$, since f is irreducible, we have $g \subseteq f$ or $h \subseteq f$. Hence f is prime.

4.15. Definition. A fuzzy subset f of a Γ-AG-groupoid is called a Γ-AG-band if all its elements are idempotent.

4.16. Lemma. The concepts of fuzzy right and fuzzy left Γ-ideals in a Γ-AG-band coincide.

Proof. Let f be a fuzzy right Γ-ideal of S and $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} f(a\alpha b) &= f((a\alpha a)\beta b) \\ &= f((b\alpha a)\beta a), \text{ by left invertive law} \\ &\geq f(b\alpha a), \text{ since } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\geq f(b). \end{aligned}$$

Hence $f(a\alpha b) \geq f(b)$ and f is a fuzzy left Γ-ideal of S.

Conversely, assume that f is a fuzzy left Γ-ideal. Then for $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$, we have

$$\begin{aligned} f(a\alpha b) &= f((a\alpha a)\beta b) \\ &= f((b\alpha a)\beta a), \text{ by left invertive law} \\ &\geq f(a), \text{ since } f \text{ is fuzzy left } \Gamma\text{-ideal.} \end{aligned}$$

Hence f is a fuzzy right Γ-ideal of S.

4.17. Lemma. Every fuzzy Γ-ideal of a Γ-AG-band is idempotent.

Proof. Straightforward.

4.18. Theorem. Every fuzzy Γ-ideal of a Γ-AG-band is prime if and only if $FI(S)$ is totally ordered under inclusion.

Proof. Suppose that every fuzzy Γ-ideal of S is prime. Let f and g be fuzzy Γ-ideals of S, then $f \wedge g$ is a fuzzy Γ-ideal of S and hence prime. Thus

$$\begin{aligned} f\Gamma g &\leq f \wedge g \\ &\Longrightarrow f \leq f \wedge g \text{ or } g \leq f \wedge g \\ &\Longrightarrow f \leq g \text{ or } g \leq f. \end{aligned}$$

Hence $FI(S)$ is totally ordered under inclusion.

Conversely, suppose that $FI(S)$ is totally ordered by inclusion. Let f, g, and h be fuzzy Γ-ideal of S such that $f\Gamma g \leq h$. Since $FI(S)$ is totally ordered under inclusion so either $f \leq g$ or $g \leq f$. Assume that $f \leq g$, since S is a band, so every fuzzy Γ-ideal is idempotent and hence

$$\begin{aligned} f &= f\Gamma f \leq f\Gamma g \leq h \\ &\Longrightarrow f \leq h. \end{aligned}$$

Hence f is prime.

5. Conclusion

In this paper we have characterized Γ-AG-groupoids by the properties of fuzzy Γ-ideals and prime, semiprime fuzzy Γ-ideals. In our future work, we have planed to define fuzzy radical of Γ-AG-groupoids and to discuss its related properties. Also we will try to define prime fuzzy bi-Γ-ideals in Γ-AG-groupoids and will discuss those Γ-AG-groupoids for which every fuzzy bi-Γ-ideals is idempotent.

References

[1] J. Ahsan, K. Y. Li and M. Shabir, *Semigroups characterized by their fuzzy bi-ideals*, The J. Fuzzy Math., 10(2)(2002), 441-449.

[2] W. E. Barnes, *On the Γ-rings of Nobusawa,* Pacific J. Math., 18(1966), 411-422.

[3] G. L. Booth and N. J. Groenwald, *On uniformly strongly prime gamma rings*, Bull Austral Math Soc., 37(1988), 437-445.

[4] T. K. Dutta and T. Chanda, *Structures of fuzzy ideals of Γ-ring*, Bull. Malays. Math. Sci. Soc., (2) 28(1)(2005), 9-18.

[5] Y. B. Jun and C. Y. Lee, *Fuzzy Γ-rings*, Pusan Kyongnam Math. J., 84(1981), 264-269.

[6] Y. B. Jun, *On fuzzy prime ideals of Γ-rings*, Soochow J. Math., 21(1)(1995), 41-48.

[7] S. M. Hong and Y. B. Jun, *A note on fuzzy ideals in Γ-ring,* Bull. Honam. Math. Soc., 12 (1995), 39-48.

[8] M. A. Kazim and M. Naseerudin, *On almost semigroups*, Alig. Bull. Math., 2(1972), 1-7.

[9] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Pauli XXVIII, 1(1979), 17-21.

[10] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets System, 5(1981), 203-215.

[11] N. Kuroki, *On fuzzy semigroups*, Inform. Sci., 53(1991), 203-236.

[12] Q. Mushtaq and S. M. Yusuf, *On locally associative LA-semigroups,* J. Nat. Sci. Math., 1, 19(1979), 57-62.

[13] M. K. Sen, *On Γ-semigroups*, Proceeding of International Symposium on Algebra and Its Applications. Decker Publication, New York, 30(1981), 1-8.

[14] M. K. Sen and N. K. Saha, *On Γ-semigroups I,* Bull. Cal. Math. Soc., 78(1986), 180-186.

[15] T. Shah and I. Rehman, *On Γ-ideals and Γ-bi-ideals in Γ-AG-groupoids*, International Journal of Algebra, 4(2010), no. 6, 267-276.

[16] T. Shah and I. Rehman, *On M-systems in Γ-AG-groupoids,* Proc. Pakistan Acad. Sci., 47(1)(2010), 33-39.

[17] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8(1965), 338-353.