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Bayesian estimation and prediction for the generalized Lindley distribution under asymmetric loss function

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Abstract

The paper develops the Bayesian estimation procedure for the generalized Lindley distribution under squared error and general entropy loss functions in case of complete sample of observations. For obtaining the Bayes estimates, both non-informative and informative priors are used. Monte Carlo simulation is performed to compare the behaviour of the proposed estimators with the maximum likelihood estimators in terms of their estimated risks. Discussion is further extended to Bayesian prediction problem based on an informative sample where an attempt is made to derive the prediction intervals for future observations. Numerical illustrations are provided based on a real data example.

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1. Introduction

The one parameter exponential distribution is the most popular distribution for analysing the lifetime data due to its simplicity and has also been widely applied in many areas. But its applicability is restricted to constant hazard rate. It seems quite unrealistic to assume time independent hazard rate for any real-life system. [15] introduced a distribution in the context of Bayes theorem, as a counter example of fudicial statistics named as Lindley distribution which overcomes the drawback of exponential distribution as it does not permit the constant hazard rate. The various papers have been published on the parameter estimation for Lindley distribution under different set-up. [11] have provided a comprehensive treatment of the mathematical properties of Lindley distribution. [14] have discussed the procedures of reliability estimation for progressively type-II censored

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sample under both classical and Bayesian set-up when the data follows the Lindley distribution. They have also suggested that this distribution can be used as a life time model.

The generalizations of exponential distribution are Weibull and Gamma distributions which are most frequently used lifetime models in survival analysis. However, these distributions have their own limitations. The Weibull distributions are more popular than the gamma because the survival function of the latter cannot be expressed in a closed form and one needs numerical integration. But, the Weibull distribution also has its own disadvantages. [2] have pointed out that the maximum likelihood estimators of the Weibull parameters may not behave properly for the whole parameter space. The above three distributions (exp, Gamma, and Weibull) are inappropriate when the failure rate is indicated to be uni-modal or bath-tub shaped.

Recently, [16] have introduced a two parameter generalization of Lindley distribution as an extended model for modelling of bathtub data alternative to gamma, lognormal, Weibull, and exponentiated exponential distributions. The generalized Lindley distribution has the following probability density function (PDF)

$$f(x) = \frac{\alpha\lambda^2}{(1+\lambda)}(1+x)\left[1-\frac{1+\lambda+\lambda x}{(1+\lambda)}\right]^{\alpha-1}e^{-\lambda x} \tag{1.1}$$

It is denoted by $GLD(\alpha, \lambda)$. The corresponding CDF is given by

$$F(x) = \left[1-\frac{1+\lambda+\lambda x}{(1+\lambda)}e^{-\lambda x}\right]^{\alpha} \tag{1.2}$$

For $\alpha = 1$, it reduces to one parameter Lindley distribution (see Figure 1).

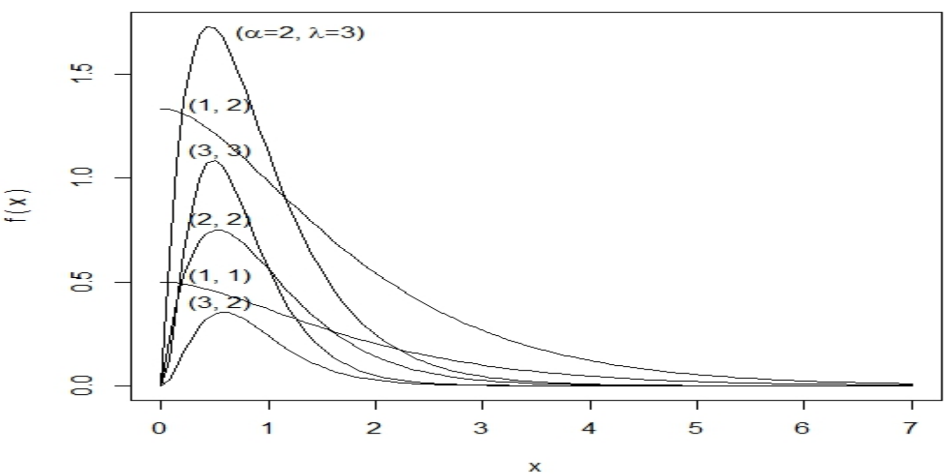

Figure 1: Density function for different values of α and λ

[16] have discussed different properties and the inferential procedure for this distribution under classical paradigm. They have also checked in many ways the goodness-of-fit of this distribution to real data set over gamma, Weibull, and log-normal distributions.

In Bayesian approach, a prior distribution for the parameter is considered and there is no clear cut rule by which one can say that one prior is better than any other. The prior information is purely subjective assessment of an expert before any data have been observed. Also [3] argues that when information is not in compact form the Bayesian analysis uses non-informative prior consideration.

For some distributions, the posterior density is not in tractable form as it contains integrals. The Bayes estimation cannot be performed explicitly and becomes quite complicated. In such a situation, some approximate methods are inevitable such as numerical integration or analytic approximation techniques. Alternatively, many authors use Monte Carlo morkov chain (MCMC) methods namely Gibbs sampling [22] and Metropolis Hasting [13, 4] algorithms to simulate the deviates from posterior density so that sample based inference can be done.

It is often seen that usual squared error loss function (SELF) is considered for Bayesian procedure which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. But it may be illogical to assume symmetric loss function. In some situation overestimation is more serious than underestimation or vice-versa. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation. A more appropriate asymmetric loss function, general entropy loss function (GELF) is introduced by [5], defined as

$$L\left(\theta, \hat{\theta}\right) = \left(\frac{\hat{\theta}}{\theta}\right)^{C} - C \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \tag{1.3}$$

Where, $\hat{\theta}$ is the estimate of θ and C is the loss parameter which reflects the departure from symmetry. The loss parameter C allows different shapes of this loss function. For $C > 0$, a positive error $\left(\hat{\theta} > \theta\right)$causes more serious consequences than a negative error and vice versa. The Bayes estimate $\hat{\theta_G}$ of θ under GELF is given by

$$\hat{\theta}_G = \left[E_\pi\left(\theta^{-C}|\underset{\sim}{x}\right)\right]^{(-1/C)} \tag{1.4}$$

provided that the expectation $E_\pi\left(\theta^{-C}|\underset{\sim}{x}\right)$ exits and is finite. Here, E_π denote the expectations with respect to posterior pdf of θ. The Bayes estimate of θ under SELF is the posterior mean of θ. For $C = -1$, equation (4) provides the Bayes estimator under SELF for θ.

The performance of the Bayes estimators depends upon the prior distribution and the loss functions to be considered. Many authors have been studying the performance of the Bayes estimator under various assumptions regarding the prior belief and loss functions (symmetric and asymmetric). [21] and [7] have obtained the Bayes estimators under different loss functions for inverse Gaussian distribution and generalized exponential distribution respectively. [20] have proposed to obtain the Bayes estimator for the parameter of exponentiated gamma distribution under general entropy loss function as they have found that the Bayes estimator obtained under GELF have the smaller risk than considered existing estimators.

Prediction of future observations based on information from existing data is concerned topic in Statistics, arising in many contexts in a natural way. [9] have developed the Bayesian procedure to the prediction problems of future observables. Prediction problems can be generally classified into two types. 1.) One sample prediction 2.) Two sample prediction. In earlier case, the variable to be predicted comes from the same sequence of variables observed and is dependent on observed data, and arises in case of censored data. In two sample prediction, the variable to be predicted comes from another independent future sample follow the same distribution function. Bayesian prediction for future observations based on certain distributions has been discussed by several authors [see 8, 23, 19, 1, 17, 10, 18, 12, and references cited therein].

This distribution is not considered earlier in literature through Bayesian setup. Therefore in this paper, our aim is to perform the Bayesian estimation for the unknown parameters of the generalized Lindley distribution under different forms of the gamma priors in case of symmetric and asymmetric loss functions. We have also discussed the two sample prediction problems for considered model.

The rest of the paper is organized as follows. In section 2, the maximum likelihood estimators under type II censored data, along with its asymptotic properties are discussed. In section 3, for obtaining the Bayes estimates for α and λ the MCMC techniques have been utilized. To implement the Gibbs algorithm full condition posteriors for α and λ are derived. Bayesian prediction is discussed in section 4. We have also constructed the predictive bounds for future observations. In section 5, a simulation study is conducted to check the behaviour of our proposed estimators. In section 6, a real data set is analyzed. Finally, conclusions have been given in section 7.

2. Maximum likelihood estimates

Suppose that $\{x_1, x_2, ..., x_n\}$ is the independent and identical (IID) random sample of size n obtained from (1). Then the likelihood equation is given by

$$L\left(\underset{\sim}{x}|\alpha,\lambda\right) = \left[\frac{\alpha\lambda^2}{(1+\lambda)}\right]^n e^{-\lambda\sum\limits_{i=1}^{n} x_i} \prod_{i=1}^{n}(1+x_i)\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right]^{\alpha-1} \tag{2.1}$$

Then, the log likelihood function can be written as

$$LogL = n\log\left[\frac{\alpha\lambda^2}{(1+\lambda)}\right] - \lambda\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log(1+x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right] \tag{2.2}$$

The MLEs of α and λ can be obtained as the simultaneous solution of the following equations

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right] = 0 \tag{2.3}$$

$$\frac{2n}{\lambda} - \frac{n}{(1+\lambda)} + \frac{\lambda(\alpha-1)}{(1+\lambda)^2}\sum_{i=1}^{n}\frac{(2+x_i+\lambda(1+x_i))}{\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right]}x_i e^{\lambda x_i} \tag{2.4}$$

In order to solve the above equations, we can apply any suitable iterative procedure such as Newton RaphsonÔÇÖs method.
Under some regularity conditions, the MLEs $(\hat{\alpha},\hat{\lambda})$ is approximately bi-variate normal with mean $(\hat{\alpha},\hat{\lambda})$ and covariance matrix $I^{-1}\left(\hat{\alpha},\hat{\lambda}\right)$ i.e.$(\hat{\alpha},\hat{\lambda}) \sim N_2\left(\left[\hat{\alpha}\ \hat{\lambda}\right]', I^{-1}\left(\hat{\alpha},\hat{\lambda}\right)\right)$. where $I\left(\hat{\alpha},\hat{\lambda}\right)$ is the observed FisherÔÇÖs information matrix, and defined as

$$I\left(\hat{\alpha},\hat{\lambda}\right) = \begin{bmatrix} -\frac{dlogL}{d\alpha^2} & -\frac{dlogL}{d\lambda d\alpha} \\ -\frac{dlogL}{d\alpha d\lambda} & -\frac{dlogL}{d\lambda^2} \end{bmatrix}_{(\hat{\alpha},\hat{\lambda})}$$

The diagonal elements of $I^{-1}\left(\hat{\alpha},\hat{\lambda}\right)$ provides the asymptotic variances for the parameters α and λ respectively. Then two-sided $100(1-\gamma)\%$ normal approximation confidence interval of α and λ can be defined as $\left\{\hat{\alpha} \mp Z_{\gamma/2}\sqrt{var(\hat{\alpha})}\right\}$ and $\left\{\hat{\lambda} \mp Z_{\gamma/2}\sqrt{var(\hat{\lambda})}\right\}$ respectively.

3. Bayes estimation

In this section, we have developed the Bayesian estimation procedure for the parameters of $GLD(\alpha,\lambda)$ assuming independent informative priors for the unknown parameters. It is assumed that the parameters α and λ are independent. The variation in the parameters α and λ have been incorporated in the analysis through Gamma priors having the following density function

$$g(\lambda) = \frac{a_1^{b_1}}{\Gamma b_1} e^{-a_1\lambda}\lambda^{b_1-1}; \lambda, a_1, b_1 > 0 \tag{3.1}$$

$$g(\alpha) = \frac{a_2^{b_2}}{\Gamma b_2} e^{-a_2\alpha}\alpha^{b_2-1}; \alpha, a_2, b_2 > 0 \tag{3.2}$$

Where, a_1, a_2, b_1, and b_2 are the hyper-parameters. Then, the joint posterior PDF of α and λ is defined as

$$\pi\left(\alpha,\lambda \mid \underset{\sim}{x}\right) = \frac{L\left(\underset{\sim}{x}|\alpha,\lambda\right) g(\alpha)g(\lambda)}{\int_0^\infty\int_0^\infty L\left(\underset{\sim}{x}|\alpha,\lambda\right) g(\alpha)g(\lambda)d\alpha d\lambda} \tag{3.3}$$

The Gibbs sampling techniques is used for simulating the sample from the joint posterior density. The full conditional posterior density, for implementing the Gibbs algorithm, of each of unknowns is obtained by regarding all other parameters in (11) as known. Then, we have

$$\pi_1\left(\alpha \mid \lambda, \underset{\sim}{x}\right) \propto \alpha^{n+b_2-1}e^{-\alpha a_2}\prod_{i=1}^{n}\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}e^{-\lambda x_i}\right]^{\alpha-1} \tag{3.4}$$

$$\pi_2\left(\lambda \mid \alpha, \underset{\sim}{x}\right) \propto \frac{\lambda^{2n+b_1-1}}{(1+\lambda)^n}e^{-\lambda(a_1+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}e^{-\lambda x_i}\right]^{\alpha-1} \tag{3.5}$$

The Gibbs algorithm consist the following steps

(1) Start with j=1, and initial values of $\{\alpha^{(0)}, \lambda^{(0)}\}$.

(2) Generate new draws for α and λ as follows:

   (a) $\alpha^{(j)} \sim \pi_1\left(\alpha \mid \lambda^{(j-1)}, \underset{\sim}{x}\right)$

   (b) $\lambda^{(j)} \sim \pi_2\left(\lambda \mid \alpha^{(j)}, \underset{\sim}{x}\right)$

   In this step, the Metropolis algorithm is used to generate deviates from the full conditional posterior densities by choosing any arbitrary proposal distribution.

(3) Repeat step 2 for all j= 1, 2,..., M and obtained $(\alpha_1,\lambda_1),(\alpha_2,\lambda_2),...(\alpha_M,\lambda_M)$.

Once we have the posterior sample, we can use discrete formulas, applied to these samples to approximate the integrals of interest. By this sampling approach, the Bayes estimates of α and λ with respect to the general entropy loss function are as follows

$$\hat{\alpha}_G = \left[E_\pi\left(\alpha^{(-C)} \mid \underset{\sim}{x}\right)\right]^{(-1/C)} \approx \left(\frac{1}{M-M_0}\sum_{k=1}^{M-M_0}\alpha_k^{-C}\right)^{(-1/C)}$$

$$\hat{\lambda}_G = \left[E_\pi\left(\lambda^{(-C)} \mid \underset{\sim}{x}\right)\right]^{(-1/C)} \approx \left(\frac{1}{M-M_0}\sum_{k=1}^{M-M_0}\lambda_k^{-C}\right)^{(-1/C)}$$

Where, M_0 is the burn-in-period of Markov Chain. The HPD credible intervals for α and λ can be constructed by using algorithm given in [6]. Let $\{(\alpha_{(i)}, \lambda_{(i)}); i = 1, 2, ..., M\}$ be the corresponding ordered MCMC sample of $\{(\alpha_i, \lambda_i); i = 1, 2, ..., M\}$. Then construct all the $100(1-\psi)\%$ credible intervals for of α and λ as

$$(\alpha_{[1]}, \alpha_{[M(1-\psi)]}), ..., (\alpha_{[M\psi]}, \alpha_{[M]}) \text{ and } (\lambda_{[1]}, \lambda_{[M(1-\psi)]}), ..., (\lambda_{[M\psi]}, \lambda_{[M]}).$$

Here, $[x]$ denotes the largest integer less than or equal to x. Then the HPD credible interval is that interval which has the shortest length.

4. Bayesian Prediction

In this section, we have discussed the Bayesian prediction of future ordered sample on the basis of observed data. Let $y_{(1)}, y_{(2)}, ..., y_{(m)}$ be another sample (called future sample) of size m, independent of the informative sample $x_{(1)}, x_{(2)}, ..., x_{(m)}$ from the same distribution. The density function of $y_{(s)}$ th ordered future sample is

(4.1)
$$p\left(y_{(s)} \mid \alpha, \lambda\right) = \frac{m!}{(s-1)!(m-s)!}\left[F\left(y_{(s)} \mid \alpha, \lambda\right)\right]^{s-1}\left[1 - F\left(y_{(s)} \mid \alpha, \lambda\right)\right]^{m-s} f\left(y_{(s)} \mid \alpha, \lambda\right)$$

By substituting (1) and (2) in (14), we get

(4.2)
$$p\left(y_{(s)} \mid \alpha, \lambda\right) = \frac{m!\alpha\lambda^2}{(s-1)!(m-s)!}\frac{(1+y_{(s)})}{(1+\lambda)}e^{-\lambda y_{(s)}} \sum_{j=0}^{m-s}(-1)^j\binom{m-s}{j}\left(1 - \frac{1+\lambda+\lambda y_{(s)}}{(1+\lambda)}e^{-\lambda y_{(s)}}\right)^{\alpha(s+j)-1}$$

The Bayesian predictive density of $y_{(s)}th$ ordered future sample can be obtained as

(4.3)
$$p\left(y_{(s)} \mid \underset{\sim}{x}\right) = \int_0^\infty\int_0^\infty p\left(y_{(s)} \mid \alpha, \lambda\right)\pi\left(\alpha, \lambda \mid \underset{\sim}{x}\right) d\alpha d\lambda$$

It is to be noted that $p\left(y_{(s)} \mid \underset{\sim}{x}\right)$ cannot be expressed in closed form and hence it cannot be evaluated analytically. The MCMC sample $\{(\alpha_i, \lambda_i); i = 1, 2, ..., M\}$ obtained from $\pi\left(\alpha, \lambda \mid \underset{\sim}{x}\right)$ using Gibbs sampling has been utilized to obtain the consistent estimate of $p\left(y_{(s)} \mid \underset{\sim}{x}\right)$. It can be obtained as

(4.4)
$$\hat{p}\left(y_{(s)} \mid \underset{\sim}{x}\right) = \frac{1}{M}\sum_{i=1}^{M} p\left(y_{(s)} \mid \alpha_i, \lambda_i\right)$$

Thus, we can obtain the two sided $100(1-\psi)\%$ prediction interval (L, U) for future sample by solving the following two equations:

$$P\left(Y_{(s)} > U \mid \underset{\sim}{x}\right) = \tfrac{\psi}{2}$$

and

$$P\left(Y_{(s)} > L \mid \underset{\sim}{x}\right) = 1 - \frac{\psi}{2}$$

It is not possible to obtain the solutions analytically. We need to apply suitable numerical techniques for solving these non-linear equations.

5. Simulation Study

This section contains the simulation study which is carried out to compare the behaviour of the different Bayes estimators and Maximum likelihood estimators of the shape and scale parameters. The comparisons are made on the basis of simulated risk (expectation of loss function over sample space) and width of the confidence/HPD intervals. The risk function of the estimators cannot be expressed in closed form. Therefore, the risks of the estimators are estimated on the basis of simulated sample. For this purpose, we generate 1000 samples of size n (small sample size n = 20, moderate sample size n = 30, and large sample size n = 50) from generalized Lindley distribution by using inversion method suggested by [16]. The estimators $\hat{\alpha}_M$ and $\hat{\lambda}_M$denote the MLEÔÇÖs of the parameters α and λ respectively.$(\hat{\alpha}_G, \hat{\lambda}_G)$ and $(\hat{\alpha}_s, \hat{\lambda}_s)$ are the corresponding Bayes estimators under GELF and SELF respectively. The risks of the estimators under GELF and SELF are denoted by $R_G(\cdot)$ and $R_s(\cdot)$ respectively. We have looked into many ways to view the clear picture of the performance of the considered competing estimators.

(1) For varying sample size when parameters are fixed (see Tables 1 and 2).
(2) For different values of hyper parameters along with increasing sample size when both shape and scale parameters are fixed (see Tables 1 and 2).
(3) For the choice of the loss parameter C of GELF when sample size and model parameters are fixed (see Figures 2, 3, 4 and 5).
(4) For variation in the model parameters for fixed sample size, loss parameter, and hyper parameters (see Tables 3, 4, 5 and 6).

The MCMC sample have been utilize for Bayesian analysis so that sample based inference can be done. We check the convergence of the MCMC sample and it is found that Markov chain converges rapidly with any arbitrary initial starting. All computational algorithms are calculated with help of R software.

In case of non-informative prior, we take $a_1 = a_2 = b_1 = b_2 = 0$. For informative prior, the hyper parameters can be chosen easily if we take prior mean (say, ν) equals to the true value of the parameter with varying prior variance (say, η). The prior variance indicates the confidence of our prior guess. A large prior variance shows less confidence in prior guess and resulting prior distribution is relatively flat. On other hand, small prior variance indicates greater confidence in prior guess. Another case is also considered where prior mean is taken away from the true value of the parameter with moderate variance.

From the Table 1, it is observed that the risks of all the estimators decrease as sample size increases in all the considered cases. The width of the confidence/HPD intervals also decreases as sample size increases. As confidence of prior guess decreases, the risks of the estimators and width of the HPD intervals both increases. We can also conclude that the maximum likelihood estimator is less efficient than the Bayes estimator since MLE shows the larger Risk/MSE in all the cases.

The risks of all the estimators under GELF increases as loss parameter C increase in all the considered cases, although the increase is more for maximum likelihood estimators. The behaviour of all considered estimator remain same for all combinations of model parameters and hyper parameters. ThatÔÇÖs why; we are presenting selected figures. When over estimation is more serious than under estimation i.e. $C > 0$, the risks of $\hat{\alpha}_G$ and $\hat{\lambda}_G$ under GELF are found to be least among the considered competing estimators (see Figure 3). When under estimation is more serious than over estimation i.e. $C < 0$, the magnitudes of the risks of the estimators obtained under SELF and GELF for both parameters are more or less same but least than that of MLEs (see Figure 2).

Under SELF, the risks of $\hat{\alpha}_G$ and $\hat{\alpha}_G$ are going to be smaller than that of $\hat{\alpha}_s$ and $\hat{\alpha}_s$ when $C > 0$. When under estimation is more serious than over estimation, the Bayes estimators obtained under GELF have smaller risk than that of obtained under SELF for $C < -1$, and for $C > 1$, the Bayes estimators obtained under GELF become less efficient than Bayes estimators obtained under SELF (see Figures 4 and 5).

For comparing the performance of the estimators for different combination of model parameters, we fix $C = 1.5$ when over estimation is more serious than under estimation and $C = -0.5$ in case of under estimation is more serious than over estimation.

Table 3, 4, and 5 shows the performance of the proposed estimators for the variations of the model parameters α and λ under for fixed values of $n = 20$ and C. it is to be noticed that as the value of α increases the magnitude of the risk of the estimators of α increases but reveres trend have found in the magnitude of the risk of the estimators of λ for fixed values of λ under both loss functions. Further, it is also noticed that as λ increases for fixed α the magnitude of the risk of α decreases but increment in the magnitude of the risk of λ is noticed under both functions for any given values of n and C. The similar trends are found in case of the width of the confidence/HPD intervals (see Table 6).

6. Real data analysis

In this section, a real data set is analyzed to verify our proposed estimation procedure. The data set of relief times of twenty patients has taken from [16]. It has been observed by [16] that the generalized Lindley distribution can be effectively used to analyse this data set. The relief times of the twenty patients are as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1,1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The maximum likelihood estimates, Bayes estimates under SELF along with their confidence/HPD intervals and Bayes estimates under GELF are presented in Table 7. The MLEs of shape and scale parameters are same as obtained by [16]. The width of the HPD interval is smaller than that of classical interval for both parameters. For this data set, Bayesian analysis is carried out in case of non-informative priors. The MCMC iterations of α and λ are plotted in Figure 6(a) and 6(b) respectively. Trace plots are indicating that the MCMC samples are well mixed and stationary achieved. The $3-D$ histogram of the posterior samples of α and λ is plotted in Figure 7. This figure indicates that posterior distributions of α and λ are symmetric. [16] have checked the goodness-of-fit to the real data set using maximum likelihood estimates. For demonstrating the goodness-of-fit of the this real data set for the Bayes estimates, We have applied Kolmogorov-Smirnov (KS) one sample test and KS distances are presented in the Table 9. The empirical CDF and fitted CDF (for maximum likelihood estimates and Bayes estimates) have been plotted in figure 10. It is observed that the goodness-of-fit to the real data set is quite acceptable even with the Bayes estimates.

We also predict the first three and last observations of independent future sample of size $m(=5, 10, 15, 20, and\ 30)$ corresponding to real data set. Bayesian point predictions and along with their confidence bounds are presented in Table 8. It is to be noticed that the width of the predictive bounds increases as s increases. The density function (16) for first ordered statistics of future sample is plotted for different values of m in Figure 8. Figures 9(a), 9(c), and 9(e) are the MCMC trace plots of $y_{(1)}, y_{(2)}, and\ y_{(3)}$ future sample for $m = 5$. The posterior densities of simulated $y_{(1)}, y_{(2)}, and\ y_{(3)}$ are plotted in Figures 9(b), 9(d), and 9(f) respectively.

7. Discussion

In this paper, we have considered the problem of estimation of the parameters of the generalized Lindley distribution under Bayesian paradigm. On the basis of compression of simulated risks of estimators, it is found that Bayes estimators perform better than maximum likelihood estimation and Bayes estimator under GELF is also more efficient than the Bayes estimator under SELF in most of the situation. From the above mentioned discussion, we may conclude that the Bayes procedure discussed in this paper can be recommended for their use.

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Table 1. Average estimates (risks under SELF) for fixed $\alpha = 2$ and $\lambda = 2$.

n	Parameters	MLE	Bayes				
			$a_1 = b_1 = 0$	$\eta = 2$	$\eta = 2$	$\eta = 2$	$\eta = 3$
			$a_2 = b_2 = 0$	$\nu = 8$	$\nu = 0.5$	$\nu = 100$	$\nu = 8$
20	α	2.41643	2.33918	2.31967	2.15601	2.33746	2.3568
		(1.0155)	(0.8966)	(0.8144)	(0.2945)	(0.8891)	(0.8444)
	λ	2.17122	2.13785	2.13387	2.0932	2.13749	2.1516
		(0.2511)	(0.2346)	(0.2256)	(0.1432)	(0.2339)	(0.2303)
30	α	2.28203	2.23343	2.22504	2.13827	2.23275	2.25006
		(0.6173)	(0.5657)	(0.5341)	(0.2692)	(0.563)	(0.5494)
	λ	2.12204	2.10035	2.09862	2.07749	2.1002	2.11056
		(0.1543)	(0.1469)	(0.1434)	(0.1057)	(0.1467)	(0.1458)
50	α	2.16842	2.14088	2.13817	2.10441	2.14079	2.15339
		(0.3091)	(0.2924)	(0.2842)	(0.1961)	(0.2920)	(0.2907)
	λ	2.07563	2.06292	2.06238	2.05392	2.06292	2.0696
		(0.0862)	(0.0835)	(0.0824)	(0.0688)	(0.0835)	(0.0834)

Table 2. The average 95% confidence intervals and HPD credible intervals when $\alpha = 2$ and $\lambda = 2$.

n	Parameters	MLE	Bayes				
			$a_1 = b_1 = 0$	$\eta = 2$	$\eta = 2$	$\eta = 2$	$\eta = 3$
			$a_2 = b_2 = 0$	$\nu = 8$	$\nu = 0.5$	$\nu = 100$	$\nu = 8$
20	α	0.66531	1.43837	1.43402	1.43266	1.43881	1.47635
		4.16755	3.28692	3.25047	2.92185	3.28363	3.28279
	λ	1.29705	1.65981	1.6584	1.65404	1.65913	1.67981
		3.04538	2.61849	2.61175	2.53719	2.61718	2.62601
30	α	0.95444	1.5371	1.53588	1.53209	1.53742	1.5638
		3.60962	2.95623	2.94134	2.77382	2.95495	2.96393
	λ	1.42057	1.7162	1.71586	1.71469	1.71665	1.73011
		2.82352	2.48552	2.48254	2.44366	2.48559	2.49306
50	α	1.20733	1.62981	1.62974	1.63008	1.63024	1.64592
		3.12951	2.6680	2.66209	2.59496	2.66732	2.67594
	λ	1.54166	1.77032	1.77072	1.77143	1.77076	1.77881
		2.6096	2.35619	2.35533	2.33853	2.3564	2.36166

Table 3. Average estimates and (risks of the estimators under SELF) for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

α	λ	$R_s(\hat{\alpha_M})$	$R_s(\hat{\alpha_s})$	$R_s(\hat{\alpha_G})$		$R_s(\hat{\lambda_M})$	$R_s(\hat{\lambda_s})$	$R_s(\hat{\lambda_G})$	
				$C = 1.5$	$C = -0.5$			$C = 1.5$	$C = -0.5$
1	1	1.18926	1.1496	1.09209	1.13836	1.10603	1.08415	1.05623	1.0787
		(0.2413)	(0.2068)	(0.1676)	(0.1985)	(0.0881)	(0.0808)	(0.0746)	(0.0794)
	2	1.18016	1.14091	1.08628	1.13022	2.22522	2.1779	2.11736	2.16609
		(0.2222)	(0.1895)	(0.1551)	(0.1821)	(0.3907)	(0.3494)	(0.3193)	(0.3429)
	3	1.17671	1.13645	1.08309	1.12601	3.35374	3.27324	3.17764	3.25459
		(0.2146)	(0.1804)	(0.1480)	(0.1735)	(0.9502)	(0.8122)	(0.7374)	(0.7961)
2	1	2.51594	2.39503	2.24553	2.36759	1.08876	1.06803	1.0493	1.0648
		(1.6743)	(1.2070)	(0.9205)	(1.1514)	(0.0635)	(0.0564)	(0.0532)	(0.0558)
	2	2.48721	2.37191	2.23164	2.34598	2.18462	2.1422	2.10237	2.13519
		(1.5061)	(1.1019)	(0.8525)	(1.0532)	(0.2723)	(0.2394)	(0.2247)	(0.2368)
	3	2.47464	2.3591	2.2233	2.3854	3.28637	3.21833	3.15587	3.23048
		(1.4331)	(1.0434)	(0.8121)	(1.0944)	(0.6494)	(0.5572)	(0.5209)	(0.5652)
3	1	3.93621	3.6472	3.40706	3.59098	1.08289	1.06056	1.04617	1.05653
		(5.4476)	(2.9496)	(2.2087)	(2.7431)	(0.0557)	(0.0463)	(0.0442)	(0.0454)
	2	3.88236	3.61435	3.38874	3.55775	2.17115	2.1265	2.09612	2.11717
		(4.8483)	(2.7246)	(2.0681)	(2.5149)	(0.2354)	(0.1962)	(0.1861)	(0.1912)
	3	3.85709	3.60264	3.39322	3.5421	3.26405	3.19776	3.15249	3.18051
		(4.5762)	(2.6409)	(2.0425)	(2.4092)	(0.5567)	(0.4619)	(0.4382)	(0.4458)

Table 4. Table 4: Risks of the estimators of α under GELF for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

		$C = -0.5$			$C = 1.5$		
α	λ	$R_G(\hat{\alpha_M})$	$R_G(\hat{\alpha_s})$	$R_G(\hat{\alpha_G})$	$R_G(\hat{\alpha_M})$	$R_G(\hat{\alpha_s})$	$R_G(\hat{\alpha_G})$
1	1	0.01478	0.01393	0.01374	0.19217	0.17097	0.14699
	2	0.01404	0.01317	0.01299	0.17968	0.15908	0.13774
	3	0.01374	0.01275	0.01259	0.17467	0.15278	0.13257
2	1	0.02052	0.01825	0.01793	0.29859	0.23729	0.19487
	2	0.01931	0.01717	0.01687	0.27468	0.21975	0.18232
	3	0.01876	0.01654	0.01684	0.26415	0.20996	0.17491
3	1	0.02514	0.01999	0.01958	0.39748	0.2604	0.21068
	2	0.02362	0.01883	0.01846	0.36345	0.24328	0.19875
	3	0.02289	0.01823	0.01787	0.39748	0.2604	0.21068

Table 5. Risks of the estimators of λ under GELF for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

		$C = -0.5$			$C = 1.5$		
α	λ	$R_G(\hat{\lambda_M})$	$R_G(\hat{\lambda_s})$	$R_G(\hat{\lambda_G})$	$R_G(\hat{\lambda_M})$	$R_G(\hat{\lambda_s})$	$R_G(\hat{\lambda_G})$
1	1	0.00773	0.0075	0.00748	0.08351	0.07823	0.07452
	2	0.00834	0.00793	0.00791	0.09137	0.08371	0.07912
	3	0.00884	0.00813	0.0081	0.09784	0.08617	0.08102
2	1	0.00593	0.0056	0.00559	0.06212	0.0565	0.05448
	2	0.00626	0.00586	0.00585	0.06608	0.05957	0.05719
	3	0.00656	0.00601	0.00603	0.06966	0.06141	0.05879
3	1	0.00531	0.00473	0.00472	0.05497	0.04725	0.0458
	2	0.00554	0.00493	0.00492	0.05783	0.04967	0.04801
	3	0.00578	0.00507	0.00506	0.05497	0.04725	0.04580

Table 6. The average 95% confidence intervals and HPD credible intervals for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

α	λ	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_s$	$\hat{\lambda}_s$
1	1	0.43419, 1.94432	0.61290, 1.59915	0.72157, 1.59942	0.78885, 1.38292
	2	0.45081, 1.90950	1.19498, 3.25545	0.72461, 1.57792	1.56223, 2.80175
	3	0.45795, 1.89547	1.75541, 4.95207	0.72605, 1.56817	2.32549, 4.23455
2	1	0.60628, 4.42559	0.66304, 1.51448	1.40878, 3.44031	0.82630, 1.31148
	2	0.66147, 4.31296	1.30680, 3.06245	1.41862, 3.37987	1.64299, 2.64491
	3	0.68665, 4.26262	1.93453, 4.63822	1.42277, 3.34938	2.45219, 3.99012
3	1	0.58566, 7.28676	0.68251, 1.48326	2.11234, 5.28990	0.84815, 1.27310
	2	0.69299, 7.07173	1.35003, 2.99225	2.12769, 5.19606	1.68985, 2.56415
	3	0.58566, 7.28676	0.68251, 1.48326	2.11234, 5.28990	0.84815, 1.27310

Table 7. MLE and Bayes estimates along with their confidence /HPD intervals for real data set.

	MLE	Bayes		
		SELF	GELF	
			$C=-0.5$	$C=1.5$
α	27.876 -9.714 = 0, 65.4664	26.5648 15.4246, 38.9235	26.2139	24.7238
λ	2.5395 1.6581, 3.4209	2.51002 2.1715, 2.8020	2.5074	2.49682

Table 8. Bayesian point prediction of future observations and corresponding predictive bounds for varying m and s for real data set

Size (m)	S	SELF	GELF		Predictive bounds
			$C=-0.5$	$C=1.5$	
5	1	1.28808	1.28085	1.25130	0.908644, 1.652350
	2	1.54413	1.53781	1.51235	1.179738, 1.951884
	3	1.75999	1.75403	1.73012	1.380534, 2.174262
	5	2.53550	2.51722	2.44541	1.715948, 3.367726
10	1	1.15339	1.14781	1.12496	0.836256, 1.415345
	2	1.33655	1.33221	1.31459	1.047185, 1.634530
	3	1.48345	1.47947	1.46325	1.189963, 1.781339
	10	2.82503	2.80789	2.74140	2.019081, 3.720739
15	1	1.10226	1.09658	1.07293	0.794486, 1.359548
	2	1.21866	1.21436	1.19681	0.936517, 1.506165
	3	1.35761	1.35421	1.34043	1.077485, 1.653755
	15	2.99531	2.97918	2.91633	2.165752, 3.874245
20	1	1.05936	1.03511	1.05464	0.794045, 1.315908
	2	1.19595	1.19213	1.17651	0.928652, 1.454710
	3	1.30105	1.29783	1.28471	1.041361, 1.567923
	20	3.14707	3.13062	3.06668	2.326170, 4.067767

Table 9. Kolmogorov-Smirnov test statistics summary for real data set

MLE	SELF	GELF		$D_{(20,0.05)}$
		C=-0.5	C=1.5	
0.1377	0.1389	0.1359	0.1224	0.294

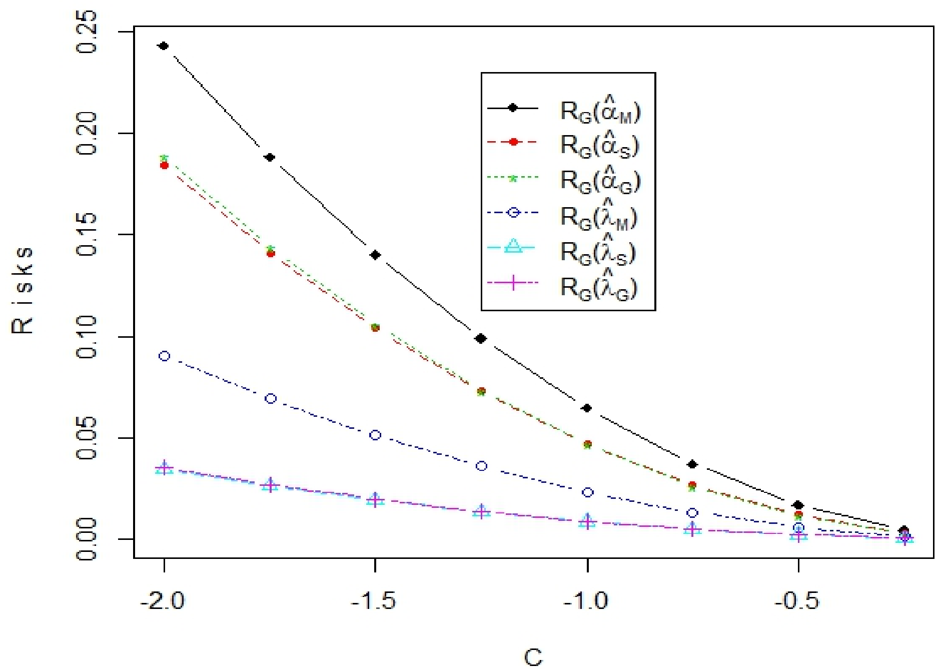

Figure 2. Risks plot of the estimators of α and λ under GELF ($C < 0$) for fixed $n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

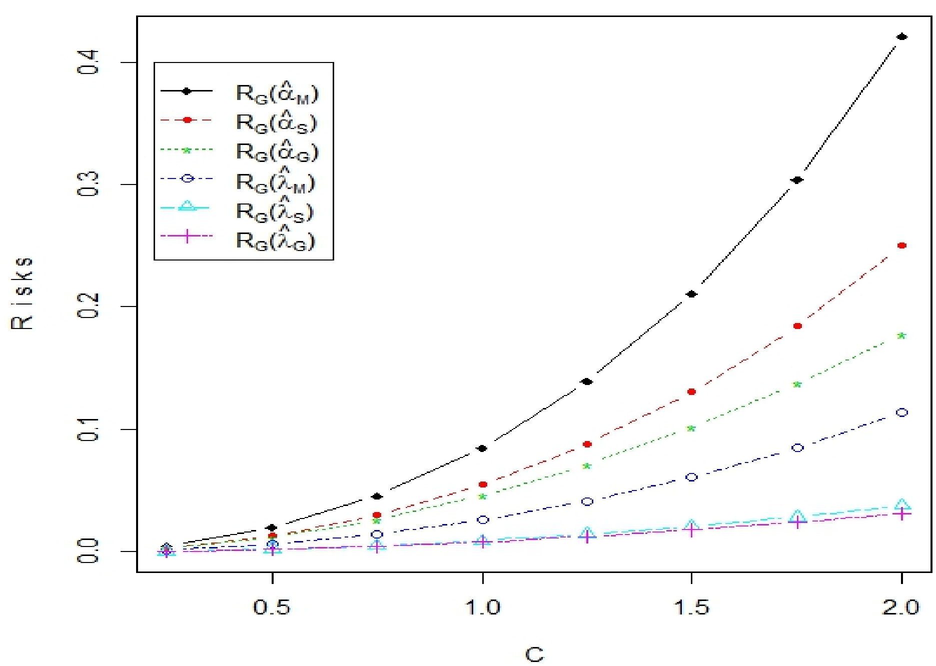

Figure 3. Risks plot of the estimators of α and λ under GELF ($C > 0$) for fixed $n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

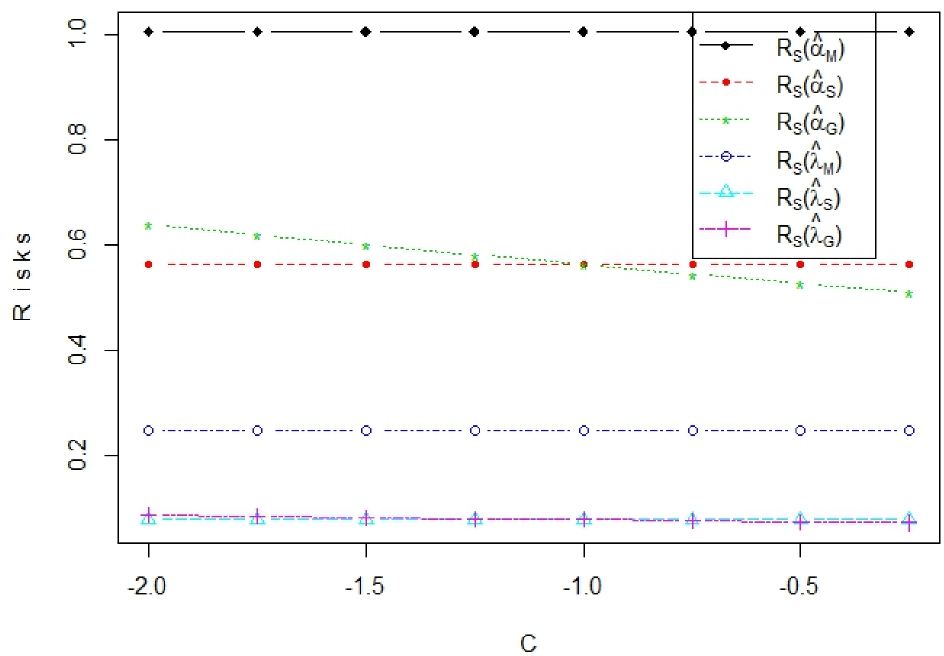

Figure 4. Risks plot of the estimators of α and λ under SELF for fixed $C < 0,\ n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

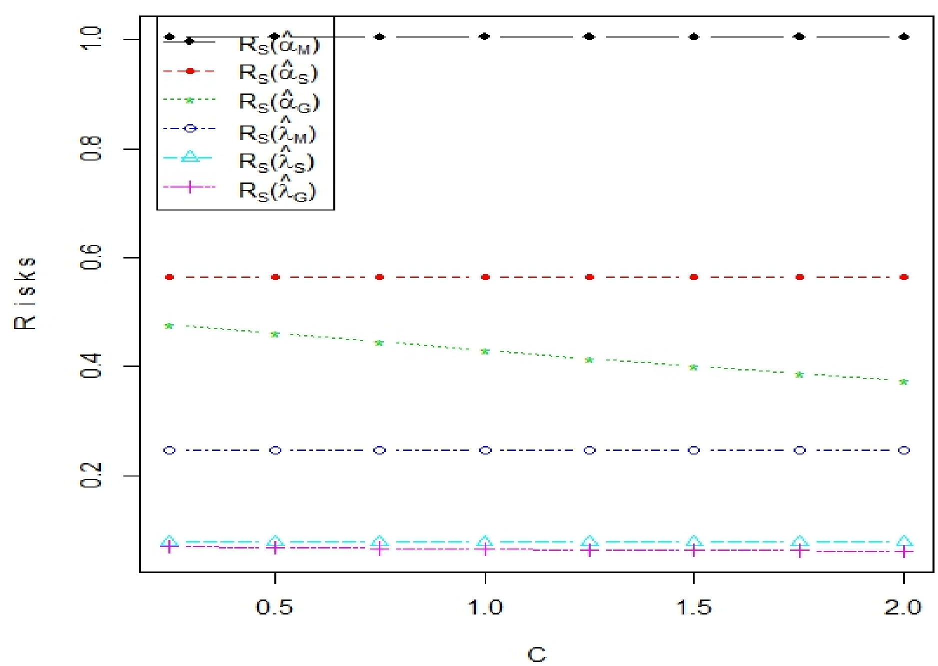

Figure 5. Risks plot of the estimators of α and λ under SELF for fixed $C > 0,\ n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

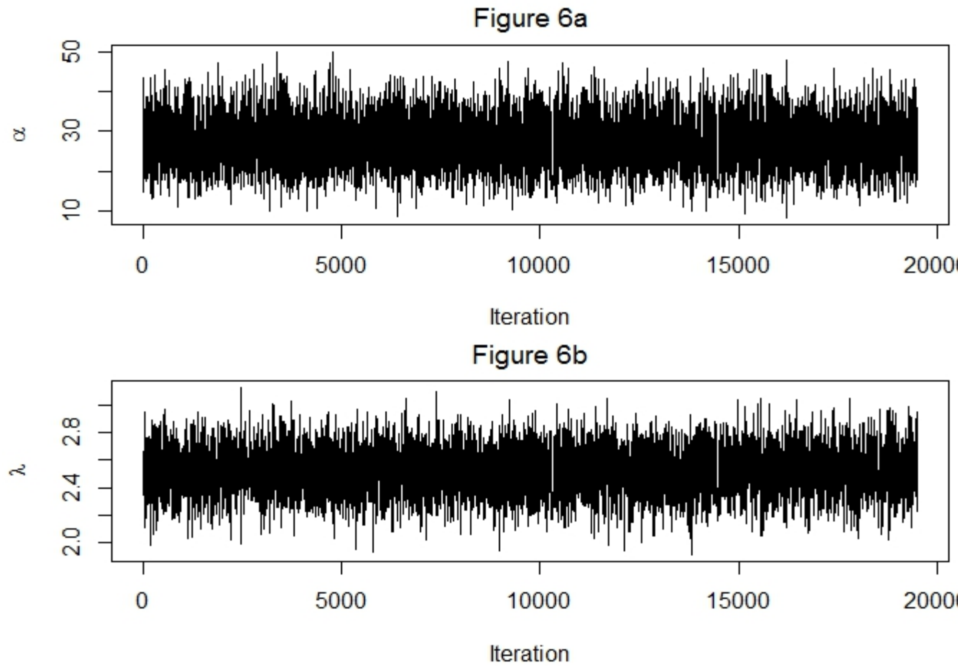

Figure 6. MCMC trace plots of simulated samples (a) for α and (b) for λ for real data set.

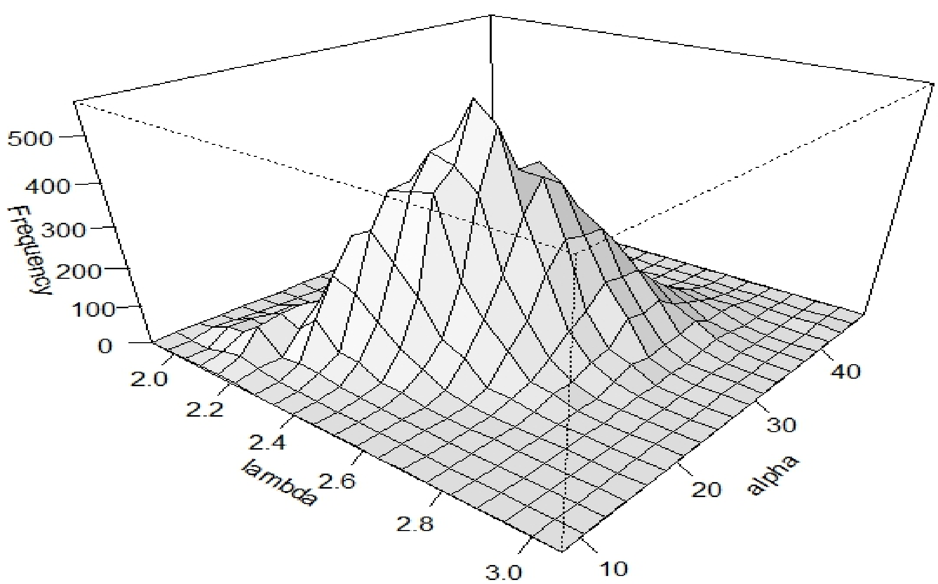

Figure 7. 3-D histogram of simulated α and λ for real data set.

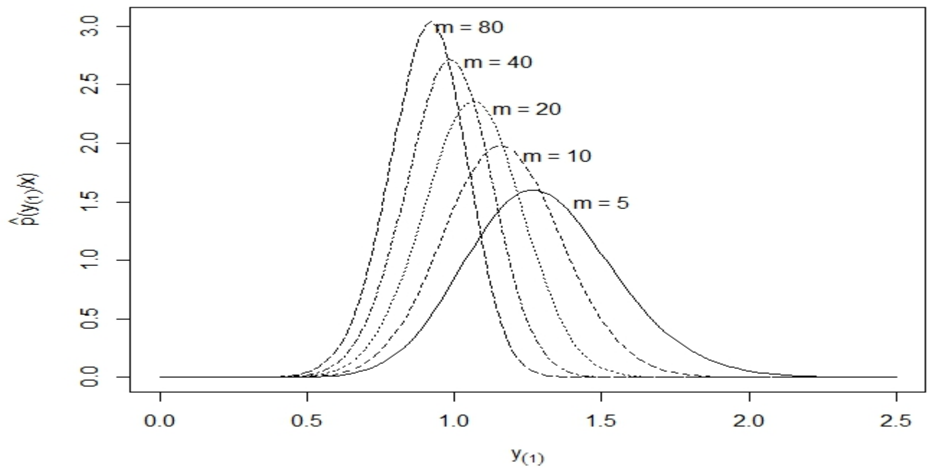

Figure 8. Density function plot of the first future observation with varying m for real data set.

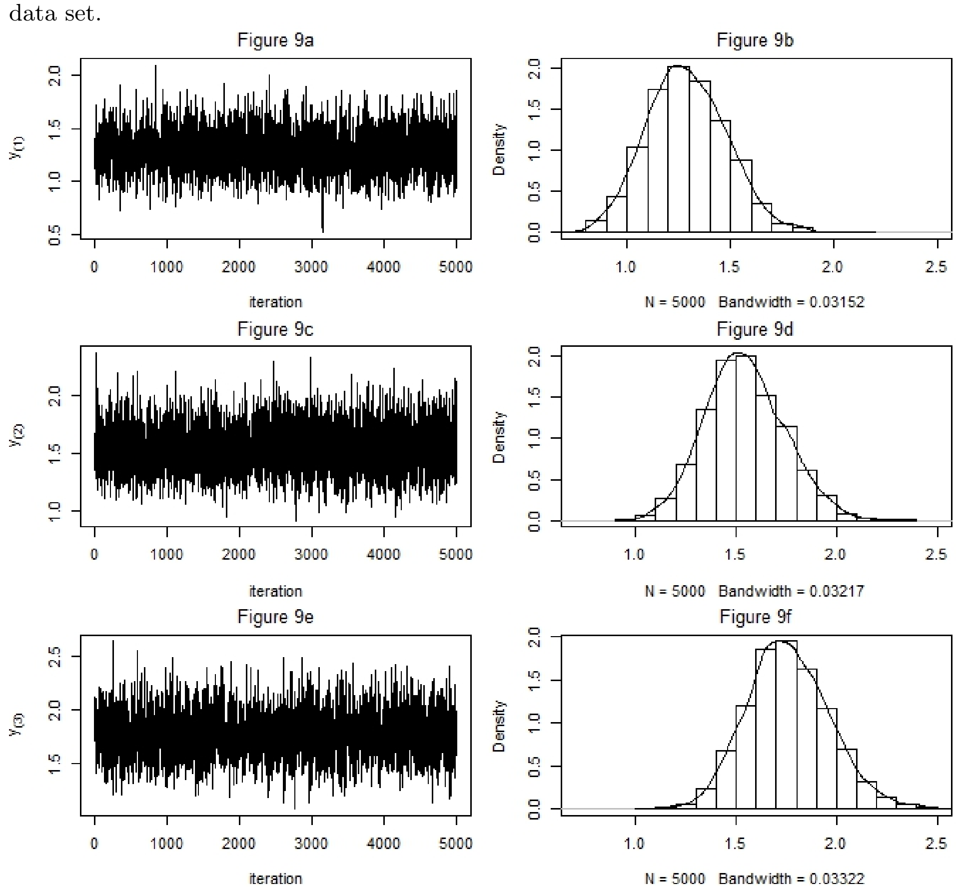

Figure 9. Trace and density plots of simulated $y_{(1)}$, $y_{(2)}$, and $y_{(3)}$ the first three future observations when $m = 5$ for real data set.

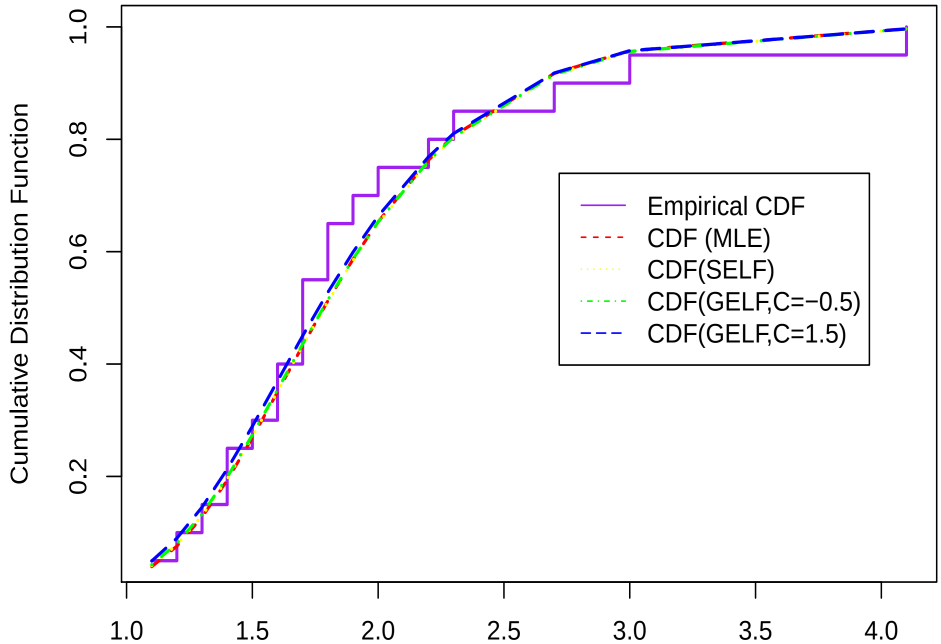

Figure 10. Empirical and fitted distribution function for real data set.