Hacettepe Journal of Mathematics and Statistics
Volume 43 (4) (2014), 679 – 687

Some conditional and unconditional expectation identities for the multivariate normal with non-zero mean

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Abstract

We give formulas for the conditional and unconditional expectations of products of multivariate Hermite and modified Hermite polynomials, each with a multivariate normal argument. A unified approach is given that covers both of these polynomials, each associated with a covariance matrix. This *extended* Hermite polynomial is associated with a matrix which is the difference between two covariance matrices, in other words, with any symmetric matrix.

Received 11/01/2012 : Accepted 22/02/2013

Keywords: Conditional expectation; Multivariate Hermite polynomial; Multivariate normal.

1. Introduction

Conditional expectation identities have a fundamental role in contingency table analysis and its applications, see, for example, Lancaster (1957). A well known identity is that $\mathbb{E}[H_n(X_2) \mid X_1] = \rho^n H_n(X_1)$, where $H_n(\cdot)$ is a Hermite polynomial and (X_1, X_2) has the standard bivariate normal distribution with correlation coefficient ρ. Conditional expectation identities are also useful for characterizing distributions, see Gupta and Ahsanullah (2004). Applications of conditional expectation identities are numerous. Some recent applications include: multivariate input processes (Biller and Ghosh, 2006); mechanisms that modulate the transfer of spiking correlations (Rosenbaum and Josić, 2011); linear-feedback sum-capacity for Gaussian multiple access channels (Ardestanizadeh *et al.*, 2012).

In this short note, we derive conditional expectation identities involving a product of multivariate Hermite and/or modified Hermite polynomials with multivariate normal arguments. This is done by extending the family of these polynomials to a polynomial associated with a matrix whose eigenvalues may be both positive and negative. Our identities generalize that due to Lancaster (1957).

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We illustrate the main results of this short note by several examples, see Example 1.1 and Examples 2.1 to 2.5. These examples give new expressions for expectations of Hermite polynomials with random arguments and expectations of products of Hermite polynomials with random arguments. Such expectations crop up in many theoretical as well as applied areas. We mention: non-central limit theorems for non-linear functionals of Gaussian fields (Dobrushin and Major, 1979, Section 4); combinatorial problems (Azor *et al.*, 1982); Wiener analysis of binary hysteresis systems (Nakayama and Omori, 1982); level crossings for regularized Gaussian processes (Berzin *et al.*, 1998); central limit theorems for functionals of level overshoot by a Gaussian field with dependence (Jeon (1998), see, for example, equation (4)); nonlinear time series analysis (Terdik (1999), see, for example, equation (2.20)); locating human faces in a cluttered scene (Rajagopalan *et al.* (2000), see, for example, equation (3)); influence of the order of input expansions in spectral stochastic finite element methods (Gaignaire *et al.* (2006), see, for example, equations (4)-(5)); stochastic finite element based on stochastic linearization for stochastic nonlinear ordinary differential equations with random coefficients (Saleh *et al.*, 2006, Section 3). Hence, the expressions given in the examples can be very useful.

Let $\mathbb{N}_+$ and $\mathbb{R}$ denote the set of non-negative integers and the set of real numbers. Suppose that

$$X = N_V \sim \mathcal{N}_p(0, V), \tag{1.1}$$

a p-dimensional normal random variable with zero means and covariance V. If $V > 0$, that is, if V is positive-definite, then its density is

$$\phi_V(x) = (2\pi)^{-p/2}\det(V)^{-1/2}\exp\left(-x'V^{-1}x/2\right)$$

for $x \in \mathbb{R}^p$. For $n \in \mathbb{N}_+^p$, $t \in \mathbb{R}^p$, $x \in \mathbb{R}^p$ and $D_j = \partial/\partial x_j$, set $n! = \prod_{j=1}^p n_j!$, $t^n = \prod_{j=1}^p t_j^{n_j}$, $(-D)^n = \prod_{j=1}^p (-D_j)^{n_j}$ and define the nth *multivariate Hermite polynomial* as

$$H_n(x, V) = \phi_V(x)^{-1}(-D)^n\phi_V(x) = \exp(q/2)(-D)^n\exp(-q/2)$$

for $q = x'V^{-1}x$ and $n \in \mathbb{N}_+^p$. This is shown in Withers (2000) to be given simply by

$$H_n(x, V) = \mathbb{E}\ \left[V^{-1}(x + \mathrm{i}X)\right]^n,$$

where $\mathrm{i} = \sqrt{-1}$, see also Withers and McGavin (2003). Its exponential generating function (egf) is

$$\sum_{n\in\mathbb{N}_+^p} H_n(x, V)t^n/n! = \mathbb{E}\left\{\exp\left[t'V^{-1}(x + \mathrm{i}X)\right]\right\} = \exp\left(t'V^{-1}x - t'V^{-1}t/2\right) \tag{1.2}$$

for $x, t \in \mathbb{R}^p$. The nth *modified multivariate Hermite polynomial* is defined by

$$H_n^*(x, V) = \phi_V(x)D^n\phi_V(x)^{-1} = \mathbb{E}\left\{\left[V^{-1}(x + X)\right]^n\right\}$$

for $n \in \mathbb{N}_+^p$. This is just $H_n(x, V)$ with all its signs positive. Its egf is

$$\sum_{n\in\mathbb{N}_+^p} H_n^*(x, V)t^n/n! = \mathbb{E}\left[\exp\left(t'V^{-1}(x + X)\right)\right] = \exp\left(t'V^{-1}x + t'V^{-1}t/2\right) \tag{1.3}$$

for $x, t \in \mathbb{R}^p$. There is a problem with notation: $H_n(x, V^{-1}) = \mathbb{E}[(Vx + \mathrm{i}X)^n]$ and its modified form exist for $V \geq 0$ (positive semi-definite), not just for $V > 0$ (positive definite). Some authors get around this by using V rather than V^{-1} as the second argument: see page 273 of Willink (2005).

We prefer to work with what we shall call the *extended Hermite polynomial*

$$h_n(x, C) = \mathbb{E}\left[(x + Y + \mathrm{i}Z)^n\right]$$

for $n \in \mathbb{N}_+^p$, $x \in \mathbb{R}^p$ and $C = A - B \in \mathbb{R}^{p\times p}$, where $Y \sim \mathcal{N}_p(0, A)$ and $Z \sim \mathcal{N}_p(0, B)$ are independent. That this polynomial only depends on (A, B) through C, follows from its egf,

$$\sum_{n\in\mathbb{N}_+^p} h_n(x, C)t^n/n! = \exp\left(t'x + t'Ct/2\right)$$

for $t \in \mathbb{R}^p$. So, $h_n(x, 0) = x^n$.

1.1. Theorem. *Using the egf, it follows that*

$$h_n\left(x_1 + x_2, C_1 + C_2\right)/n! = h_n\left(x_1, C_1\right)/n! \ \otimes h_n\left(x_2, C_2\right)/n!,$$

$$h_n\left(x, A - B\right)/n! = h_n\left(x_1, A\right)/n! \ \otimes h_n\left(x_3, -B\right)/n!,\ x = x_1 + x_3. \tag{1.4}$$

For $A > 0$, $B > 0$ we shall see that this is essentially a convolution of a Hermite polynomial and a modified Hermite polynomial. Here, $a_n \otimes b_n = \sum_{0\le k\le n,\ k\in\mathbb{N}_+^r} a_k b_{n-k}$ is the convolution of a_n and b_n in $\mathbb{N}_+^r$. The obvious choice of x_1 is x. In this case, $h_n(x_3, -B) = h_n(0, -B) = \mathrm{i}^n\mathbb{E}[Z^n]$ is essentially just a moment of a multivariate normal.

The choice of A, B is not unique. We can write $C = H'\Lambda H$, where $H'H = I$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_p) = \mathrm{diag}(\Lambda_1, 0, -\Lambda_3)$, where $\lambda_1 \ge \cdots \ge \lambda_p$ and $\Lambda_j > 0$ for $j = 1, 3$. That is, Λ_1 consists of the positive eigenvalues of C and $-\Lambda_3$ consists of the negative eigenvalues of C. Then the obvious choice of A, B is the *minimal choice*,

$$A = H'\mathrm{diag}\left(\Lambda_1, 0, 0\right)H,\ B = H'\mathrm{diag}\left(0, 0, \Lambda_3\right)H. \tag{1.5}$$

If $C = 0$, that is $A = B$, then $Y + \mathrm{i}Z \sim \mathcal{C}N_p(0, V)$, the complex normal distribution with *complex covariance* $\mathbb{E}[(YY' + ZZ')] = 2A$. Its real moments are all zero: $\mathbb{E}[(Y + \mathrm{i}Z)^n] = 0$ for $n \in \mathbb{N}_+^p$.

By (1.2) and (1.3) we have the other special cases

$$h_n(x, V) = \mathbb{E}\left[(x + X)^n\right] \text{ if } V \ge 0, \tag{1.6}$$

$$= H_n^*\left(V^{-1}x, V^{-1}\right) \text{ if } V > 0, \tag{1.7}$$

$$h_n(x, -V) = \mathbb{E}\left[(x + \mathrm{i}X)^n\right] \text{ if } V \ge 0, \tag{1.8}$$

$$= H_n\left(V^{-1}x, V^{-1}\right) \text{ if } V > 0. \tag{1.9}$$

Equivalently when $V > 0$, $H_n^*(x, V^{-1}) = h_n(Vx, V)$ and $H_n(x, V^{-1}) = h_n(Vx, -V)$. So, the class of functions $h_n(x, C)$ includes x^n, $H_n(x, V)$, $H_n^*(x, V)$ as well as the mixed case, where C has both positive and negative eigenvalues.

Note that for $p = 1$, $n \in \mathbb{N}_+$, $x \in \mathbb{R}$ and $N \sim \mathcal{N}(0, 1)$, $h_n(x, 1) = \mathbb{E}[(x + N)^n] = H_n^*(x)$ and $h_n(x, -1) = \mathbb{E}[(x + \mathrm{i}N)^n] = H_n(x)$, where $H_n(x) = H_n(x, 1)$ and $H_n^*(x) = H_n^*(x, 1)$ are the usual univariate Hermite and modified Hermite polynomials. Also

$$H_n\left(x, \sigma^{-2}\right) = \sigma^n H_n(\sigma x) = \sigma^n h_n(\sigma x, -1), \tag{1.10}$$

$$H_n^*\left(x, \sigma^{-2}\right) = \sigma^n H_n^*(\sigma x) = \sigma^n h_n(\sigma x, 1). \tag{1.11}$$

We now partition X of (1.1) as $\binom{X_1}{X_2}$ with $X_j \in \mathbb{R}^{p_j}$, so that $p_1 + p_2 = p$. We denote conditional expectation by $\mathbb{E}^{X_1}[f(X)] = \mathbb{E}[f(X)|X_1]$. Partition V into $(V_{jk} : j, k = 1, 2)$, where V_{jk} is a $p_j \times p_k$ block matrix, and set $V_{2\cdot 1} = covar(X_2|X_1) = V_{22} - V_{21}V_{11}^{-1}V_{12}$ if $p_1 > 0$, and $V_{2\cdot 1} = covar(X_2) = V_{22}$ if $p_1 = 0$. The latter corresponds to not conditioning. We assume that $V_{2\cdot 1} > 0$, that is, X_2 is not just a multiple of X_1.

Now consider $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{r\times p}$ with $\alpha_j \in \mathbb{R}^{r\times p_j}$ so that $\alpha X = \sum_{j=1}^2 \alpha_j X_j$. We have the following.

1.2. Theorem. *Suppose that* $Y = \binom{Y_1}{Y_2} = X + \mu \sim \mathcal{N}_p(\mu, V)$ *with* $Y_j \in \mathbb{R}^{p_j}$, $j = 1, 2$. *Partition* μ *in the same way. Set*

$$\delta = \left(\alpha_1 + \alpha_2 V_{21}V_{11}^{-1}\right)\left(Y_1 - \mu_1\right),\ F_0 = \alpha_2 V_{2\cdot 1}\alpha_2',\ F = F_0 + C. \tag{1.12}$$

Then

$$\mathbb{E}^{Y_1}\left[h_n\left(\alpha Y, C\right)\right] = h_n\left(\alpha\mu + \delta, F\right). \tag{1.13}$$

We now give some special cases with $\mu = 0$, that is $Y = X$ of (1.1), in terms of the Hermite polynomials H_n, H_n^*. Taking $C = 0$ gives

$$\mathbb{E}^{X_1}\left[(\alpha X)^n\right] = H_n^*\left(F_0^{-1}\delta, F_0^{-1}\right) \tag{1.14}$$

for $F_0 > 0$, that is, for $r \leq p_2$ and α_2 of rank r.

By (1.6) for $C \geq 0$, $h_n(x, C) = \mathbb{E}[(x + N_C)^n]$ so that by (1.13), the left hand side of (1.13) is equal to $h_n(\delta, F) = \mathbb{E}[(\delta + N_F)^n]$. So, by (1.7),

$$\mathbb{E}^{X_1}\left[H_n^*\left(C^{-1}\alpha X, C^{-1}\right)\right] = H_n^*\left(F^{-1}\delta, F^{-1}\right) \tag{1.15}$$

for $C > 0$. By (1.8) for $D = -C \geq 0$, $h_n(x, C) = \mathbb{E}[(x + \mathrm{i}N_D)^n]$ so that by (1.13), the left hand side of (1.13) is equal to $h_n(\delta, F) = \mathbb{E}[(\delta + iN_G)^n]$ if $G = -F = D - F_0 \geq 0$. So, by (1.9),

$$\mathbb{E}^{X_1}\left[H_n\left(D^{-1}\alpha X, D^{-1}\right)\right] = H_n\left(G^{-1}\delta, G^{-1}\right) \tag{1.16}$$

for $G = D - F_0 > 0$.

Taking $p_1 = 0$ gives $\mathbb{E}[h_n(\alpha Y, C)] = h_n(\alpha\mu, F)$, where $F = C + \alpha V_{22}\alpha'$. For example, taking $C > 0$, then $\mathbb{E}[H_n^*(C^{-1}\alpha Y, C^{-1})] = H_n^*(F^{-1}\alpha\mu, F^{-1})$.

1.3. Example. Take $r = 1$ and set $\beta = \alpha' \in \mathbb{R}^p$ and $\beta_j = \alpha_j' \in \mathbb{R}^{p_j}$. Set $v = \beta_2' V_{2\cdot 1}\beta_2$. Then by (1.14), (1.15) at $C = 1$, (1.11), (1.16) at $C = -1$, and (1.10),

$$\begin{aligned}
&(1.17) & \mathbb{E}^{X_1}\left[\left(\beta' X\right)^n\right] &= \sigma^n H_n^*(\delta/\sigma) \text{ for } \sigma^2 = v > 0,\\
&(1.18) & \mathbb{E}^{X_1}\left[H_n^*\left(\beta' X\right)\right] &= \sigma^n H_n^*(\delta/\sigma) \text{ for } \sigma^2 = 1 + v,\\
&(1.19) & \mathbb{E}^{X_1}\left[H_n\left(\beta' X\right)\right] &= \sigma^n H_n(\delta/\sigma) \text{ for } \sigma^2 = 1 - v > 0,\\
&& &= \sigma^n H_n^*(\delta/\sigma) \text{ for } \sigma^2 = v - 1 > 0,\\
&& &= \delta^n \text{ for } v = 1.
\end{aligned}$$

For example, taking $p_1 = 0$ and setting $v = \beta' V\beta$ and $N \sim \mathcal{N}(0, 1)$ gives

$$\begin{aligned}
\mathbb{E}\left[\left(\beta' X\right)^n\right] &= \sigma^n \mathbb{E}\left[N^n\right] \text{ for } \sigma^2 = v > 0,\\
\mathbb{E}\left[H_n^*\left(\beta' X\right)\right] &= \sigma^n \mathbb{E}\left[N^n\right] \text{ for } \sigma^2 = 1 + v,\\
\mathbb{E}\left[H_n\left(\beta' X\right)\right] &= \sigma^n \mathbb{E}\left[(\mathrm{i}N)^n\right] \text{ for } \sigma^2 = 1 - v > 0,\\
&= \sigma^n \mathbb{E}\left[N^n\right] \text{ for } \sigma^2 = v - 1 > 0,\\
&= \delta_{0n} \text{ for } v = 1,
\end{aligned}$$

where $\delta_{jk} = 1$ or 0 for $j = k$ or $j \neq k$. Consider the standardized bivariate normal, $p_j \equiv 1$, $V = \binom{1\rho}{\rho 1}$, where $|\rho| < 1$. Then in (1.17)-(1.19), $\delta = (\beta_1 + \rho\beta_2)X_1$ and σ^2 is given by $(1 - \rho^2)\beta_2^2$, $1 + (1 - \rho^2)\beta_2^2$, $1 - (1 - \rho^2)\beta_2^2$, respectively. If also $\beta_2 = 1$ then one can take $\sigma = \rho$ in (1.19) since ρN has the same distribution as $|\rho| N$ for $N \sim \mathcal{N}(0, 1)$, giving $\mathbb{E}^{X_1}[H_n(\beta_1 X_1 + X_2)] = \rho^n H_n((\beta_1/\rho + 1)X_1)$.

In the notation of (1.1), we can write $N_{C_1} = H' N_{L_1}$, $N_{C_3} = H' N_{L_3}$, where $L_1 = \mathrm{diag}(\Lambda_1, 0, 0)$ and $L_3 = \mathrm{diag}(0, 0, \Lambda_3)$. In our result, (1.13), the form of $h_n(x, C)$ is determined by the eigenvalues of C, while the form of $h_n(\delta, F)$ is determined by the eigenvalues of F. We now show the following.

1.4. Theorem. *An eigenvalue of F of (1.12) is either an eigenvalue of C or it satisfies*

$$\det\left(V_{2\cdot 1}^{-1} - J_\lambda\right) = 0, \tag{1.20}$$

where $J_\lambda = \alpha_2'(\lambda I_r - C)^{-1}\alpha_2$.

In Section 2, we extend our result, (1.13), to products of extended Hermite polynomials. Some conclusions and future work are noted in Section 3. The proofs of all main results are provided in the appendix.

2. An extension to products

We now give an extension of Theorem 1.2 to products. Suppose that for $1 \le k \le K$, $n_k \in \mathbb{N}_+^{r_k}$, $t_k \in \mathbb{R}^{r_k}$, C_k in $\mathbb{R}^{r_k \times r_k}$, $\alpha_k = (\alpha_{k1}, \alpha_{k2}) \in \mathbb{R}^{r_k \times p}$, $\alpha_{kj} \in \mathbb{R}^{r_k \times p_j}$.

2.1. Theorem. *Suppose that $Y = \binom{Y_1}{Y_2} \sim \mathcal{N}_p(\mu, V)$ with $Y_j \in \mathbb{R}^{p_j}$, $j = 1, 2$. Partition μ in the same way. Set $\delta_k = (\alpha_{k1} + \alpha_{k2} V_{21} V_{11}^{-1})(Y_1 - \mu_1)$, $\widetilde{\delta}_k = \alpha_k \mu + \delta_k$, $\widetilde{\delta}' = (\widetilde{\delta}_1', \ldots, \widetilde{\delta}_K')$, $n' = (n_1, \ldots, n_K)$, $\gamma' = (\alpha_{12}', \ldots, \alpha_{K2}')$, $\Lambda = \mathrm{diag}(C_1, \ldots, C_K)$ and $D = \Lambda + \gamma V_{2\cdot 1} \gamma'$. Then*

$$\mathbb{E}^{Y_1}\left[\prod_{k=1}^{K} h_{n_k}(\alpha_k Y, C_k)\right] = h_n\left(\widetilde{\delta}, D\right). \tag{2.1}$$

2.2. Example. Find $a_n = \mathbb{E}[\prod_{k=1}^{K} H_{n_k}(N)]$ for $N \sim \mathcal{N}(0,1)$. So, $\mu = p_1 = 0$, $p_2 = V = \alpha_k = 1$, $C_k = -1$, $\widetilde{\delta} = 0$, $\gamma = 1_K$, the K-vector of ones, $D = 1_K 1_K' - I_K$.
If $K = 1$ then $D = 0$ so $a_n = \delta_{n0}$.
If $K = 2$ then $D = \binom{01}{10}$ and $q = t_1 t_2$. The coefficient of t^n in $\exp(q)$ is $\delta_{n_1 n_2}/n_1!$ so that $a_n = h_n(0, D) = \delta_{n_1 n_2} n_1!$. So, $\{H_n(x)/n!^{1/2}\}$ are orthonormal with respect to $\phi(x)$, as is well-known.
If $K = 3$ then $q = t_1 t_2 + t_2 t_3 + t_3 t_1$, $\exp(q) = \sum_{m_1, m_2, m_3 = 0}^{\infty} t^n / m!$ at $n_1 = m_2 + m_3$, $n_2 = m_1 + m_3$, $n_3 = m_1 + m_2$. Set $|n| = \sum_{k=1}^{3} n_k$. So, $a_n = 0$ if $|n|$ is odd, while if $|n|$ is even, then $a_n = n!/m!$ at $2m_1 = n_2 + n_3 - n_1$, $2m_2 = n_1 + n_3 - n_2$, $2m_3 = n_1 + n_2 - n_3$.
If $K = 4$ then $q = \sum_{1 \le j < k \le 4} t_j t_k$ has six terms. Also

$$\exp(q) = \sum_{m_{12}, \ldots, m_{34} = 0}^{\infty} t^n / (m_{12}! \cdots m_{34}!)$$

at $n_j = \sum_{k \ne j} m_{jk}$, where $m_{kj} = m_{jk}$, $j = 1, \ldots, 4$. This gives four equations in six ms, so we can make m_{12} and m_{13} arbitrary, giving

$$\begin{aligned} &m_{14} = n_1 - m_{12} - m_{13},\ 2m_{23} = n_2' + n_3' - n_4', \\ &2m_{24} = n_2' + n_4' - n_3',\ 2m_{34} = n_3' + n_4' - n_2', \end{aligned} \tag{2.2}$$

where $n_j' = n_j - m_{1j}$. So,

$$a_n = n! \sum_{m_{12}, m_{13}} 1/\left[m_{12}! \cdots m_{34}!\right]_{(2.2)}.$$

For example, take $n_1 = 1$. Then $(m_{12}, m_{13}) = (0,0)$, $(0,1)$ or $(1,0)$. So, setting $n_{23} = n_2 + n_3 - n_4$, $n_{24} = n_2 + n_4 - n_3$, $n_{34} = n_3 + n_4 - n_2$, $M_{jk} = (n_{jk} - 1)/2$ and $N_{jk} = (n_{jk} + 1)/2$, we have $a_n/n! = 1/N_{23}! M_{24}! M_{34}! + 1/M_{23}! N_{24}! M_{34}! + 1/M_{23}! M_{24}! N_{34}!$.

2.3. Example. Find $a_n = \mathbb{E}[\prod_{k=1}^{K} H_{n_k}^*(N)]$ for $N \sim \mathcal{N}(0,1)$. The parameters are as for Example 2.1 except that $C_k = 1$, $D = 1_K 1_K' + I_K$ and $q = \sum_{k=1}^{K} t_k^2 + \sum_{1 \le j < k \le K} t_j t_k$.
If $K = 1$ then $q = t^2$ and $\exp(q) = \sum_{m=0}^{\infty} t^{2m}/m!$, giving $a_{2m+1} = 0$ and $a_{2m} = (2m)!/m!$.
If $K = 2$ then $q = t_1^2 + t_2^2 + t_1 t_2$ and $\exp(q) = \sum_{m_1, m_2, m_3 = 0}^{\infty} t^n / m!$ at $n_j = 2m_j + m_3$.
So, $a_n = 0$ if $|n|$ is odd, while if $|n|$ is even and $n_1 \le n_2$, then

$$a_n = n! \sum_{k=0}^{\min(n_1, n_2)} 1/m_1! m_2! m_3! \Big|_{2m_1 = n_1 - k,\ 2m_2 = n_2 - k},$$

writing k for m_3. For example,

$$a_{0n_2} = \mathbb{E}\left[H^*_{n_2}(N)\right] = n_2!/\left(n_2/2\right)!,$$
$$a_{1n_2} = \mathbb{E}\left[NH^*_{n_2}(N)\right] = n_2!/\left(\left[n_2-1\right]/2\right)!,\ n_2 \geq 1,$$
$$a_{3n_2} = \mathbb{E}\left[\left(N^3+3N\right)H^*_{n_2}(N)\right] = 3!n_2!\left\{1/\left(\left[n_2-1\right]/2\right)! + 1/3!\left(\left[n_2-3\right]/2\right)!\right\},\ n_2 \geq 3.$$

2.4. Example. Find $a_n = \sigma^n \mathbb{E}[H^*_{n_1}(N/\sigma_1)\ H_{n_2}(N/\sigma_2)] = \mathbb{E}[h_{n_1}(N,\sigma_1^2)\ h_{n_2}(N,-\sigma_2^2)]$. So, $K=2$, $C_1=\sigma_1^2$, $C_2=-\sigma_2^2$, $D = 1_2 1_2' + \mathrm{diag}(\sigma_1^2, -\sigma_2^2)$ and $q=\sum_{j=1}^2 c_j t_j^2 + t_1 t_2$, where $c_1 = (1+\sigma_1^2)/2$ and $c_2=(1-\sigma_2^2)/2$. So, in a derivation similar to that of Example 2.2 for $K=2$ one obtains $a_n=0$ for $|n|$ odd, while if $|n|$ is even and $n_1 \leq n_2$, then

$$a_n = n! \sum_{k=0}^{\min(n_1,n_2)} c_1^{m_1} c_2^{m_2}/m_1!m_2!m_3! \Big|_{2m_1=n_1-k,\ 2m_2=n_2-k}. \tag{2.3}$$

For example,

$$a_{0n_2} = n_2! c_2^{n_2/2}/\left(n_2/2\right)!,$$
$$a_{1n_2} = n_2! c_2^{(n_2-1)/2}/\left(\left[n_2-1\right]/2\right)!,\ n_2 \geq 1,$$
$$a_{3n_2} = 3!n_2!\left\{c_1 c_2^{(n_2-1)/2}/\left(\left[n_2-1\right]/2\right)! + c_2^{(n_2-3)/2}/3!\left(\left[n_2-3\right]/2\right)!\right\},\ n_2 \geq 3.$$

2.5. Example. Find $a_n = \sigma^n \mathbb{E}[H_{n_1}(N/\sigma_1)\cdots H_{n_K}(N/\sigma_K)] = \mathbb{E}[h_{n_1}(N,C_1)\cdots h_{n_K}(N,C_K)]$, where $C_k = -\sigma_k^2$, $\mu=0$, $V=1$, $\gamma = 1_K$, $D = 1_K 1_K' - \mathrm{diag}(\sigma_k^2)$, $q = \sum_{j=1}^K c_j t_j^2 + \sum_{1\leq j<k\leq K} t_j t_k$ and $c_k = (1-\sigma_k^2)/2$. So,

$$\exp(q) = \sum_m \prod_{k=1}^K \left(c_k t_k^2\right)^{m_k}/m_k!\ \sum_M \prod_{j<k} \left(t_j t_k\right)^{M_{jk}}/M_{jk}!,\ n_k = 2m_k + \sum_{j\neq k} M_{jk}.$$

If $K=2$ then $a_n = 0$ if $|n|$ is odd, while if $|n|$ is even and $n_1 \leq n_2$, then a_n is given by (2.3).

By Theorem 2.1, Examples 2.1-2.4 can be extended by (i) replacing N by $\mu+N$; and (ii) replacing expectation by conditional expectation.

According to (1.4), the form of h_n depends on the positive and negative parts of D, and these are determined by the positive and negative eigenvalues of D, which we now obtain. Suppose that x is an eigenvector of D with eigenvalue λ. Then $Dx=\lambda x$ implies $\gamma V_{2\cdot 1}\gamma' x = (\lambda I - \Lambda)x$. So, either λ is an eigenvalue of Λ, that is an eigenvalue of C_k for some $k \in \{1,\ldots,K\}$, or $\det(\lambda I - \Lambda) \neq 0$ and $z = V_{2\cdot 1}\gamma' x$ satisfies $z = V_{2\cdot 1}\gamma'(\lambda I - \Lambda)^{-1}\gamma z$, which implies that

$$\det\left(V_{2\cdot 1}^{-1} - J_\lambda\right) = 0, \tag{2.4}$$

where $J_\lambda = \sum_{k=1}^K \alpha_{k2}'(\lambda I_{r_k} - C_k)^{-1}\alpha_{k2}$. That is, the r eigenvalues of D are given by (2.4) and the eigenvalues of $\{C_k\}$. The number of roots of (2.4) depends on the number of distinct $\{C_k\}$. We illustrate this with an important example that takes up the rest of this section.

2.6. Example. Suppose that $r_k \equiv 1$, that is, the arguments of the h_{n_k} functions in (2.1) are all scalar. Consider the case when $C_k = -1$, 0 or 1. Then

$$J_\lambda = \sum_{j=-1}^{1} (\lambda - j)^{-1} w_j, \tag{2.5}$$

where $w_j = \sum_{C_k=j} \alpha_{k2}'\alpha_{k2} \geq 0 \in \mathbb{R}^{p_2\times p_2}$ and $w = \sum_{k=1}^K w_j > 0$, assuming that $\alpha_{k2} \neq 0$ for $k=1,\ldots,K$. (We can always assume this, since if $\alpha_{k2}=0$, then the corresponding term in the product on the left hand side of (2.1) can be factored out.) There are a number of cases to consider when evaluating the roots of (2.4).

Consider the situation, where $p_2 = 1$. Then $\{w_j\}$ are scalar and the equation (2.4) for the eigenvalues of D of (2.1) becomes the cubic $v = \sum_{j=-1}^{1} (\lambda - j)^{-1} w_j$, where $v = V_{2\cdot 1}^{-1}$.

The case $w_{-1} > 0 = w_0 = w_1$: The only root is $\lambda = w/v - 1$.

The case $w_0 > 0 = w_{-1} = w_1$: The only root is $\lambda = w/v$.

The case $w_1 > 0 = w_{-1} = w_0$: The only root is $\lambda = w/v + 1$.

The case $w_{-1} > 0$, $w_0 > 0$, $w_1 = 0$: There are two roots, $\lambda = (w - v \pm \epsilon_1^{1/2})/(2v)$, where $\epsilon_1 = (w - v)^2 + 4vw_0 > 0$. So, one root is positive and the other negative.

The case $w_{-1} > 0$, $w_1 > 0$, $w_0 = 0$: There are two roots, $\lambda = (w \pm \epsilon_0^{1/2})/(2v)$, where $\epsilon_0 = w^2 + 4v(v + w_1 - w_{-1}) = (w - 2v)^2 + 8vw_1 > 0$. If $v + w_1 < w_{-1}$ then both roots are positive. If $v + w_1 > w_{-1}$ then one root is positive and the other negative.

The case $w_0 > 0$, $w_1 > 0$, $w_{-1} = 0$: There are two positive roots, $\lambda = (v + w \pm \epsilon_{-1}^{1/2})/(2v)$, where $\epsilon_{-1} = (v + w)^2 - 4vw_0 = (w - v)^2 + 4vw_1 > 0$.

The general case $w_{-1} > 0$, $w_0 > 0$, $w_1 > 0$: There are three roots, those of $v\lambda^3 - w\lambda^2 + (w_{-1} - w_1 - v)\lambda + w_0 = 0$. So, one or two roots are positive and the other two or one are negative.

3. Conclusions

We have given new expressions for conditional and unconditional expectations of products of multivariate Hermite polynomials with multivariate normal arguments. This required development of an extended Hermite polynomial. Some possible applications of the expressions are noted.

Future work is to extend the results to matric variate Hermite polynomials and complex variate Hermite polynomials. The future work could also consider multivariate non-normal arguments, matric variate non-normal arguments and complex variate non-normal arguments.

Appendix: Proofs

Proof of Theorem 1.2: We prove (1.13). We use the well-known representation

$$X_2 = N_A + BX_1,\ A = V_{2\cdot 1},\ B = V_{21} V_{11}^{-1}, \tag{A.1}$$

where $N_A \sim \mathcal{N}(0, A)$ is independent of X_1. (This works since it implies that $X_2 \sim \mathcal{N}(0, V_{22})$, $\mathbb{E}[X_2 X_1'] = V_{21}$, and X is normal.) So, for $\alpha = (\alpha_1, \alpha_2)$, $\alpha_j \in \mathbb{R}^{r \times p_j}$, $\alpha X = \sum_{j=1}^{2} \alpha_j X_j = \delta + \alpha_2 N_A$, where $\delta = (\alpha_1 + \alpha_2 B) X_1$. (1.13) now follows since the egf of $\mathbb{E}^{Y_1}[h_n(\alpha Y, C)]$ is $\mathbb{E}^{Y_1}\left[\exp\left(t'\alpha Y + t'Ct/2\right)\right] = \exp\left[t'(\alpha\mu + \delta) + t'Ft/2\right]$.

Proof of Theorem 1.1: Using the minimal choice of A, B of (1.5), for $t \in \mathbb{R}^p$ set $u = Ht$ so that $t = H'u$, $t'Ct = u'\Lambda u = u_1'\Lambda_1 u_1 - u_3'\Lambda_3 u_3$, where $u_j = H_j t$, partitioning H' as $(H_1' H_2' H_3')$. For any $x_1 \in \mathbb{R}^p$, set $x_3 = x - x_1$,

$$q_1 = t'x_1 + u_1'\Lambda_1 u_1/2 = t'x_1 + u'V_1 u/2 = t'x_1 + t'C_1 t/2,$$
$$q_3 = t'x_3 - u_3'\Lambda_3 u_3/2 = t'x_3 - u'V_3 u/2 = t'x_3 - t'C_3 t/2,$$

say with $C_j = H_j'\Lambda_j H_j \geq 0$. So, $\exp(q_1)$ is the egf of $h_n(x_1, C_1) = \mathbb{E}[(x_1 + N_{C_1})^n]$ and $\exp(q_3)$ is the egf of $h_n(x_3, -C_3) = \mathbb{E}[(x_3 + \mathrm{i}N_{C_3})^n]$. But $\exp(q_1 + q_3)$ is the egf of $h_n(x, C)$. So, we obtain

$$h_n(x, C) = \sum_{n_1 + n_3 = n} \binom{n}{n_1} h_{n_1}(x_1, C_1)\, h_{n_3}(x_3, -C_3),$$

that is, $h_n(x, C)/n!$ can be written as (1.4), the convolution of $h_n(x_1, C_1)/n$ and $h_n(x_3, -C_3)/n!$. □

Proof of Theorem 1.3: For λ an eigenvalue of F with eigenvector x, $Fx = Cx + \alpha_2 z$, where $z = V_{2\cdot 1}\alpha_2' x \in \mathbb{R}^{p_2}$. In the special case that $\alpha_2' x = 0$, that is, $z = 0$ then λ is an eigenvalue of C with eigenvector x and $\alpha_2' x = 0$, implying p_2 constraints on x so that $p_2 < r$ and we can take $r - p_2$ orthogonal solutions for the eigenvalues x from the nullspace $\{x : \alpha_2' x = 0\}$. But if $z \neq 0$ then $\lambda x = Fx = Cx + \alpha_2 z$ implies that $z = V_{2\cdot 1}\alpha_2'(\lambda I_r - C)^{-1}\alpha_2 z$ with $z \neq 0$ so that (1.20) holds. □

Proof of Theorem 2.1: For A of (A.1), $\alpha_k Y = \widetilde{\delta}_k + \alpha_{k2} N_A$. So, for $t' = (t_1', \ldots, t_K') \in \mathbb{R}^r$, $r = \sum_{k=1}^K r_k$, the egf of $\mathbb{E}^{Y_1}[\prod_{k=1}^K h_{n_k}(\alpha_k Y, C_k)]$ as a function of $(n_1, \ldots, n_K) \in \mathbb{R}^r$ is

$$\text{(A.2)} \qquad \mathbb{E}^{Y_1}\left[\exp\left\{\sum_{k=1}^K \left(t_k'\alpha_k Y + t_k' C_k t_k/2\right)\right\}\right] = \exp(q),$$

where $q = \sum_{k=1}^K q_k$, $q_k = t_k'\widetilde{\delta}_k + t_k' D_k t_k/2$ and $D_k = \alpha_{k2} V_{2\cdot 1}\alpha_{k2}' + C_k$. The exponent in the left hand side of (A.2) is

$$\sum_{k=1}^K \left[t_k'\left(\widetilde{\delta}_k + \alpha_{k2} N_A\right) + t_k' C_k t_k/2\right] = \sum_{k=1}^K \left[t_k'\widetilde{\delta}_k + t_k' C_k t_k/2\right] + u' N_A,$$

where $u = \sum_{k=1}^K \alpha_{k2}' t_k = \gamma' t$ and $t' = (t_1', \ldots, t_K')$. So,

$$q = \sum_{k=1}^K \left(t_k'\widetilde{\delta}_k + t_k' C_k t_k/2\right) + u' A u/2 = t'\widetilde{\delta} + t' D t/2.$$

This completes the proof. □

Acknowledgments

The authors would like to thank the Editor and the referee for careful reading and for their comments which greatly improved the paper.

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