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Slant Helix Curves and Acceleration Centers

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Abstract

In this study, an alternative one-parameter motion to Frenet motion of a rigid-body in 3-dimensional Euclidean space $\mathbb{E}^3$ is given by moving the coordinate frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ instead of the Frenet frame $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ along a unit speed curve $\alpha(t)$, where $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{W}$ correspond, respectively, to unit principal normal vector field, derivative vector field of the unit principal normal vector field and Darboux vector field of the unit speed curve $\alpha(t)$. Also the concepts fixed axode, striction curve, instantaneous pole points, acceleration pole points (or acceleration centers) and instant screw axis (ISA) of this alternative one-parameter motion are studied.

1. INTRODUCTION

The theory of curves and motion of a rigid-body in 3-dimensional Euclidean space $\mathbb{E}^3$ are the most two fundamental areas in differential geometry. These areas have some applications in computer animation, rigid-body (i.e., robot) kinematics, mechanism, etc. An important reference for rigid-body kinematics is the study about Frenet and Bishop motions of Bottema and Roth, see [1]. Also, there are other studies about these areas, see [2-7].

The aim of this paper is to give and analyze some concepts (e.g., instant screw axis (ISA), instantaneous pole points, acceleration pole points, axode in the fixed space, striction curve) about an alternative one-parameter motion of a rigid-body in $\mathbb{E}^3$ obtained by moving the frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ along a unit speed curve $\alpha(t)$, where $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{W}$ correspond, respectively, to unit principal normal vector field, derivative vector field of the unit principal normal vector field and Darboux vector field of the unit speed curve $\alpha(t)$.

The concepts instantaneous pole points and acceleration pole points up to second order of the alternative one-parameter motion $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$ of a rigid-body are given by using the determinants of the derivative matrices $\boldsymbol{\mathcal{A}}'$, $\boldsymbol{\mathcal{A}}''$ and $\boldsymbol{\mathcal{A}}'''$, and the concept instant screw axis (ISA) of this alternative one-parameter rigid-body motion is given by using rank$\boldsymbol{\mathcal{A}}'$ and rank$\boldsymbol{\mathcal{A}}''$, where $\boldsymbol{\mathcal{A}}$ corresponds to the rotation matrix of the alternative rigid-body motion.

Since the instantaneous pole points, first-order acceleration pole points and second-order acceleration pole points of the alternative one-parameter rigid-body motion $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$ exist only, respectively, when $det\boldsymbol{\mathcal{A}}'\neq 0$, $det\boldsymbol{\mathcal{A}}''\neq 0$ and $det\boldsymbol{\mathcal{A}}'''\neq 0$, it is examined whether or not these conditions are satisfied by taking the unit speed base curve $\alpha(t)$ as cylindrical helix (or general helix), slant helix, constant precession slant helix, C-slant helix or constant precession C-slant helix.

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2. PRELIMINARIES

In this section, firstly some basic concepts of *curves* (e.g., *unit speed curves, principal-direction curves, cylindrical helices (or general helices), slant helices, C–slant helices*) and *ruled surfaces* (e.g., *central point*, *striction curve*) in $\mathbb{E}^3$ will be given. Afterwards definitions of *one-parameter motion* and its *instant screw axis (ISA), instantaneous pole points* and *acceleration pole points* in $\mathbb{E}^3$ will be given.

2.1. Curves in $\mathbb{E}^3$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary curve in $\mathbb{E}^3$ defined on an open interval I. If $\langle \alpha'(t), \alpha'(t) \rangle = 1$ for all $t \in I$, then α is called *unit speed curve* (or *parameterized by the arc-length function* t), where $\langle , \rangle$ denotes the standard scalar product on $\mathbb{R}^3$.

Let α be a unit speed curve, then α is called *cylindrical helix* (or *general helix*) if its unit tangent vector field $\boldsymbol{T} = \alpha'$ makes a constant angle $\theta \in \mathbb{R}$ with a fixed direction unit vector $\vec{\boldsymbol{u}}$ along α, i.e., if $\langle \boldsymbol{T}, \vec{\boldsymbol{u}} \rangle = cos\theta$ is constant along α. Thus, the following theorem can be given.

Theorem 1. A unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is called *cylindrical helix* (or *general helix*) if and only if the ratio τ/κ is constant, where $\kappa = \|\alpha''\| > 0$ being the curvature and $\tau = \langle \alpha' \times \alpha'', \alpha''' \rangle / \kappa^2$ the torsion of α. By the last equation "×" denotes the standard cross product on $\mathbb{R}^3$, see [8].

The unit speed curve α is called *slant helix* if its unit principal normal vector field $\boldsymbol{N} = \alpha''/\|\alpha''\|$ makes a constant angle $\theta \in \mathbb{R}$ with a fixed direction unit vector $\vec{\boldsymbol{u}}$ along α, i.e., if $\langle \boldsymbol{N}, \vec{\boldsymbol{u}} \rangle = cos\theta$ is constant along α. Thus, the following theorem can be given.

Theorem 2. A unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is called *slant helix* if and only if the value of

$$\sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'$$

is constant, where $\kappa > 0$ being the curvature and τ the torsion of α, see [9].

The unit speed curve α is called C–*slant helix* if its derivative vector field of the unit principal normal vector field $\boldsymbol{C} = \boldsymbol{N}'/\|\boldsymbol{N}'\|$ makes a constant angle $\theta \in \mathbb{R}$ with a fixed direction unit vector $\vec{\boldsymbol{u}}$ along α, i.e., if $\langle \boldsymbol{C}, \vec{\boldsymbol{u}} \rangle = cos\theta$ is constant along α. Thus, the following theorem can be given.

Theorem 3. A unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is called C–*slant helix* if and only if the value of

$$\frac{f^2}{(f^2 + g^2)^{3/2}} \left(\frac{g}{f}\right)'$$

is constant for

$$f = \sqrt{\kappa^2 + \tau^2}, \;\; g = \frac{\kappa^2}{\kappa^2 + \tau^2} \left(\frac{\tau}{\kappa}\right)' = \sigma f,$$

where $\kappa > 0$ being the curvature and τ the torsion of α, see [10].

From Theorem 2 and Theorem 3 it can be said that a unit speed curve α is called *slant helix* if and only if the value of $f/g = 1/\sigma$ is constant.

An integral curve of the principal vector field $\boldsymbol{N}$ of a unit speed curve α is called the *principal-direction curve* of α. This curve will be denoted by $\beta(t) = \int \boldsymbol{N}(t)dt$ in this study and has unit speed, see [11].

2.2. Ruled Surfaces in $\mathbb{E}^3$

Given a differentiable one-parameter family of lines $\{\psi(t), w(t)\}$, which means that to each $t \in I$ corresponds a point $\psi(t) \in \mathbb{R}^3$ and a vector $0 \neq w(t) \in \mathbb{R}^3$ so that both $\psi(t)$ and $w(t)$ depend differentiably on t, the parametrized surface

$$\boldsymbol{s}(t,\mu) = \psi(t) + \mu w(t), \quad t \in I, \quad \mu \in \mathbb{R},$$

is called the *ruled surface* generated by the family $\{\psi(t), w(t)\}$. The lines L_t, which pass through $\psi(t)$ and are parallel to $w(t)$, are called the *rulings*, and the regular curve $\psi(t)$ is called the *directrix* (or *base curve*) of the surface $\boldsymbol{s}(t,\mu)$, see [12].

If there exist a common perpendicular to two constructive rulings in the ruled surface, then the foot of the common perpendicular on the main rulings is called a *central point*. The locus of the central points is called the *striction curve*, see [13]. By taking $w(t)$ as unit and $\|w'(t)\| \neq 0$, then the parametrization of the striction curve on the ruled surface $\boldsymbol{s}(t,\mu)$ can be given by the equation

$$\tilde{\psi}(t) = \psi(t) - \frac{\langle \psi'(t), w'(t) \rangle}{\|w'(t)\|^2} w(t).$$

As a result the following theorem can be given.

Theorem 4. Let $\boldsymbol{s}(t,\mu) = \psi(t) + \mu w(t)$ be a ruled surface for all $t \in I$ and $\mu \in \mathbb{R}$. Then the striction curve $\tilde{\psi}(t)$ on the ruled surface $\boldsymbol{s}(t,\mu)$ coincides with the directrix $\psi(t)$ of $\boldsymbol{s}(t,\mu)$, i.e., $\tilde{\psi}(t) = \psi(t)$, if and only if $\langle \psi'(t), w'(t) \rangle = 0$, see [14].

2.3. One-Parameter Motion in $\mathbb{E}^3$

A *one-parameter motion* of a rigid-body in E^3 can be generated by the transformation

$$F: \mathbb{E}^3 \to \mathbb{E}^3$$

defined by

$$\boldsymbol{X} \mapsto F(\boldsymbol{X}) = \boldsymbol{Y} = \boldsymbol{\mathcal{A}}\boldsymbol{X} + \boldsymbol{\mathcal{C}}$$

in which $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the *position vectors*, which correspond to 3×1 column matrix, of a moving point X measured in 3-dimensional moving space E^3 and reference space (or fixed space) $\mathbb{E}^3$, respectively. $\boldsymbol{\mathcal{A}}$ is the *rotation vector*, which corresponds to 3×3 positive orthogonal matrix (i.e., $\boldsymbol{\mathcal{A}}^{\mathrm{T}} = \boldsymbol{\mathcal{A}}^{-1}$ and $det\boldsymbol{\mathcal{A}} = 1$) and $\boldsymbol{\mathcal{C}}$ the *displacement vector* of the origin in $\mathbb{E}^3$, which corresponds to 3×1 column matrix, both depend upon the motion parameter t. Any point X of E^3 describes a curve in $\mathbb{E}^3$. $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ are smooth functions of the motion parameter t. It will be assumed that for the initial time $t = 0$ the origins of E^3 and $\mathbb{E}^3$ coincide, see [1, 15].

The matrix representation corresponding to the one-parameter rigid-body motion $\boldsymbol{Y} = \boldsymbol{\mathcal{A}}\boldsymbol{X} + \boldsymbol{\mathcal{C}}$ can be given as

$$\begin{bmatrix} \boldsymbol{Y} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{A}} & \boldsymbol{\mathcal{C}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}.$$

The *angular-velocity tensor* $\boldsymbol{\Omega}$ of the one-parameter rigid-body motion $\boldsymbol{Y} = \boldsymbol{\mathcal{A}}\boldsymbol{X} + \boldsymbol{\mathcal{C}}$ is defined as

$$\boldsymbol{\Omega} \equiv \boldsymbol{\mathcal{A}}'\boldsymbol{\mathcal{A}}^{\mathrm{T}},$$

which is anti-symmetric. Hence $\boldsymbol{\Omega}$ can be expressed in terms of the components of its axial vector $\boldsymbol{\zeta}$, which is named *angular-velocity vector* or *Darboux vector*, i.e.,

$$\boldsymbol{\zeta} = vect(\boldsymbol{\Omega}), \quad \boldsymbol{\Omega} = \frac{\partial \boldsymbol{\zeta} \times \mathbf{v}}{\partial \mathbf{v}},$$

where $\mathbf{v}$ is any cartesian vector. Due to the anti-simetry of $\boldsymbol{\Omega}$, the velocity field in a rigid-body is *helicoidal* that means there exist a line of the body whose points have a velocity parallel to $\boldsymbol{\zeta}$. This line is called the *instant screw axis* and is denoted by ISA, see [2].

Theorem 5. If $\mathcal{A} \in SO(3)$ and $rank\boldsymbol{\mathcal{A}}' = 2$, then the following statements can be given for the direction of ISA, see [16]:

(1) It is stationary if and only if $rank\boldsymbol{\mathcal{A}}'' = 2$.
(2) It is not stationary if and only if $rank\boldsymbol{\mathcal{A}}'' = 3$.

The *angular-acceleration tensor* $\boldsymbol{\mathcal{W}}$ of the one-parameter rigid-body motion $\boldsymbol{Y} = \boldsymbol{\mathcal{A}}\boldsymbol{X} + \boldsymbol{C}$ is defined as

$$\boldsymbol{\mathcal{W}} \equiv \boldsymbol{\Omega}^2 + \boldsymbol{\Omega}'.$$

Since $\boldsymbol{\Omega}^2$ is symmetric and $\boldsymbol{\Omega}'$ is anti-symmetric, $\boldsymbol{\mathcal{W}}$ is not anti-symmetric. Therefore, the acceleration field in a rigid-body is not helicoidal, see [2].

Determinant of $\boldsymbol{\mathcal{W}}$ can be calculated directly by using $\boldsymbol{\zeta}$, see [2]:

$$det\boldsymbol{\mathcal{W}} = -\|\boldsymbol{\zeta} \times \boldsymbol{\zeta}'\|^2.$$

The relationship between the determinants of $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{A}}''$ can be given by the following theorem.

Theorem 6. Let $\boldsymbol{Y} = \boldsymbol{\mathcal{A}}\boldsymbol{X} + \boldsymbol{C}$ be a one-parameter motion of a rigid-body in $\mathbb{E}^3$, $\boldsymbol{\mathcal{W}}$ the angular-acceleration tensor and $\boldsymbol{\zeta}$ the angular-velocity vector (or Darboux vector) of this motion. Then,

$$det\boldsymbol{\mathcal{A}}'' = det\boldsymbol{\mathcal{W}} = -\|\boldsymbol{\zeta} \times \boldsymbol{\zeta}'\|^2.$$

Proof. From the equation $\boldsymbol{\Omega} = \boldsymbol{\mathcal{A}}'\boldsymbol{\mathcal{A}}^{\mathrm{T}}$ it will be obtained $\boldsymbol{\mathcal{A}}' = \boldsymbol{\Omega}\boldsymbol{\mathcal{A}}$. Taking the first derivative of the equation $\boldsymbol{\mathcal{A}}' = \boldsymbol{\Omega}\boldsymbol{\mathcal{A}}$ let to $\boldsymbol{\mathcal{A}}'' = (\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}')\boldsymbol{\mathcal{A}}$ that means $\boldsymbol{\mathcal{A}}'' = \boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{A}}$. Since $det\boldsymbol{\mathcal{A}} = 1$, it can be written $det\boldsymbol{\mathcal{A}}'' = det\boldsymbol{\mathcal{W}} = -\|\boldsymbol{\zeta} \times \boldsymbol{\zeta}'\|^2$.

The representation of the *axode* in the fixed space $\mathbb{E}^3$ can be given by the equation

$$\boldsymbol{\mathcal{Y}} = \left(\boldsymbol{C} + \frac{\boldsymbol{\Omega} \times \boldsymbol{C}'}{\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle}\right) + \mu\left(\frac{\boldsymbol{\Omega}}{\|\boldsymbol{\Omega}\|}\right),$$

where μ is an arbitrary scalar, see [8]. It is obvious that $\boldsymbol{\mathcal{Y}}$ represents a ruled surface. By taking

$$\varphi = \boldsymbol{C} + \frac{\boldsymbol{\Omega} \times \boldsymbol{C}'}{\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle}, \quad \widehat{\boldsymbol{\Omega}} = \frac{\boldsymbol{\Omega}}{\|\boldsymbol{\Omega}\|},$$

the parametrization of the striction curve

$$\tilde{\varphi}(t) = \varphi(t) - \frac{\langle \varphi'(t), \widehat{\boldsymbol{\Omega}}'(t) \rangle}{\|\widehat{\boldsymbol{\Omega}}'(t)\|^2}\widehat{\boldsymbol{\Omega}}(t)$$

on the ruled surface $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}}(t)$, which is generated by the family $\{\varphi(t), \widehat{\boldsymbol{\Omega}}(t)\}$, will be coincide with the directrix $\varphi(t)$ of $\boldsymbol{\mathcal{Y}}(t)$ if and only if $\langle \varphi'(t), \widehat{\boldsymbol{\Omega}}'(t) \rangle = 0$.

2.4. Instantaneous Pole Points and Acceleration Pole Points in $\mathbb{E}^3$

Derivative of the one-parameter motion $\boldsymbol{Y} = \boldsymbol{\mathcal{A}X} + \boldsymbol{\mathcal{C}}$ of a rigid-body in $\mathbb{E}^3$ with respect to t, yields to the equation

$$\boldsymbol{Y}' = \boldsymbol{\mathcal{A}}'\boldsymbol{X} + \boldsymbol{\mathcal{C}}' + \boldsymbol{\mathcal{A}X}',$$

where $\boldsymbol{Y}'$ is called the *absolute velocity*, $\boldsymbol{\mathcal{A}}'\boldsymbol{X} + \boldsymbol{\mathcal{C}}'$ is called the *sliding velocity* and $\boldsymbol{\mathcal{A}X}'$ is called the *relative velocity* of the point X.

The *sliding velocity* of a fixed point X can be given by the equation, see [17],

$$\boldsymbol{Y}' = \boldsymbol{\mathcal{A}}'\boldsymbol{X} + \boldsymbol{\mathcal{C}}'.$$

Any solution vector $\boldsymbol{X} = -(\boldsymbol{\mathcal{A}}')^{-1}\boldsymbol{\mathcal{C}}'$ at every t-instant of the equation

$$\boldsymbol{\mathcal{A}}'\boldsymbol{X} + \boldsymbol{\mathcal{C}}' = 0$$

is the position vector of the point X, which can be considered as a fixed point on fixed space $\mathbb{E}^3$ and moving space E^3, at the same time t. These points are called the *instantaneous pole points* at every t-instant, see [17].

The *first-order sliding acceleration* (or the *second-order velocity*) of a fixed point X can be given by the equation, see [17],

$$\boldsymbol{Y}'' = \boldsymbol{\mathcal{A}}''\boldsymbol{X} + \boldsymbol{\mathcal{C}}''.$$

Any solution vector $\boldsymbol{X} = -(\boldsymbol{\mathcal{A}}'')^{-1}\boldsymbol{\mathcal{C}}''$ at every t-instant of the equation

$$\boldsymbol{\mathcal{A}}''\boldsymbol{X} + \boldsymbol{\mathcal{C}}'' = 0$$

is the position vector of the point X and these points are called the *first-order acceleration pole points* (or the *acceleration center of order 1*) at every t-instant, see [17].

The *second-order sliding acceleration* (or the *third-order velocity*) of a fixed point X can be given by the equation, see [11],

$$\boldsymbol{Y}''' = \boldsymbol{\mathcal{A}}'''\boldsymbol{X} + \boldsymbol{\mathcal{C}}'''.$$

Any solution vector $\boldsymbol{X} = -(\boldsymbol{\mathcal{A}}''')^{-1}\boldsymbol{\mathcal{C}}'''$ of the equation

$$\boldsymbol{\mathcal{A}}'''\boldsymbol{X} + \boldsymbol{\mathcal{C}}''' = 0$$

is the position vector of the point X and these points are called the *second-order acceleration pole points* (or the *acceleration center of order 2*) at every t-instant, see [17].

3. FRENET MOTION OF A BASE CURVE

In this section, definitions of the *Frenet motion* of a curve in $\mathbb{E}^3$ with its some basic concepts (e.g., *constant precession slant helices*, *axode* in fixed space, *striction curve*) will be given.

A unit speed curve $\alpha : I \to \mathbb{E}^3$, which is parameterized by arc-length parameter $t \in I$, is called *Frenet curve* if $\alpha''(t) \neq 0$ that means if it has non-zero curvature. Let α be a Frenet curve and $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ the Frenet frame. Then by moving the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ along α, which is fixed in $\mathbb{E}^3$, generates a kind of a one-parameter motion, which is called *Frenet motion*, of a rigid-body in $\mathbb{E}^3$. In this motion the moving frame O_{xyz} moves with O along α while rotating so that the x and y axes coincide, respectively, with the tangent and principal normal of α. Hence, the following equations can be given:

$$\boldsymbol{T} = \alpha', \ \ \boldsymbol{N} = \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|}, \ \ \boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N},$$

where $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$ correspond, respectively, to *unit tangent vector field*, *principal normal vector field* and *binormal vector field* of α. Also $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$ are unit vector fields that are mutually orthogonal at each point of α. In Frenet motion $\boldsymbol{\mathcal{C}}$ corresponds to α and $\boldsymbol{\mathcal{A}}$ corresponds to the matrix $[\boldsymbol{T} \quad \boldsymbol{N} \quad \boldsymbol{B}]$. Thus, the Frenet motion can be given as

$$F(\boldsymbol{X}) = \boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t).$$

Moreover, the following *Frenet formulas* can be given by taking the first derivative of the vector fields $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$:

$$\boldsymbol{T}' = \kappa\boldsymbol{N}, \ \ \boldsymbol{N}' = -\kappa\boldsymbol{T} + \tau\boldsymbol{B}, \ \ \boldsymbol{B}' = -\tau\boldsymbol{N},$$

where $\kappa = \|\boldsymbol{T}'\|$ and $\tau = -\langle \boldsymbol{B}', \boldsymbol{N} \rangle$ are the smooth functions on I and are, respectively, called the *curvature* and *torsion* according to the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ of α, see [8, 18]. These Frenet formulas can be expressed in matrix form as

$$\boldsymbol{\mathcal{A}}' = \begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}.$$

If a rigid-body moves along the unit speed curve α, then the motion of the body consists of translation along α and rotation about α. In Frenet motion the rotation is determined by a vector $\boldsymbol{W} = (\tau\boldsymbol{T} + \kappa\boldsymbol{B})/\sqrt{\kappa^2 + \tau^2}$ which satifies $\boldsymbol{T}' = \boldsymbol{W} \times \boldsymbol{T}$, $\boldsymbol{N}' = \boldsymbol{W} \times \boldsymbol{N}$ and $\boldsymbol{B}' = \boldsymbol{W} \times \boldsymbol{B}$. This vector is called the *Darboux vector* (or *angular-velocity vector*) of the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ of α.

The unit speed curve α is called *constant precession slant helix* if its Darboux vector $\boldsymbol{W}$ revolves about a fixed line in space with constant angle and constant speed. Thus, the following theorem can be given.

Theorem 7. A unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is called *constant precession slant helix* if the following equations hold for the curvature $\kappa > 0$ and the torsion τ of α for all $t \in I$:

$$\kappa(t) = -\lambda sin(\mu t),$$
$$\tau(t) = \lambda cos(\mu t),$$

where $\lambda > 0$ and μ are constants, see [19].

The representation of the *axode* in the fixed space $\mathbb{E}^3$ of the Frenet motion $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ can be given by using the equation

$$\boldsymbol{\mathcal{Y}}(t) = \left(\alpha(t) + \frac{\boldsymbol{W}(t) \times \alpha'(t)}{\langle \boldsymbol{W}(t), \boldsymbol{W}(t) \rangle}\right) + \mu\left(\frac{\boldsymbol{W}(t)}{\|\boldsymbol{W}(t)\|}\right)$$

as

$$\boldsymbol{\mathcal{Y}}(t)=\left(\alpha(t)+\frac{\kappa(t)}{\kappa^2(t)+\tau^2(t)}\boldsymbol{N}(t)\right)+\mu\left(\frac{\tau(t)\boldsymbol{T}(t)+\kappa(t)\boldsymbol{B}(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}\right),$$

where μ is an arbitrary scalar. By taking

$$\gamma(t)=\alpha(t)+\frac{\kappa(t)}{\kappa^2(t)+\tau^2(t)}\boldsymbol{N}(t),\ \ \widehat{\boldsymbol{W}}(t)=\frac{\tau(t)T(t)+\kappa(t)\boldsymbol{B}(t)}{\kappa^2(t)+\tau^2(t)},$$

it will be obtained

$$\langle\gamma'(t),\widehat{\boldsymbol{W}}'(t)\rangle=0.$$

Thus, the following corollary can be given.

Corollary 1. Let

$$\boldsymbol{\mathcal{Y}}(t)=\left(\alpha(t)+\frac{\boldsymbol{W}(t)\times\alpha'(t)}{\langle\boldsymbol{W}(t),\boldsymbol{W}(t)\rangle}\right)+\mu\left(\frac{\boldsymbol{W}(t)}{\|\boldsymbol{W}(t)\|}\right)$$

be the axode in the fixed space $\mathbb{E}^3$ of the Frenet motion $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$. Then the striction curve on the ruled surface $\boldsymbol{\mathcal{Y}}(t)$ will coincides with the directrix

$$\alpha(t)+\frac{\boldsymbol{W}(t)\times\alpha'(t)}{\langle\boldsymbol{W}(t),\boldsymbol{W}(t)\rangle}$$

of $\boldsymbol{\mathcal{Y}}(t)$.

4. FRENET MOTION OF THE PRINCIPAL-DIRECTION CURVE OF A BASE CURVE

In this section, firstly definition of an alternative one-parameter motion of a rigid-body in $\mathbb{E}^3$ will be given by moving the coordinate frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ instead of the Frenet frame $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ along a unit speed curve $\alpha(t)$, see Fig. 1. This alternative motion corresponds to Frenet motion of the principal-direction curve $\beta(t)=\int\boldsymbol{N}(t)dt$ of $\alpha(t)$. Afterwards, some basic concepts (e.g., constant *precession C-slant helices*, *instant screw axis (ISA)*, *instantaneous pole points*, *acceleration pole points*, *axode* in the fixed space, striction curve) of this alternative motion will be given.

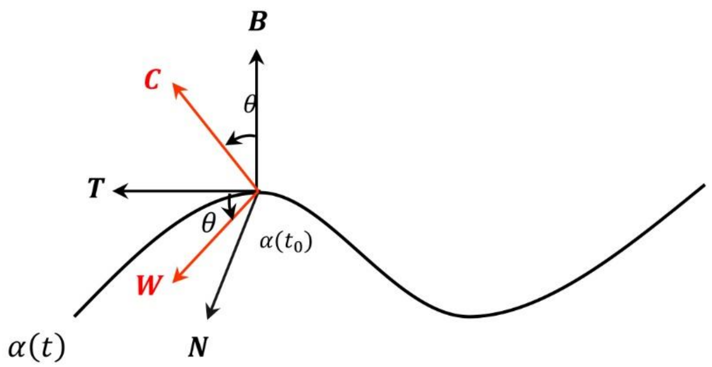

Figure 1. *The relationship between the Frenet frame* $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ *and the frame* $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ *of the curve* $\alpha(t)$ *at the point* $\alpha(t_0)$

As it can be seen from Figure 1, the frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ at any point $\alpha(t_0)$ of the curve $\alpha(t)$ will be obtained by rotating $\boldsymbol{T}$ and $\boldsymbol{B}$ of the Frenet frame $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ at the point $\alpha(t_0)$ by a constant angle $\theta \in \mathbb{R}$ on the rectifying plane, which is the plane of $\boldsymbol{T}$ and $\boldsymbol{B}$.

An alternative approach to define a one-parameter motion of a rigid-body in $\mathbb{E}^3$ along the unit speed curve α, can be given by taking $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ as the coordinate frame and matrix $[\boldsymbol{N} \quad \boldsymbol{C} \quad \boldsymbol{W}]$ as corresponding to $\boldsymbol{\mathcal{A}}$ in the equation $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$. In this motion the moving frame O_{xyz} moves with O along α while rotating so that the x and y axes coincide, respectively, with the principal normal and derivative of the principal normal of α. Hence, the following equations can be given:

$$\boldsymbol{N}=\frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|},\ \ \boldsymbol{C}=\frac{\boldsymbol{N}'}{\|\boldsymbol{N}'\|}=\frac{-\kappa\boldsymbol{T}+\tau\boldsymbol{B}}{\sqrt{\kappa^2+\tau^2}},\ \ \boldsymbol{W}=\boldsymbol{N}\times\boldsymbol{C}=\frac{\tau\boldsymbol{T}+\kappa\boldsymbol{B}}{\sqrt{\kappa^2+\tau^2}},$$

where $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{W}$ correspond, respectively, to *unit principal normal vector field*, *derivative vector field of the unit principal normal vector field* and *Darboux vector field* of α. Also $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{W}$ are unit vector fields that are mutually orthogonal at each point of α. Moreover, the following alternative frame formulas can be given by taken the first derivative of the vector fields $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{W}$, see [10]:

$$\boldsymbol{N}'=f\boldsymbol{C},\ \ \boldsymbol{C}'=-f\boldsymbol{N}+g\boldsymbol{W},\ \ \boldsymbol{W}'=-g\boldsymbol{C},$$

These alternative frame formulas can be expressed in matrix form as

$$\boldsymbol{\mathcal{A}}'=\begin{bmatrix}\boldsymbol{N}'\\ \boldsymbol{C}'\\ \boldsymbol{W}'\end{bmatrix}=\begin{bmatrix}0 & f & 0\\ -f & 0 & g\\ 0 & -g & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{N}\\ \boldsymbol{C}\\ \boldsymbol{W}\end{bmatrix}.$$

This alternative one-parameter rigid-body motion $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$ can be handled as Frenet motion by taking the principal-direction curve $\beta(t)=\int \boldsymbol{N}(t)dt$ instead of $\alpha(t)$. In this case, the coordinate frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ corresponds to Frenet frame of $\beta(t)$. Thus, the following corollary can be given.

Corollary 2. Let $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$ be the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$ and $\beta(t)=\int \boldsymbol{N}(t)dt$ be the principal-direction curve of $\alpha(t)$. Then the following results hold:

(1) $\boldsymbol{N}$, $\boldsymbol{C}$ and $\boldsymbol{W}$ correspond, respectively, to *unit tangent vector field*, *principal normal vector field* and *binormal vector field* of β.

(2) $f=\sqrt{\kappa^2+\tau^2}$ and $g=f\sigma=(\kappa^2/(\kappa^2+\tau^2))(\tau/\kappa)'$ correspond, respectively, to the *curvature* and *torsion* according to the Frenet frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ of β.

(3) $\boldsymbol{D}=g\boldsymbol{N}+f\boldsymbol{W}$ corresponds to the *Darboux vector* (or *angular-velocity vector*) of the Frenet frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ of β, satisfying $\boldsymbol{N}'=\boldsymbol{D}\times\boldsymbol{N}$, $\boldsymbol{C}'=\boldsymbol{D}\times\boldsymbol{C}$ and $\boldsymbol{W}'=\boldsymbol{D}\times\boldsymbol{W}$.

(4) α is slant helix if and only if β is *cylindrical helix* (or *general helix*).

The unit speed curve α is called *constant precession* C*–slant helix* if its Darboux vector $\boldsymbol{D}$ revolves about a fixed line in space with constant angle and constant speed. Thus, the following theorem can be given.

Theorem 8. A unit speed curve $\alpha : I\subset\mathbb{R}\to\mathbb{E}^3$ is called *constant precession* C*–slant helix* if the following equations hold for the curvature $f>0$ and the torsion g according to the Frenet frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ of the principal-direction curve $\beta(t)=\int \boldsymbol{N}(t)dt$ of α for all $t\in I$:

$$f(t)=-\lambda sin(\mu t),$$
$$g(t)=\lambda cos(\mu t),$$

where $\lambda>0$ and μ are constants, see [10].

The representation of the *axode* in the fixed space $\mathbb{E}^3$ of the alternative one-parameter rigid-body motion can be given by using the equation

$$\boldsymbol{\mathcal{Y}}(t)=\left(\alpha(t)+\frac{\boldsymbol{D}(t)\times\alpha'(t)}{\langle\boldsymbol{D}(t),\boldsymbol{D}(t)\rangle}\right)+\mu\left(\frac{\boldsymbol{D}(t)}{\|\boldsymbol{D}(t)\|}\right)$$

as

$$\boldsymbol{\mathcal{Y}}(t)=\left(\alpha(t)+\frac{\kappa(t)\boldsymbol{N}(t)-g(t)\boldsymbol{B}(t)}{f^2(t)+g^2(t)}\boldsymbol{N}(t)\right)+\mu\left(\frac{g(t)\boldsymbol{N}(t)+f(t)\boldsymbol{W}(t)}{\sqrt{f^2(t)+g^2(t)}}\right),$$

where μ is an arbitrary scalar. By taking

$$\delta(t)=\alpha(t)+\frac{\kappa(t)\boldsymbol{N}(t)-g(t)\boldsymbol{B}(t)}{f^2(t)+g^2(t)}\boldsymbol{N}(t),\quad \widehat{\boldsymbol{D}}(t)=\frac{g(t)\boldsymbol{N}(t)+f(t)\boldsymbol{W}(t)}{\sqrt{f^2(t)+g^2(t)}},$$

the solution conditions of the equation

$$\begin{aligned}\langle\delta'(t),\widehat{\boldsymbol{D}}'(t)\rangle=&\frac{g^2(t)f'(t)-g(t)g'(t)f(t)}{\left(f^2(t)+g^2(t)\right)^2}\left(\tau(t)-\frac{\kappa(t)g'(t)}{f^2(t)+g^2(t)}\right.\\&\left.+\frac{2\big(g(t)g'(t)+f(t)f'(t)\big)g(t)\kappa(t)}{\left(f^2(t)+g^2(t)\right)^2}\right)\\&+\frac{g'(t)f^2(t)-f(t)f'(t)g(t)}{\left(f^2(t)+g^2(t)\right)^2}\left(\frac{\tau(t)g(t)+\kappa'(t)}{f^2(t)+g^2(t)}+\frac{2\big(g(t)g'(t)+f(t)f'(t)\big)\kappa(t)}{\left(f^2(t)+g^2(t)\right)^2}\right)\\&=0\end{aligned}$$

is not obvious and can be studied in a further work. But a special solution condition, which is associated with this work, can be given by the following corollary.

Corollary 3. Let

$$\boldsymbol{\mathcal{Y}}(t)=\left(\alpha(t)+\frac{\boldsymbol{D}(t)\times\alpha'(t)}{\langle\boldsymbol{D}(t),\boldsymbol{D}(t)\rangle}\right)+\mu\left(\frac{\boldsymbol{D}(t)}{\|\boldsymbol{D}(t)\|}\right)$$

be the axode in the fixed space $\mathbb{E}^3$ of the one-parameter rigid-body motion $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$ obtained by moving the coordinate frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$. Then by taking $\alpha(t)$ as slant helix the striction curve on the ruled surface $\boldsymbol{\mathcal{Y}}(t)$ coincides with the directrix

$$\alpha(t)+\frac{\boldsymbol{D}(t)\times\alpha'(t)}{\langle\boldsymbol{D}(t),\boldsymbol{D}(t)\rangle}$$

of $\boldsymbol{\mathcal{Y}}(t)$.

4.1. Instantaneous Pole Points and Acceleration Pole Points

By taking $\boldsymbol{Y}(t)=\boldsymbol{\mathcal{A}}(t)\boldsymbol{X}+\alpha(t)$ as the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N},\boldsymbol{C},\boldsymbol{W}\}$ along the unit speed curve α, $\boldsymbol{\mathcal{A}}'$ was obtained as

$$\boldsymbol{\mathcal{A}}' = \begin{bmatrix} \boldsymbol{N}' \\ \boldsymbol{C}' \\ \boldsymbol{W}' \end{bmatrix} = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}.$$

Hence $det\boldsymbol{\mathcal{A}}' = 0$ that means the solution of the equation $\boldsymbol{\mathcal{A}}'(t)\boldsymbol{X} + \alpha'(t) = 0$ is not unique, so this motion has not any instantaneous pole points at every t-instant.

The first derivative of $\boldsymbol{\mathcal{A}}'$ leads to

$$\boldsymbol{\mathcal{A}}'' = \begin{bmatrix} \boldsymbol{N}'' \\ \boldsymbol{C}'' \\ \boldsymbol{W}'' \end{bmatrix} = \begin{bmatrix} -f^2 & f' & fg \\ -f' & -(f^2 + g^2) & g' \\ fg & -g' & -g^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}.$$

Hence

$$det\boldsymbol{\mathcal{A}}'' = -\left(g^2\left(\frac{f}{g}\right)'\right)^2 = -\left(g^2\left(\frac{1}{\sigma}\right)'\right)^2$$

where $g \neq 0$ which means $\sigma \neq 0$ (or α is not a cylindrical helix). Consequently, the following theorems can be given.

Theorem 9. Let $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ be the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$. Then $\alpha = \alpha(t)$ is called slant helix (i.e., σ is constant) if and only if $det\boldsymbol{\mathcal{A}}'' = 0$.

Theorem 10. The one-parameter rigid-body motion $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha = \alpha(t)$, has only one first-order acceleration pole point, which corresponds to the position vector $\boldsymbol{X} = -(\boldsymbol{\mathcal{A}}'')^{-1}\alpha''$ at every t-instant, if and only if the curve α is not a slant helix (i.e., σ is not constant).

Theorem 11. Let $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ be the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$. Then it will be obtained $rank\boldsymbol{\mathcal{A}}' = 2$ and $det\boldsymbol{\mathcal{A}}'' = -(g^2(1/\sigma)')^2$. Thus, the following two statements can be given for the direction of the ISA.

(1) By taking σ as constant, it will be obtained $det\boldsymbol{\mathcal{A}}'' = 0$ and $rank\boldsymbol{\mathcal{A}}'' = 2$. In this case, the direction of the ISA will be stationary.

(2) By taking σ as not constant, it will be obtained $det\boldsymbol{\mathcal{A}}'' \neq 0$ and $rank\boldsymbol{\mathcal{A}}'' = 3$. In this case, the direction of the ISA will not be stationary.

An alternative way to calculate the value of $det\boldsymbol{\mathcal{A}}''$ can be given by the following theorem.

Theorem 12. Let $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ be the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$ and $\boldsymbol{D}$ the Darboux vector (or angular-velocity vector) of the Frenet frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ of $\beta(t) = \int \boldsymbol{N}(t)dt$. Then $det\boldsymbol{\mathcal{A}}'' = -\|\boldsymbol{D} \times \boldsymbol{D}'\|^2$.

The first derivative of $\boldsymbol{\mathcal{A}}''$ leads to

$$\boldsymbol{\mathcal{A}}''' = \begin{bmatrix} \boldsymbol{N}''' \\ \boldsymbol{C}''' \\ \boldsymbol{W}''' \end{bmatrix} = \begin{bmatrix} -3ff' & (-f^3 + f'' - fg^2) & (2f'g + fg') \\ (-f'' + f^3 + fg^2) & (-3ff' - 3gg') & (-f^2g - g^3 + g'') \\ (2fg' + f'g) & (f^2g - g'' + g^3) & -3gg' \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}.$$

Hence

$$det\boldsymbol{\mathcal{A}}''' = 3g^2\left(\frac{f}{g}\right)'\left(2(ff'+gg')(fg'-f'g)-(f^2+g^2)(fg''-f''g)\right)$$
$$+3\left(g^2\left(\frac{f}{g}\right)'\right)'(f'g''-f''g'),$$

where $g \neq 0$ which means $\sigma \neq 0$. The solution conditions of the equation $det\boldsymbol{\mathcal{A}}''' = 0$ is not obvious and can be studied in a further work. But the special solutions associated with this work can be given by the following theorems.

Theorem 13. Let $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ be the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$. If the value of $f/g = 1/\sigma$ is not constant (i.e., if $\alpha = \alpha(t)$ is not a slant helix), then $\alpha = \alpha(t)$ is called constant precession C–slant helix if $det\boldsymbol{\mathcal{A}}''' = 0$.

Theorem 14. Let $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ be the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$. If $\alpha = \alpha(t)$ is not a constant precession C–slant helix, then α is called slant helix if $det\boldsymbol{\mathcal{A}}''' = 0$.

Theorem 15. The one-parameter rigid-body motion $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along the unit speed curve $\alpha(t)$, has only one second-order acceleration pole point, which corresponds to the position vector $\boldsymbol{X} = -(\boldsymbol{\mathcal{A}}''')^{-1}\alpha'''$ at every t-instant, if and only if $det\boldsymbol{\mathcal{A}}''' \neq 0$.

Example. Consider the curve

$$\alpha(t) = \frac{1}{5}\left(\frac{4sin(3t)}{3}, 2sin(2t) - \frac{sin(8t)}{8}, -2cos(2t) + \frac{cos(8t)}{8}\right)$$

where $t \in (\pi/3\,, 2\pi/3)$. The derivative of $\alpha = \alpha(t)$, i.e., the unit tangent vector field of α, is

$$\boldsymbol{T} = \alpha'(t) = \frac{1}{5}\big(4cos(3t), 4cos(2t) - cos(8t), 4sin(2t) - sin(8t)\big)$$

with $\|\alpha'(t)\| = 1$, so α has unit speed. The unit principal normal vector field $\boldsymbol{N}$, derivative vector field of the unit principal normal vector field $\boldsymbol{C}$ and Darboux vector field $\boldsymbol{W}$ of α can be calculated, respectively, as

$$\boldsymbol{N} = \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|} = \frac{1}{5}\big(3, -4cos(5t), -4sin(5t)\big),$$
$$\boldsymbol{C} = \frac{\boldsymbol{N}'}{\|\boldsymbol{N}'\|} = \big(0, sin(5t), -cos(5t)\big),$$
$$\boldsymbol{W} = \boldsymbol{N} \times \boldsymbol{C} = \frac{1}{5}\big(4, 3cos(5t), 3sin(5t)\big).$$

Thus the matrix $[\boldsymbol{N} \quad \boldsymbol{C} \quad \boldsymbol{W}]$ corresponding to $\boldsymbol{\mathcal{A}}$ of the one-parameter motion $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$ of a rigid-body in $\mathbb{E}^3$, obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along α, can be given by the following relation:

$$\boldsymbol{A} = [\boldsymbol{N} \quad \boldsymbol{C} \quad \boldsymbol{W}] = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4cos(5t)}{5} & sin(5t) & \frac{3cos(5t)}{5} \\ -\frac{4sin(5t)}{5} & -cos(5t) & \frac{3sin(5t)}{5} \end{bmatrix}$$

with $det\boldsymbol{A} = 1$.

The curvature and torsion of α can be calculated, respectively, as

$$\kappa = \|\boldsymbol{T}'\| = -4sin(3t), \;\; \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\kappa^2} = 4cos(3t),$$

that means α is constant precession slant helix. Hence it will be obtained

$$\sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' = \frac{3}{4},$$

which means α is slant helix.

The curvature and torsion of the principal-direction curve $\beta(t) = \int \boldsymbol{N}(t)dt$ of α can be calculated, respectively, as

$$f = \sqrt{\kappa^2 + \tau^2} = 4, \;\; g = f\sigma = 3,$$

so the matrix corresponding to $\boldsymbol{A}'$ can be given as

$$\boldsymbol{\mathcal{A}}' = \begin{bmatrix} \boldsymbol{N}' \\ \boldsymbol{C}' \\ \boldsymbol{W}' \end{bmatrix} = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ -4 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}$$

with $det\boldsymbol{\mathcal{A}}' = 0$, which means the solution of the equation $\boldsymbol{\mathcal{A}}'(t)\boldsymbol{X} + \alpha'(t) = 0$ is not unique. So the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along $\alpha(t)$ has not any instantaneous pole points at every t-instant.

The matrix corresponding to $\mathbf{A}''$ can be given as

$$\boldsymbol{\mathcal{A}}'' = \begin{bmatrix} \boldsymbol{N}'' \\ \boldsymbol{C}'' \\ \boldsymbol{W}'' \end{bmatrix} = \begin{bmatrix} -f^2 & f' & fg \\ -f' & -(f^2 + g^2) & g' \\ fg & -g' & -g^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix} = \begin{bmatrix} -16 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & -9 \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}$$

with $det\boldsymbol{\mathcal{A}}'' = 0$, which means the solution of the equation $\boldsymbol{\mathcal{A}}''(t)\boldsymbol{X} + \alpha''(t) = 0$ is not unique. So the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along $\alpha(t)$ has not any first-order acceleration pole point at every t-instant.

The Darboux vector of the Frenet frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ of $\beta(t) = \int \boldsymbol{N}(t)dt$ can be calculated with respect to the moving space E^3 as

$$\boldsymbol{D} = g\boldsymbol{N} + f\boldsymbol{W} = (5, 0, 0) = 5\boldsymbol{N}$$

and since $rank\boldsymbol{\mathcal{A}}' = rank\boldsymbol{\mathcal{A}}'' = 2$, the direction of the ISA will be stationary, see Fig. 2:

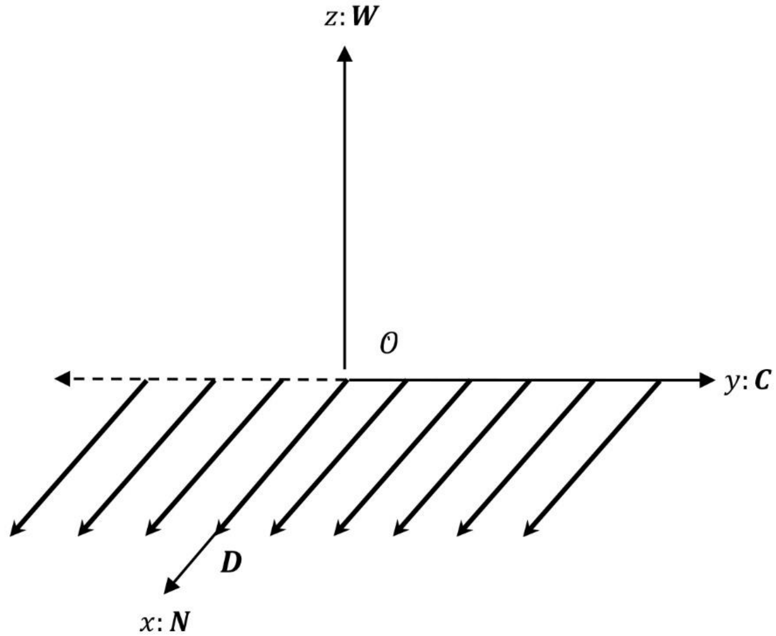

Figure 2. *Direction of the ISA*

In Figure 2, the moving frame O_{xyz} moves with O along α while rotating so that x, y and z axes coincide, respectively, with the principal normal $\boldsymbol{N}$, derivative of the principal normal $\boldsymbol{C}$ and Darboux vector $\boldsymbol{W}$ of α.

The following remarks can be given for this example:

Remark 1. The Darboux vector of the Frenet motion $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$, obtained by moving the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ along $\alpha(t)$, is $\boldsymbol{W} = (1/5)\big(4, 3cos(5t), 3sin(5t)\big)$; where the Darboux vector of the Frenet motion $\boldsymbol{Y}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{X} + \alpha(t)$, obtained by moving the Frenet frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along $\beta(t) = \int \boldsymbol{N}(t)dt$, is $\boldsymbol{D} = (5, 0, 0) = 5\boldsymbol{N}$.

Remark 2. From Theorem 5 the ISA is not stationary since $rank\boldsymbol{\mathcal{A}}' = 2$ and $rank\boldsymbol{\mathcal{A}}'' = 3$, where $\boldsymbol{\mathcal{A}}$ corresponds to the matrix $[\boldsymbol{T} \quad \boldsymbol{N} \quad \boldsymbol{B}]$; from Theorem 11 the ISA is stationary since $rank\boldsymbol{\mathcal{A}}' = rank\boldsymbol{\mathcal{A}}'' = 2$, where $\boldsymbol{\mathcal{A}}$ corresponds to the matrix $[\boldsymbol{N} \quad \boldsymbol{C} \quad \boldsymbol{W}]$.

Finally, the matrix corresponding to $\boldsymbol{A}'''$ may be given as

$$\boldsymbol{\mathcal{A}}''' = \begin{bmatrix} \boldsymbol{N}''' \\ \boldsymbol{C}''' \\ \boldsymbol{W}''' \end{bmatrix} = \begin{bmatrix} -3ff' & (-f^3 + f'' - fg^2) & (2f'g + fg') \\ (-f'' + f^3 + fg^2) & (-3ff' - 3gg') & (-fg^2 - g^3 + g'') \\ (2fg' + f'g) & (f^2g - g'' + g^3) & -3gg' \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -100 & 0 \\ 100 & 0 & -63 \\ 0 & 73 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{bmatrix}$$

with $det\boldsymbol{\mathcal{A}}''' = 0$ which means the solution of the equation $\boldsymbol{\mathcal{A}}'''(t)\boldsymbol{X} + \alpha'''(t) = 0$ is not unique. So the one-parameter rigid-body motion obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ along $\alpha(t)$ has not any second-order acceleration pole point at every t-instant.

The equation of the fixed axoid

$$\boldsymbol{\mathcal{Y}}(t)=\left(\alpha(t)+\frac{\boldsymbol{D}(t)\times\alpha'(t)}{\langle\boldsymbol{D}(t),\boldsymbol{D}(t)\rangle}\right)+\mu\left(\frac{\boldsymbol{D}(t)}{\|\boldsymbol{D}(t)\|}\right)$$

will be obtained as

$$\boldsymbol{\mathcal{Y}}(t)=\left(\frac{16sin(3t)}{15}+\frac{3\mu}{5},-\frac{2sin(2t)}{5}-\frac{9sin(8t)}{40}-\frac{4\mu cos(5t)}{5},\frac{2cos(2t)}{5}+\frac{9cos(8t)}{40}-\frac{4\mu sin(5t)}{5}\right),$$

where μ is an arbitrary scalar, see Fig. 3.

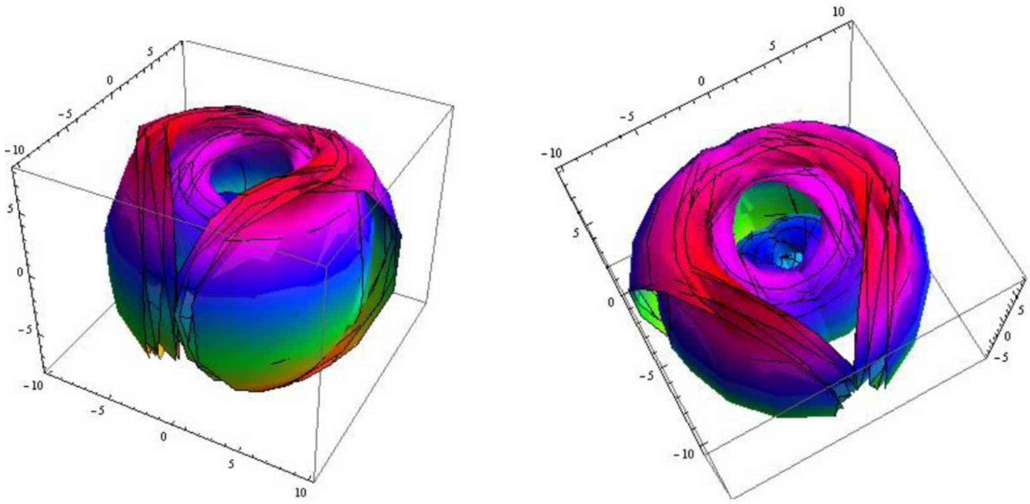

Figure 3. *Images of the fixed axoid* $\boldsymbol{\mathcal{Y}}$ *from two different perspectives*

Since α is slant helix the striction curve, see Figure 4, will be obtained as

$$\tilde{\delta}(t)=\alpha(t)+\frac{\boldsymbol{D}(t)\times\alpha'(t)}{\langle\boldsymbol{D}(t),\boldsymbol{D}(t)\rangle}=\left(\frac{16sin(3t)}{15},-\frac{2sin(2t)}{5}-\frac{9sin(8t)}{40},\frac{2cos(2t)}{5}+\frac{9cos(8t)}{40}\right)$$

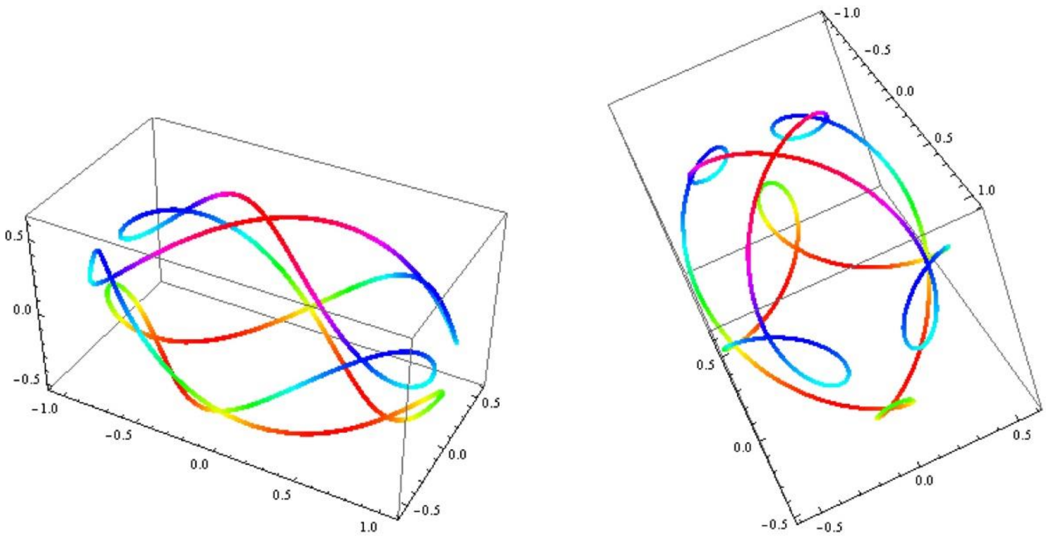

Figure 4. *Images of the striction curve* $\tilde{\delta}(t)$ *from two different perspectives*

5. CONCLUSION

In this study, the instantaneous pole points and the acceleration pole points up to second order are examined of a one-parameter motion of a rigid-body in $\mathbb{E}^3$ obtained by moving the coordinate frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ instead of the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ along a unit speed curve $\alpha(t)$. These concepts are investigated by using the determinants of the derivative matrices $\boldsymbol{\mathcal{A}}'$, $\boldsymbol{\mathcal{A}}''$ and $\boldsymbol{\mathcal{A}}'''$ since the acceleration pole points exist only if the determinants of these matrices are not equal to zero. The conditions that ensure $det\boldsymbol{\mathcal{A}}' \neq 0$, $det\boldsymbol{\mathcal{A}}'' \neq 0$ and $det\boldsymbol{\mathcal{A}}''' \neq 0$ is not obvious and need a further work, but the conditions associated with this work are given.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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