



Kumaraswamy Type I Half Logistic Family of Distributions with Applications

El-sayed El-SHERPIENY¹ , Mahmoud ELSEHETRY¹ 

¹Cairo University, Institute of Statistical Studies and Research, Department of Mathematical Statistics, Cairo, Egypt

Article Info

Received: 14/05/2017

Accepted: 11/09/2018

Keywords

Half logistic distribution

Order statistics

Maximum likelihood

method

Abstract

A new family of distributions called the Kumaraswamy type I half logistic is introduced and studied. The new family is extending well-known distributions as well as provides great flexibility to model specific real data and it is very easy in mathematical properties. Four new special models are presented. Some mathematical properties of the Kumaraswamy type I half logistic family are studied. Explicit expressions for the moments, probability weighted, quantile function, mean deviation and order statistics are investigated. Parameter estimates of the family are obtained based on maximum likelihood procedure. Two real data sets are employed to show the usefulness of the new family.

1. INTRODUCTION

The most popular traditional distributions often do not characterize and do not predict most of interesting data sets. Generated family of continuous distributions is a new improvement for creating and extending the usual classical distributions. The newly generated families have been broadly studied in several areas as well as yield more flexibility in applications. Eugene et al. [1] studied the beta-family of distributions. Zografos and Balakrishnan [2] suggested a generated family using gamma distribution which is defined as follows

$$F(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log[1-G(x)]} t^{\delta-1} e^{-t} dt, \quad (1)$$

Kumaraswamy generalized family provided by Cordeiro and de Castro [3]. Ristic and Balakrishnan, [4] proposed an alternative gamma generator for any continuous distribution. Further, some generated families were studied by several authors, for example, the kummer beta by Pescim et al. [5], exponentiated generalized by Cordeiro et al. [6], Weibull-G by Bourguignon et al. [7], exponentiated half-logistic by Cordeiro et al. [8], the type I half-logistic by Cordeiro et al. [9], the new Kumaraswamy Kumaraswamy family of generalized distributions with application has been presented by Mahmoud et al. [10], Garhy generated family of distributions introduced by Elgarhy et al. [11] and the Kumaraswamy Weibull by Hassan and Elgarhy [12], odd Burr generalized by Alizadeh et al. [13], generalized odd log-logistic by Cordeiro et al. [14], a new generalized odd log-logistic by Haghbin et al. [15], odd Lindley-G by Gomes-Silva et al. [16], odd Frechet-G by Haq and Elgarhy [17], among others.

In the current paper, we introduce a recently generated family of distributions using the half logistic distribution as a generator. This paper can be sorted as follows. In the next section, the Kumaraswamy type I half logistic- generated (KwTIHL-G) family is defined. Section 3 concerns with some general mathematical properties of the family. In Section 4, some new special models of the generated family are considered. In Section 5, estimation of the parameters of the family is implemented through maximum likelihood method. Simulation study is carried out to estimate the model parameters of distribution in

Section 6. An illustrative purpose on the basis of real data is investigated, in Section 7. Finally, concluding remarks are handled in Section 8.

2. KUMARASWAMY TYPE I HALF LOGISTIC FAMILY

The Kumaraswamy half logistic distribution is a member of the family of logistic distributions which has the following cumulative distribution function (cdf)

$$G(x) = 1 - \left\{ 1 - \left[\frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^a \right\}^b, \quad x, \lambda, a, b > 0 \quad (2)$$

The associated probability density function (pdf) is as follows

$$g(x) = ab \left[\frac{2\lambda e^{-\lambda x}}{(1 + e^{-\lambda x})^2} \right] \left[\frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^{a-1} \left\{ 1 - \left[\frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^a \right\}^{b-1}. \quad (3)$$

On the basis of cdf (1), we use the half logistic generator instead of gamma generator to obtain the Kumaraswamy type I half logistic family which is denoted by KwTIHL-G. Hence the cdf of KwTIHL-G family can be expressed as follows

$$F(x) = \int_0^{-\log[1-G(x)]} ab \left[\frac{2\lambda e^{-\lambda t}}{(1 + e^{-\lambda t})^2} \right] \left[\frac{1 - e^{-\lambda t}}{1 + e^{-\lambda t}} \right]^{a-1} \left\{ 1 - \left[\frac{1 - e^{-\lambda t}}{1 + e^{-\lambda t}} \right]^a \right\}^{b-1} dt$$

$$F(x) = 1 - \left\{ 1 - \left[\frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a \right\}^b, \quad x, \lambda, a, b > 0, \quad (4)$$

where, λ is a scale parameter, a, b are two shape parameters and $G(x)$ is a baseline cdf. The distribution function (4) provides a broadly Kumaraswamy type I half logistic generated distributions. Therefore, the pdf of the Kumaraswamy type I half logistic- generated family is as follows

$$f(x) = \frac{2ab\lambda g(x)(1 - G(x))^{\lambda-1}}{\left[1 + (1 - G(x))^\lambda \right]^2} \left[\frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^{a-1} \left\{ 1 - \left[\frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a \right\}^{b-1} \quad (5)$$

Hereafter, we denote by $X \sim \text{KwTIHL} - G$ a random variable X has pdf (5).

The survival function, hazard rate function and reserved hazard rate function are, respectively, given by

$$\bar{F}(x) = 1 - F(x) = \left\{ 1 - \left[\frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a \right\}^b,$$

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\frac{2ab\lambda g(x)(1 - G(x))^{\lambda-1}}{\left[1 + (1 - G(x))^\lambda \right]^2} \left[\frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^{a-1}}{1 - \left[\frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a},$$

and

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{2ab\lambda g(x)(1-G(x))^{\lambda-1} \left[\frac{1 - (1-G(x))^{\lambda}}{1 + (1-G(x))^{\lambda}} \right]^{a-1} \left\{ 1 - \left[\frac{1 - (1-G(x))^{\lambda}}{1 + (1-G(x))^{\lambda}} \right]^a \right\}^{b-1}}{1 - \left\{ 1 - \left[\frac{1 - (1-G(x))^{\lambda}}{1 + (1-G(x))^{\lambda}} \right]^a \right\}^b}.$$

Note that:

If $a = b = 1$ we get the type I half-logistic family of distributions (Cordeiro et al. [9]).

If $b = 1$ we get the exponentiated half-Logistic family of distributions (Cordeiro et al. [8]).

3. SOME STATISTICAL PROPERTIES

This section provides some statistical properties of KwTIHL-G family of distributions.

3.1. Quantile Function

Let X denotes a random variable has the cdf (4), the quantile function; say $Q(u)$ of X is given by

$$Q(u) = G^{-1} \left[1 - \left[\frac{1 - \left(1 - (1-u)^{\frac{1}{b}} \right)^{\frac{1}{a}}}{1 + \left(1 - (1-u)^{\frac{1}{b}} \right)^{\frac{1}{a}}} \right]^{\frac{1}{\lambda}} \right], \quad (6)$$

where, u is a uniform distribution on the interval $(0,1)$ and $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. In particular, the first quartile, the median, and the third quartile are obtained by putting $u = 0.25, 0.50$ and 0.75 respectively, in (6).

3.2. Important Representation

In this section some representations of the cdf and pdf for the Kumaraswamy type I half logistic family of distributions will be presented. The mathematical relation given below will be useful in this section.

It is well-known that, if $\beta > 0$ and $|z| < 1$ the generalized binomial theorem is written as follows

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i. \quad (7)$$

Then, by applying the binomial theorem (7) in (5), the distribution function of KwTIHL-G distribution where b is real becomes

$$f(x) = 2ab\lambda g(x)(1-G(x))^{\lambda-1} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \left[1 - (1-G(x))^{\lambda} \right]^{a(i+1)-1} \times \left[1 + (1-G(x))^{\lambda} \right]^{-(a(i+1)+1)}. \quad (8)$$

Using the generalized binomial theorem, we can write

$$\left[1 + (1 - G(x))^\lambda\right]^{-[a(i+1)+1]} = \sum_{j=0}^{\infty} (-1)^j \binom{a(i+1)+j}{j} [1 - G(x)]^{\lambda j} . \quad (9)$$

Inserting (9) and (7) in (8), the KwTIHL-G density function can be written as follows

$$f(x) = 2ab\lambda g(x) \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{a(i+1)+j}{j} \binom{a(i+1)-1}{k} (1 - G(x))^{\lambda(k+j+1)-1} .$$

Then, using binomial expansion again in the last equation, where λ is real non integer, leads to:

$$f(x) = \sum_{i,j,k,l=0}^{\infty} \eta^* g(x) (G(x))^l, \quad (10)$$

where,

$$\eta^* = 2ab\lambda (-1)^{i+j+k+l} \binom{b-1}{i} \binom{a(i+1)+j}{j} \binom{a(i+1)-1}{k} \binom{\lambda(k+j+1)-1}{l}$$

Further, an expansion for $[F(x)]^h$ is derived, for h is integer, again, the binomial expansion is worked out.

$$[F(x)]^h = \sum_{p=0}^h \sum_{q,u,v,w=0}^{\infty} W^* (G(x))^w, \quad (11)$$

where,

$$W^* = (-1)^{p+q+u+v+w} \binom{h}{p} \binom{bp}{q} \binom{aq}{u} \binom{aq+v-1}{v} \binom{\lambda(u+v)}{w} .$$

3.3. The Probability Weighted Moments

Class of moments, called the probability-weighted moments (PWMs), has been proposed by Greenwood et al. [18]. This class is used to derive estimators of the parameters and quantiles of distributions expressible in inverse form. For a random variable X the PWMs, denoted by $\tau_{r,s}$, can be calculated through the following relation

$$\tau_{r,s} = E(X^r F(x)^s) = \int_{-\infty}^{\infty} x^r f(x) F(x)^s dx \quad (12)$$

The PWMs of KwTIHL-G is obtained by substituting (10) and (11) into (12), and replacing h with s , as follows

$$\tau_{r,s} = \sum_{P=0}^s \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \eta^* w^* \int_{-\infty}^{\infty} x^r g(x) G(x)^{l+w} dx .$$

Then,

$$\tau_{r,s} = \sum_{P=0}^S \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \eta^{*w} \tau_{r,l+w},$$

where,

$$\tau_{r,l+w} = \int_{-\infty}^{\infty} x^r g(x) G(x)^{l+w} dx.$$

3.4. Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the r^{th} moment for the KwTIHL-G family. If X has the pdf (10), then r^{th} moment is obtained as follows

$$\dot{\mu}_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{i,j,k,l=0}^{\infty} \eta^* \int_{-\infty}^{\infty} x^r g(x) G(x)^l dx$$

Then,

$$\dot{\mu}_r = \sum_{i,j,k,l=0}^{\infty} \eta^* \tau_{r,l}.$$

3.5. Moment Generating Function

For a random variable X it is known that, the moment generating function is defined as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \dot{\mu}_r = \sum_{r,i,j,k,l=0}^{\infty} \frac{t^r}{r!} \eta^* \tau_{r,l}.$$

3.6. The Mean Deviation

In statistics, mean deviation about the mean and mean deviation about the median measure the amount of scattering in a population. For random variable X with pdf $f(x)$, cdf $F(x)$, the mean deviation about the mean and mean deviation about the median, are defined by

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \text{ and } \delta_2(X) = \mu - 2T(M),$$

where,

$$\mu = E(X), \quad M = \text{Median}(X)$$

and

$$T(q) = \int_{-\infty}^q x f(x) dx$$

which is the first incomplete moment.

3.7. Order Statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let X_1, X_2, \dots, X_n be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ the corresponding ordered random sample from a population of size n . According to David [19], the pdf of the k^{th} order statistic, is defined as

$$f_{X_{(k)}}(x_{(k)}) = \frac{f(x_{(k)})}{B(k, n-k+1)} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} F(x_{(k)})^{m+k-1}, \quad (13)$$

$B(\cdot, \cdot)$ stands for beta function. The pdf of the k^{th} order statistic for KwTIHL-G family is derived by substituting (10) and (11) in (13), replacing h with $m+k-1$

$$f_{X_{(k)}}(x_{(k)}) = \frac{g(x_{(k)})}{B(k, n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \sum_{P=0}^{m+k-1} C^* G(x_{(k)})^{l+w}, \quad (14)$$

where, $C^* = (-1)^m \binom{n-k}{m} \eta^* w^*$, $g(\cdot)$ and $G(\cdot)$ are the pdf and cdf of the KwTIHL-G family, respectively.

Further, the r^{th} moment of k^{th} order statistics for KwTIHL-G is defined family by:

$$E(X_{(k)}^r) = \int_{-\infty}^{\infty} x_{(k)}^r f(x_{(k)}) dx_{(k)}. \quad (15)$$

By substituting (14) in (15), leads to

$$E(X_{(k)}^r) = \frac{1}{B(k, n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \sum_{P=0}^{m+k-1} C^* \int_{-\infty}^{\infty} x_{(k)}^r g(x_{(k)}) G(x_{(k)})^{l+w} dx_{(k)}.$$

Then,

$$E(X_{(k)}^r) = \frac{1}{B(k, n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \sum_{P=0}^{m+k-1} C^* \tau_{r,l+w}.$$

4. SOME SPECIAL MODELS

In this section, we define and describe four special models of the KwTIHL generated family namely, KwTIHL-uniform, KwTIHL -Frechet, KwTIHL -exponential and KwTIHL -Lindley.

4.1. KwTIHL-Uniform Distribution

The pdf of type I half logistic-uniform KwTIHLU is derived from (5), by taking $g(x, \theta) = \frac{1}{\theta}$, $0 < x < \theta$ and $G(x, \theta) = \frac{x}{\theta}$ as the following

$$f(x) = \frac{2ab\lambda \theta^\lambda (\theta - x)^{\lambda-1} [\theta^\lambda - (\theta - x)^\lambda]^{a-1}}{[\theta^\lambda + (\theta - x)^\lambda]^{a+1}} \left\{ 1 - \frac{[\theta^\lambda - (\theta - x)^\lambda]^a}{[\theta^\lambda + (\theta - x)^\lambda]^a} \right\}^{b-1}, \quad 0 < x < \theta$$

The corresponding cdf takes the following form

$$F(x) = 1 - \left\{ 1 - \frac{[\theta^\lambda - (\theta - x)^\lambda]^a}{[\theta^\lambda + (\theta - x)^\lambda]^a} \right\}^b.$$

Moreover, the survival and the hazard rate functions are given, respectively, as follows

$$\bar{F}(x) = \left\{ 1 - \frac{[\theta^\lambda - (\theta - x)^\lambda]^a}{[\theta^\lambda + (\theta - x)^\lambda]^a} \right\}^b,$$

and

$$h(x) = \frac{2ab\lambda \theta^\lambda (\theta - x)^{\lambda-1} [\theta^\lambda - (\theta - x)^\lambda]^{a-1}}{[\theta^\lambda + (\theta - x)^\lambda]^{a+1} \left\{ 1 - \frac{[\theta^\lambda - (\theta - x)^\lambda]^a}{[\theta^\lambda + (\theta - x)^\lambda]^a} \right\}}.$$

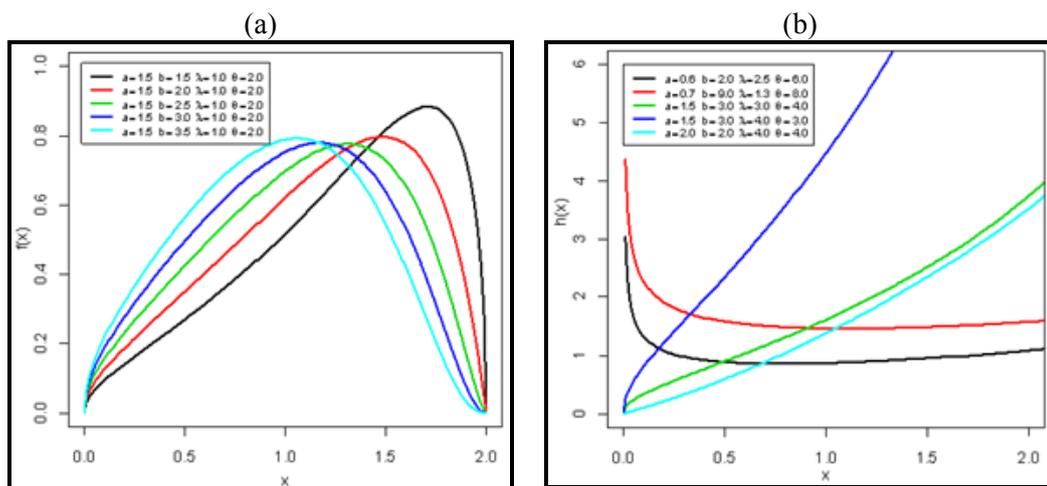


Figure 1. a) pdf of KwTIHLU distribution b) Hazard rate function of KwTIHLU distribution

4.2. KwTIHL-Frechet Distribution

Let us consider the Frechet distribution with distribution functions given by $G(x) = e^{-\left(\frac{\mu}{x}\right)^\delta}$. Then the KwTIHLF distribution has the following cdf, pdf, survival, and hazard rate functions

$$F(x) = 1 - \left\{ 1 - \frac{\left[1 - \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^a}{\left[1 + \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^a} \right\}^b, \quad x, a, b, \mu, \lambda, \delta > 0$$

$$f(x) = \frac{2ab\lambda\delta\mu^\delta e^{-\left(\frac{\mu}{x}\right)^\delta} \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^{\lambda-1} \left[1 - \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^{a-1}}{x^{\delta+1} \left[1 + \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^{a+1}} \left\{ 1 - \frac{\left[1 - \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^a}{\left[1 + \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]} \right\}^{b-1},$$

$$\bar{F}(x) = \left\{ 1 - \frac{\left[1 - \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^a}{\left[1 + \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]} \right\}^b,$$

and

$$h(x) = \frac{2ab\lambda\delta\mu^\delta e^{-\left(\frac{\mu}{x}\right)^\delta} \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^{\lambda-1} \left[\left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^{a-1}}{x^{\delta+1} \left[1 + \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^{a+1} \left\{ 1 - \frac{\left[1 - \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]^a}{\left[1 + \left[1 - e^{-\left(\frac{\mu}{x}\right)^\delta}\right]^\lambda\right]} \right\}}.$$

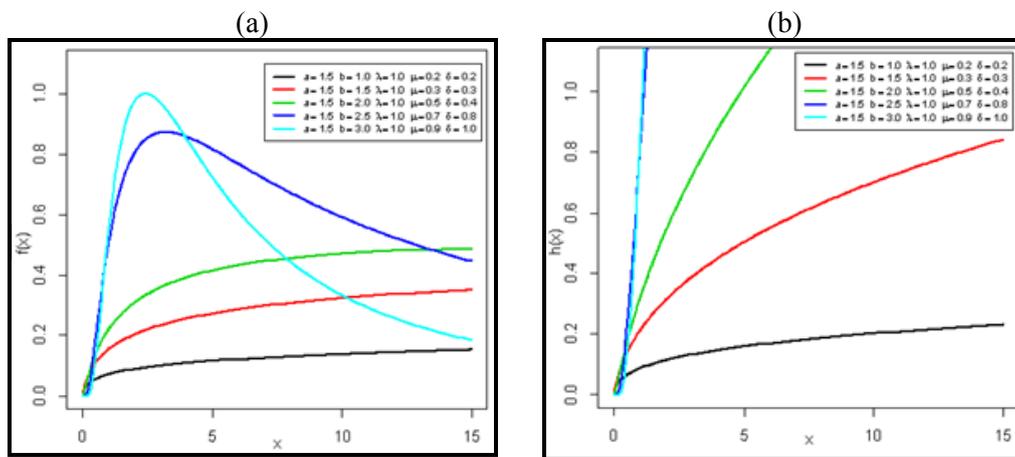


Figure 2. a) pdf of KwTIHLF distribution b) Hazard rate function of KwTIHLF distribution

4.3. KwTIHL-Exponential Distribution

The cdf and pdf of KwTIHL-Exponential (KwTIHLE) distribution are derived from (5) and (6) taking $G(x, \alpha) = 1 - e^{-\alpha x}$ as the following

$$F(x) = 1 - \left\{ 1 - \frac{\left[1 - e^{-\alpha\lambda x}\right]^a}{\left[1 + e^{-\alpha\lambda x}\right]} \right\}^b, \quad x, a, b, \alpha, \lambda > 0,$$

and,

$$f(x) = \frac{2ab\lambda\alpha e^{-\alpha\lambda x} \left[1 - e^{-\alpha\lambda x}\right]^{a-1}}{\left[1 + e^{-\alpha\lambda x}\right]^{a+1}} \left\{ 1 - \frac{\left[1 - e^{-\alpha\lambda x}\right]^a}{\left[1 + e^{-\alpha\lambda x}\right]} \right\}^{b-1}$$

Further, the survival and hazard rate functions are as follows

$$\bar{F}(x) = \left\{ 1 - \left[\frac{1 - e^{-\alpha\lambda x}}{1 + e^{-\alpha\lambda x}} \right]^a \right\}^b,$$

and

$$h(x) = \frac{2ab \lambda \alpha e^{-\alpha\lambda x} [1 - e^{-\alpha\lambda x}]^{a-1}}{[1 + e^{-\alpha\lambda x}]^{a+1} \left\{ 1 - \left[\frac{1 - e^{-\alpha\lambda x}}{1 + e^{-\alpha\lambda x}} \right]^a \right\}}.$$

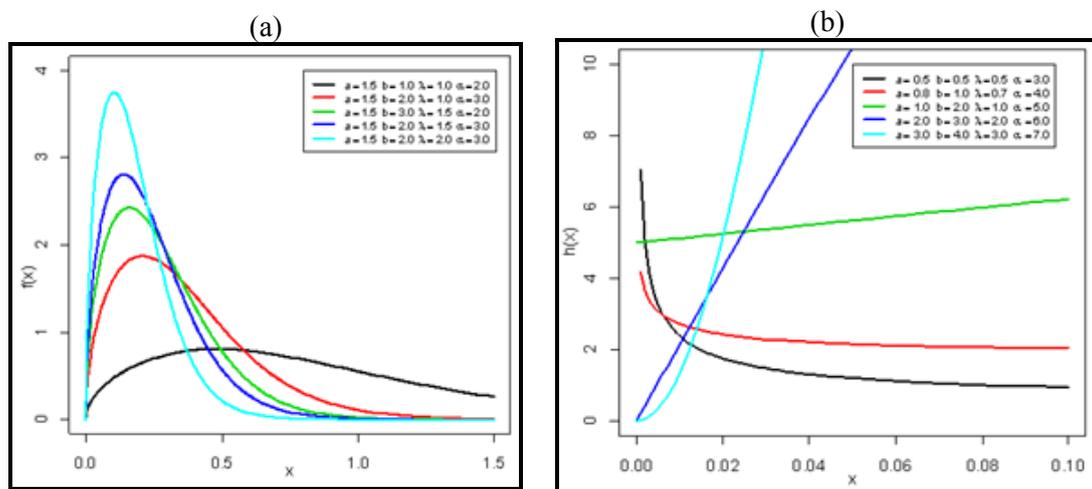


Figure 3. a) pdf of KwTIHLE distribution b) Hazard rate function of KwTIHLE distribution

The quantile function of the KwTIHLE distribution is given by

$$Q(u) = -\frac{1}{\alpha\lambda} \ln \left[\frac{1 - \left[1 - [1 - u]^{\frac{1}{b}} \right]^{\frac{1}{a}}}{1 + \left[1 - [1 - u]^{\frac{1}{b}} \right]^{\frac{1}{a}}} \right].$$

Specifically, the first quartile, the median, and the third quartile are obtained by setting $u=0.25, 0.5$ and 0.75 , respectively, in the previous equation. Also, the random variable X has KwTIHLE distribution can be generated from (7), where Q has the uniform distribution over the interval $(0,1)$. Furthermore, the analysis of the variability of the skewness and kurtosis on the shape parameters a and b can be investigated based on quantile measures. The Bowley skewness [20], denoted by B , is defined by

$$B = \frac{Q_{0.75} - 2Q_{0.50} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

The Moors kurtosis [21], denoted by M , can be defined as follows

$$M = \frac{Q_{0.875} - Q_{0.625} + Q_{0.375} - Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

The Bowley skewness and Moors kurtosis measures do not depend on the moments of the distribution and

are almost insensitive to outliers. Plots of the skewness and kurtosis for some choices of the parameter a as a function of b and for some choices of the parameter b as function of a are shown in Figures 4 and 5.

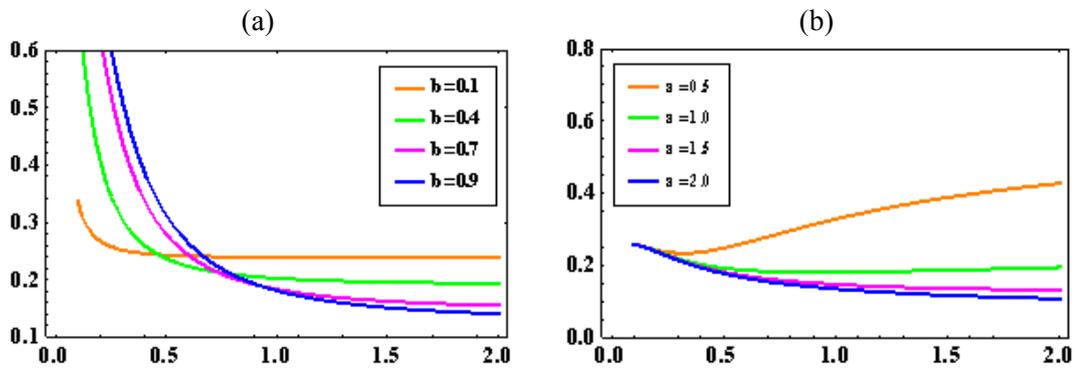


Figure 4. Bowley skewness of the KTIHLE distribution. a) As function of a for some values of b b) As function of b for some values of a .

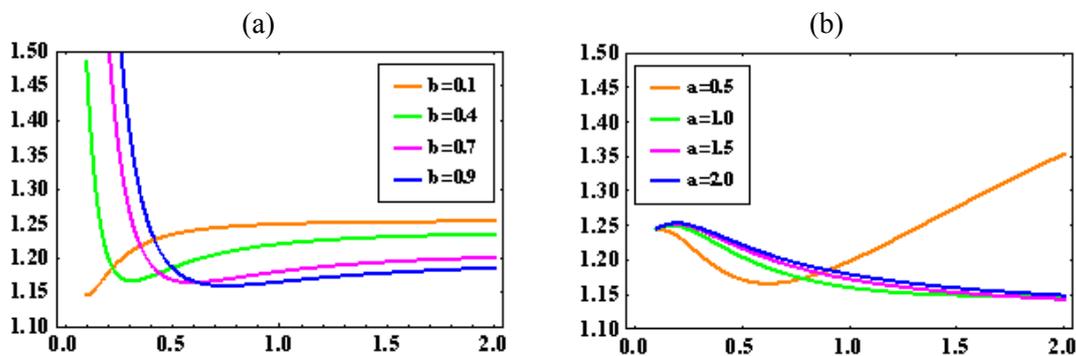


Figure 5. Moors kurtosis of the KTIHLE distribution. a) As function of a for some values of b b) As function of b for some values of a

4.4. KwTIHL-Lindley Distribution

The Lindley distribution has been suggested by Lindley. The probability density and distribution functions of quasi Lindley distribution are given by

$$G(x, \beta) = 1 - \left[1 + \frac{\beta x}{\beta + 1} \right] e^{-\beta x}, \quad g(x, \beta) = \frac{\beta^2}{\beta + 1} (1 + x) e^{-\beta x}$$

The cdf, pdf, survival and the hazard rate functions for KwTIHL- Lindley distribution (KwTIHL) are obtained from (4) and (5), respectively as

$$F(x) = 1 - \left\{ 1 - \left[\frac{1 - (w)^\lambda e^{-\beta \lambda x}}{1 + (w)^\lambda e^{-\beta \lambda x}} \right]^a \right\}^b, \quad x, a, b, \lambda > 0, \beta > -1,$$

where, $w = 1 + \frac{\beta x}{\beta + 1}$

$$f(x) = \frac{2ab \lambda \beta^2 (1 + x) e^{-\beta \lambda x} (w)^{\lambda - 1} [1 - (w)^\lambda e^{-\beta \lambda x}]^{a - 1}}{(\beta + 1) [1 + (w)^\lambda e^{-\beta \lambda x}]^{a + 1}} \left\{ 1 - \left[\frac{1 - (w)^\lambda e^{-\beta \lambda x}}{1 + (w)^\lambda e^{-\beta \lambda x}} \right]^a \right\}^{b - 1},$$

$$\bar{F}(x) = \left\{ 1 - \left[\frac{1 - (w)^\lambda e^{-\beta\lambda x}}{1 + (w)^\lambda e^{-\beta\lambda x}} \right]^a \right\}^b,$$

and,

$$h(x) = \frac{2ab\lambda\beta^2(1+x)(w)^{\lambda-1}e^{-\beta\lambda x} [1 - e^{-\beta\lambda x}(w)^\lambda]^{a-1}}{(\beta + 1)[1 + (w)^\lambda e^{-\beta\lambda x}]^{a+1} \left\{ 1 - \left[\frac{1 - (w)^\lambda e^{-\beta\lambda x}}{1 + (w)^\lambda e^{-\beta\lambda x}} \right]^a \right\}}.$$

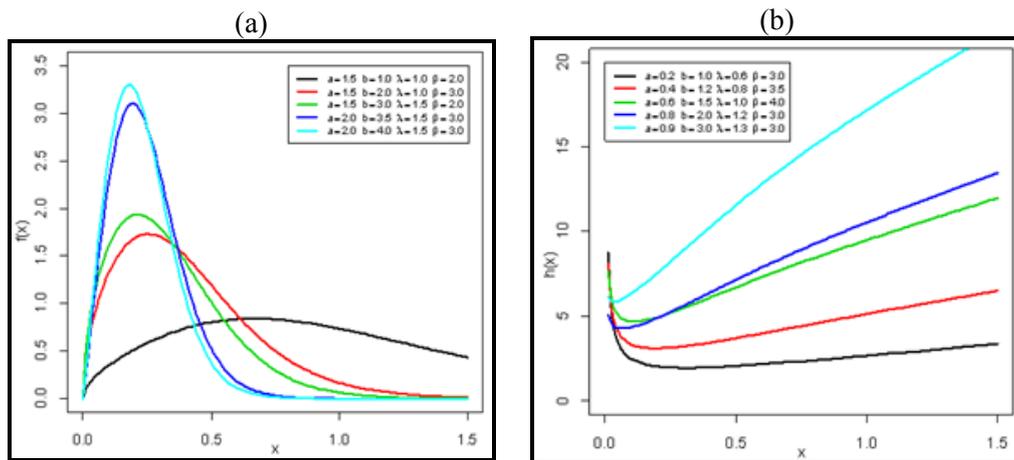


Figure 6. a) pdf of KwTIHLL distribution b) Hazard rate function of KwTIHLL distribution

5. MAXIMUM LIKELIHOOD METHOD

This section deals with the maximum likelihood estimators of the unknown parameters for the KwTIHL-G family of distributions on the basis of complete samples. Let X_1, X_2, \dots, X_n be the observed values from the KwTIHL-G family with set of parameter = $(a, b, \lambda, \xi)^T$. The log-likelihood function for parameter vector $\Phi = (a, b, \lambda, \xi)^T$ is obtained as follows

$$\begin{aligned} \ln(L, \phi) = & n \ln(2) + n \ln(a) + n \ln(b) + n \ln(\lambda) + \sum_{i=1}^n \ln[g(x_i, \xi)] + (\lambda - 1) \sum_{i=1}^n \ln[1 - G(x_i, \xi)] \\ & + (a - 1) \sum_{i=1}^n \ln \left[1 - (1 - G(x_i, \xi))^\lambda \right] - (a + 1) \sum_{i=1}^n \ln \left[1 + (1 - G(x_i, \xi))^\lambda \right] \\ & + (b - 1) \sum_{i=1}^n \ln \left[1 - \left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a \right], \end{aligned}$$

$$\begin{aligned} U_a = & \frac{n}{a} + \sum_{i=1}^n \ln \left[1 - (1 - G(x_i, \xi))^\lambda \right] - \sum_{i=1}^n \ln \left[1 + (1 - G(x_i, \xi))^\lambda \right] \\ & + (b - 1) \sum_{i=1}^n \frac{\left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a \ln \left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]}{1 - \left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a}, \end{aligned}$$

$$U_b = \frac{n}{b} + \sum_{i=1}^n \ln \left[1 - \left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a \right],$$

$$U_\lambda = \frac{n}{\lambda} + \sum_{i=1}^n \ln[1 - G(x_i, \xi)] + (a - 1) \sum_{i=1}^n \frac{(1 - G(x_i, \xi))^\lambda \ln[1 - G(x_i, \xi)]}{1 - (1 - G(x_i, \xi))^\lambda} - (a + 1) \sum_{i=1}^n \frac{(1 - G(x_i, \xi))^\lambda \ln[1 - G(x_i, \xi)]}{1 + (1 - G(x_i, \xi))^\lambda} + a(b - 1) \sum_{i=1}^n \frac{\left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^{a-1} \frac{2(1 - G(x_i, \xi))^\lambda \ln[1 - G(x_i, \xi)]}{[1 + (1 - G(x_i, \xi))^\lambda]^2}}{1 - \left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a},$$

and,

$$U_{\xi_k} = \sum_{i=1}^n \frac{g'_k(x_i, \xi)}{g(x_i, \xi)} - (\lambda - 1) \sum_{i=1}^n \frac{G'_k(x_i, \xi)}{1 - G(x_i, \xi)} + \lambda(\alpha - 1) \sum_{i=1}^n \frac{G'_k(x_i, \xi) [1 - G(x_i, \xi)]^{\lambda-1}}{1 - [1 - G(x_i, \xi)]^\lambda} + \lambda(\alpha + 1) \sum_{i=1}^n \frac{G'_k(x_i, \xi) [G(x_i, \xi)]^{\lambda-1}}{1 + [1 - G(x_i, \xi)]^\lambda} - a(b - 1) \sum_{i=1}^n \frac{\left[\frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^{a-1} \frac{2\lambda(1 - G(x_i, \xi))^{\lambda-1} G'_k(x_i, \xi)}{[1 + (1 - G(x_i, \xi))^\lambda]^2}}{[1 + (1 - G(x_i, \xi))^\lambda]^2},$$

where, $g'_k(x_i, \xi) = \frac{\partial g(x_i, \xi)}{\partial \xi_k}$ and $G'_k(x_i, \xi) = \frac{\partial G(x_i, \xi)}{\partial \xi_k}$.

Setting U_a, U_b, U_λ , and U_{ξ_k} equal to zero and solving these equations simultaneously yield the maximum likelihood estimate (MLE) $\hat{\Phi} = (\hat{a}, \hat{b}, \hat{\lambda}, \hat{\xi})$ of $\Phi = (a, b, \lambda, \xi)^T$ these equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

6. SIMULATION STUDY

It is very difficult to compare the theoretical performances of the different estimators (MLE) for the KwTIHLE distribution. Therefore, simulation is needed to compare the performances of the estimation mainly with respect to their biases and mean square errors for different sample sizes. A numerical study is performed using Mathematica 9 software. Different sample sizes are considered through the experiments at size $n = 20, 30, 50$ and 100 . In addition, the different values of the parameters a, b and α .

The experiment will be repeated 1000 times. In each experiment, the estimates of the parameters will be obtained by maximum likelihood method. The means, MSEs and biases for the different estimators will be reported from these experiments.

Table 1. MLEs, biases and MSEs for some parameter values

n	Par	Init	MLE	Bais	MSE	Init	MLE	Bais	MSE
20	a	1.5	1.5259	0.0259	0.1588	2.0	1.9029	0.0970	0.3054

	b	0.7	1.7847	1.0847	1.5737	0.5	1.3298	0.8298	1.0279
	λ	1.5	1.0021	0.4978	0.2481	1.5	0.9927	0.5072	0.2587
	α	1.2	1.0020	0.1980	0.0395	1.2	0.9922	0.2077	0.0446
30	a	1.5	1.4836	0.0163	0.1140	2.0	1.8177	0.1822	0.2143
	b	0.7	1.6631	0.9631	1.1884	0.5	1.2336	0.7336	0.6683
	λ	1.5	0.9965	0.5034	0.2537	1.5	0.9891	0.5108	0.2617
	α	1.2	0.9964	0.2035	0.0417	1.2	0.9887	0.2112	0.0454
50	a	1.5	1.4208	0.0791	0.0723	2.0	1.7518	0.2481	0.1530
	b	0.7	1.6375	0.9375	1.0190	0.5	1.1869	0.6869	0.5413
	λ	1.5	0.9970	0.5029	0.2530	1.5	0.9878	0.5122	0.2627
	α	1.2	0.9970	0.2029	0.0413	1.2	0.9875	0.2124	0.0455
100	a	1.5	1.4285	0.0714	0.0309	2.0	1.7124	0.2875	0.1210
	b	0.7	1.6099	0.9099	0.9019	0.5	1.1543	0.6543	0.4618
	λ	1.5	0.9952	0.5047	0.2549	1.5	0.9868	0.5131	0.2635
	α	1.2	0.9951	0.2048	0.0420	1.2	0.9866	0.2133	0.0457

7. APPLICATION

In this section, two real data sets are used to illustrate the potentiality of the KwTIHL family. Application of the KwTIHL distributions based on four distributions; namely, Kumaraswamy type I half logistic exponential (KwTIHLE), type I half logistic Weibull (TIHLW), Exponentiated Weibull Weibull (EWW) [22]. And Weibull Weibull (WW) [23].

Data set 1: is obtained from Hinkley [24]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul.

Data set 2: The data are obtained from Bjerkedal [25] and represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The data sets are as follows:

The estimate of the unknown parameters of each distribution is obtained by the maximum-likelihood method. In order to compare the four distribution models, various criteria were used. Criteria like; $-2 \ln L$, Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion (CAIC), Hannan information criterion(HQIC), the Kolmogorov-Smirnov (K – S), Anderson Darling (A*) and Cramer-von Mises (W*) statistics are considered for the data set.

The "best" distribution corresponds to the smallest values of $-2 \ln L$, AIC, BIC, CAIC, HQIC, K – S, A* and W* criteria.

Table 2. Shows the MLEs of the model parameters and its standard error (in parentheses) for data set 1

Model	MLEs and S. E				
KwTIHLE(a, b, α, λ)	1.82 (1.198)	25.172 (0.322)	0.31 (0.247)	0.579 (0.55607)	-
TIHLW(α, β, λ)	0.626 (0.124)	1.532 (0.293)	0.889 (0.1791)	-	-
EWW($a, \alpha, \beta, \lambda, \gamma$)	78.61 (0.14836)	79.35 (0.561)	20.486 (0.131)	0.624 (0.024)	0.014 (0.148)
WW($\alpha, \beta, \lambda, \gamma$)	39.853 (0.414)	3.154 (0.518)	0.196 (0.102)	0.5 (0.072)	

where, $SE(\hat{\alpha}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{\alpha} - \alpha)^2}$.

Table 3. Measurements for all models based on the data set 1

Distribution	$-2 \ln L$	AIC	BIC	CAIC	HQIC	$K - S$	A^*	W^*
KwTIHLE	77.244	85.244	83.153	86.844	87.037	0.06879	0.17008	0.02196
TIHLW	106.639	112.639	112.211	113.562	115.358	0.069	0.22044	0.02953
EWV	129.022	139.022	141.522	136.407	141.263	0.113	0.34353	0.05525
WW	138.194	146.194	145.623	147.794	149.819	0.07549	0.20247	0.02622

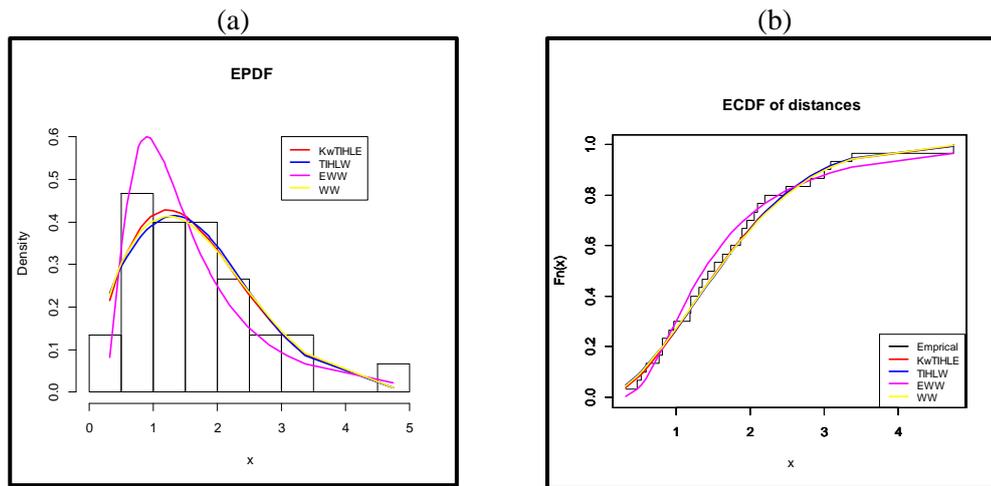


Figure 7. a) Estimated densities for the data set 1 b) Estimated cumulative densities for the data set 1

Table 4. Shows the MLEs of the model parameters and its standard error (in parentheses) for data set 2

Model	MLEs and S. E				
$KwTIHLE(a, b, \alpha, \lambda)$	1.841 (0.635)	32.646 (0.233)	0.283 (0.203)	0.531 (0.45607)	-
$TIHLW(\alpha, \beta, \lambda)$	0.952 (0.23859)	1.535 (0.035)	0.544 (0.136)	-	-
$EWV(a, \alpha, \beta, \lambda, \gamma)$	115.001 (0.0787)	125.918 (0.361)	19.125 (0.085)	0.61 (0.015)	0.013 (0.093)
$WW(\alpha, \beta, \lambda, \gamma)$	48.725 (0.27)	2.947 (0.324)	0.162 (0.063)	0.546 (0.047)	-

Table 5. Measurements for all models based on for the data set 2

Distribution	$-2 \ln L$	AIC	BIC	CAIC	HQIC	$K - S$	A^*	W^*
KwTIHLE	187.794	195.794	195.223	196.391	199.419	0.10197	0.94021	0.15696
TIHLW	354.017	360.017	359.589	360.37	362.736	0.113	1.10046	0.18381
EWV	302.076	312.076	311.363	312.972	316.608	0.134	1.39873	0.20049
WW	337.304	345.304	344.734	345.901	348.93	0.10975	1.04681	0.17318

(a)

(b)

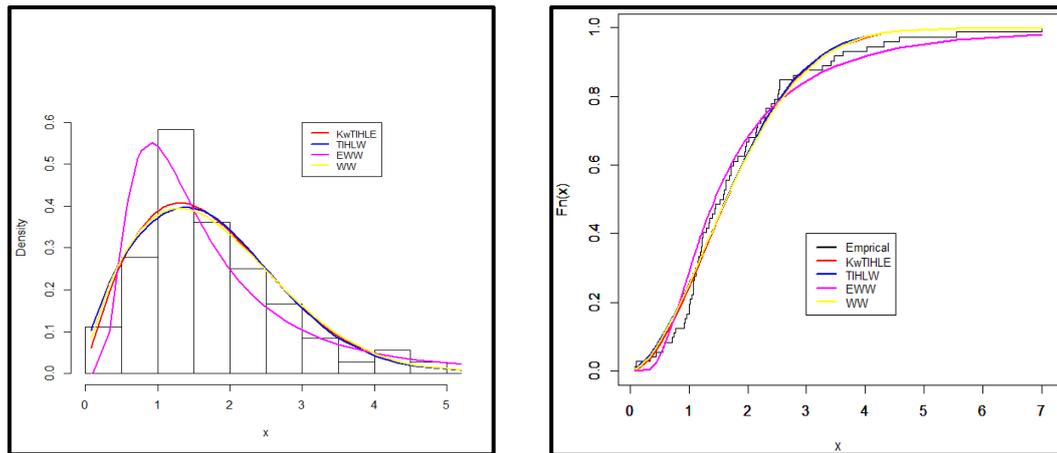


Figure 8. a) Estimated densities for second data set 2 b) Estimated cumulative densities for second data set 2

The values in Tables 3 and 5, indicate that the Kumaraswamy type I half logistic exponential distribution is a strong competitor to other distributions used here for fitting data set. A density and cdf plots compares the fitted densities of the models with the empirical histogram of the observed data Figure 7 and Figure 8. The fitted density for the Kumaraswamy type I half logistic exponential model is the closest to the empirical histogram of the other fitted models.

8. CONCLUSION

In this article, we introduced the new Kumaraswamy type I half logistic generated family of distributions. More specifically, Kumaraswamy type I half logistic generated family covers several new distributions. We wish a broadly statistical application in some area for this new generalization. Properties of the KwTIHLF were discussed, such as, expressions for the density function, moments, mean deviation, quantile function and order statistics. The maximum likelihood method is employed for estimating the model parameters. Four special models are provided. Further, the derived properties of the generated family are valid to these selected models. We fit some KwTIHL-G distributions to one real data set to demonstrate the potentiality of this family.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

- [1] Eugene, N., Lee, C., Famoye, F., "Beta-normal distribution and its applications", Communications in Statistics-Theory and methods, 31(4): 497-512, (2002).
- [2] Zografos, K., Balakrishnan, N., "On families of beta and generalized gamma-generated distributions and associated inference", Statistical Methodology, 6, 344- 362, (2009).
- [3] Cordeiro, G. M., de Castro, M., "A new family of generalized distributions", Journal of statistical computation and simulation, 81(7): 883-898, (2011).
- [4] Ristić, M. M., Balakrishnan, N., "The gamma-exponentiated exponential distribution", Journal of Statistical Computation and Simulation, 82(8): 1191-1206, (2011).
- [5] Pescim, R. R., Cordeiro, G. M., Demétrio, C. G., Ortega, E. M., Nadarajah, S., "The new class of Kummer beta generalized distributions" SORT-Statistics and Operations Research Transactions, 36(2): 153-180, (2012).

- [6] Cordeiro, G. M., Ortega, E. M., da Cunha, D. C., "The exponentiated generalized class of distributions", *Journal of Data Science*, 11(1): 1-27, (2013).
- [7] Bourguignon, M., Silva, R. B., Cordeiro, G. M., "The Weibull-G family of probability distributions", *Journal of Data Science*, 12(1): 53-68, (2014).
- [8] Cordeiro, G. M., Alizadeh, M., Ortega, E. M., "The exponentiated half-logistic family of distributions: Properties and applications", *Journal of Probability and Statistics*, (2014).
- [9] Cordeiro, G. M., Alizadeh, M., Diniz Marinho, P. R., "The type I half-logistic family of distributions", *Journal of Statistical Computation and Simulation*, 86(4): 707-728, (2016).
- [10] Mahmoud, M. R., El-Sherpieny, E. A., Ahmed, M. A., "The new Kumaraswamy Kumaraswamy family of generalized distributions with application", *Pakistan Journal of Statistics and Operation Research*, 11(2) ,(2015).
- [11] Elgarhy, M., Hassan, A. S., Rashed, M., "Garhy-generated family of distributions with application", *Mathematical Theory and Modeling*, 6, 1-15, (2016).
- [12] Hassan, A. S., Elgarhy, M., "Kumaraswamy Weibull-generated family of distributions with applications", *Advances and Applications in Statistics*, 48(3): 205, (2016a).
- [13] Alizadeh, M., Cordeiro, G. M., Nascimento, A. D. C., Lima, M. C. S., Ortega, E. M. M., "Odd-Burr generalized family of distributions with some applications", *Journal of Statistical Computation & Simulation*, 87(2): 367-389, (2017).
- [14] Cordeiro, G. M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E. M. M., and Altun, E., "The generalized odd log-logistic family of distributions: properties, regression models and applications", *Journal of Statistical Computation & Simulation*, 87(5): 908-932, (2017).
- [15] Haghbin, H., Ozel, G., Alizadeh, M., and Hamedani, G. G., "A new generalized odd log-logistic family of distributions", *Communications in Statistics - Theory and Methods*, 46 (20): 9897-9920, (2017).
- [16] Gomes-Silva, F., Ramos, M.W., Percontini, A., Venancio, R., de Brito, E., Cordeiro, G. M., "The odd Lindley-G family of distributions", *Austrian Journal of Statistics*, 46: 65-87, (2017).
- [17] Haq, M., Elgarhy, M., "The odd Fréchet-G family of probability distributions", *Journal of Statistics Applications & Probability*, 7: 185-201, (2018).
- [18] Greenwood, J. A., Landwehr, J. M., Matalas, N. C., Wallis, J. R., "Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form", *Water Resources Research*, 15(5): 1049-1054, (1979).
- [19] David, H. A., & Nagaraja, H. N., "Order statistics", John Wiley & Sons, Inc. ,(1970).
- [20] Kenney, J. F., Keeping, E., "Mathematics of Statistics", D. Van Nostr and Company ,(1962).
- [21] Moors, J. J. A. ,A quantile alternative for kurtosis. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 37(1), 25–32, (1988).
- [22] Hassan, A. S., & Elgarhy, M., "A new family of exponentiated Weibull-generated distributions" 55, 7 ,(2016b).

- [23] Abouelmagd, T. H. M., Al-mualim, S. Elgarhy, M., Afify, A. Z., Ahmad, M. "Properties of the four-parameter Weibull distribution and its applications". *Pakistan Journal of Statistics*, 330: 449-466, (2017).
- [24] Hinkley, D., "On quick choice of power transformation", *Applied Statistics*, 67-69, (1977).
- [25] Bjerkedal, T., "Acquisition of Resistance in Guinea Pigs Infected with Different Doses of Virulent Tubercle Bacilli," *American Journal of Epidemiology*, 72(1): 130-148, (1960).