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Odd Generalized Exponential Power Function Distribution: Properties and Applications

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Abstract

In this article we introduce and study a new four-parameter distribution, called the odd generalized exponential power function distribution. The proposed model is a particular case from the odd generalized exponential family. Expressions for the moments, probability weighted moments, quantile function, Bonferroni and Lorenz curves, Rényi entropy and order statistics are obtained. The model parameters are estimated via the maximum likelihood and percentiles methods of estimation. A simulation study is carried out to evaluate and compare the performance of estimates in terms of their biases, standard errors and mean square errors. Eventually, the practical importance and flexibility of the proposed distribution in modelling real data application is checked. It can be concluded that the new distribution works better than some other known distributions.

1. INTRODUCTION

Statistical distributions are very useful in describing the real world phenomena. The exponential, Pareto, power function and Weibull distributions are of interest and very attractive in lifetime literature due to their simplicity, easiness and flexible features to model various types of data in different fields. The *power function* (PF) distribution is reasonably tractable model to evaluate the reliability of real life data such as electrical components including semiconductors devices [1]. The PF is one of the most important univariate and parametric models. This distribution is derived from Pareto distribution using the inverse transformation. Also, the PF is a special case from beta distribution. As mentioned in [2] the PF distribution is better than the exponential, Weibull and log-normal distributions to examine the reliability of any electrical component. The moments of order statistics for a PF distribution have been derived in [3]. The characterizations of the PF distribution were discussed in [4]. For more information about statistical properties of the PF distribution and its applications, can be found in [5-9]. A new characterization of the PF based on lower records was discussed in [10]. Parameter estimates of the PF distribution using different estimation procedures were found in [11]. For Bayesian estimation of the PF distribution, see for example, [12,13]. Probability weighted moments and generalized probability weighted moments estimators of PF distribution were discussed in [14].

The *probability density function* (pdf) and *cumulative distribution function* (cdf) of the PF with scale parameter γ, and shape parameter α are given, respectively, by

$$g(x;\gamma,\alpha)=\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1}; \qquad 0<x<\gamma \quad ,\alpha>0, \tag{1}$$

and

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$$G(x;\gamma,\alpha)=\left(\frac{x}{\gamma}\right)^{\alpha}. \tag{2}$$

Some extensions of the PF have been discussed by several authors. For example; beta PF [15], Weibull PF [16], *Kumaraswamy PF* (KwPF) [17], *Transmuted PF* (TPF) [18], *exponentiated Kumaraswamy PF* (EKwPF) [19], exponentiated Weibull PF [20] and *transmuted Weibull PF* (TPF) [21], *McDonald PF* (McPF) [22].

Generated families of continuous distributions are recent development which provide great flexibility in modelling real data. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. Some of the generated families are listed as follows; the beta- genertaed (B-G) [23, 24], gamma-G (type 1) [25], Kumaraswamy-G [26], McDonald-G [27], gamma-G (type 2) [28], transformed-transformer-G [29], Weibull-G [30], *odd generalized exponential-G* (OGE-G) [31], Kumaraswamy Weibull-G [32], exponentiated Weibull-G [33] and additive Weibull-G [34], among others.

Our interest here, with the OGE-G family which is flexible because of the hazard rate shapes: increasing, decreasing, J, reversed-J, bathtub and upside-down bathtub. The cdf and pdf of the OGE-G are defined as follows

$$F(x;\lambda,\theta,\xi)=\left[1-\mathrm{e}^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right]^{\theta}, \tag{3}$$

and

$$f(x;\lambda,\theta,\xi)=\frac{\lambda\theta g(x;\xi)}{\left(\bar{G}(x;\xi)\right)^{2}}e^{-\lambda[\frac{G(x;\xi)}{\bar{G}(x;\xi)}]}\left[1-\mathrm{e}^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right]^{\theta-1};\ x,\lambda,\theta>0, \tag{4}$$

where, $g(x;\xi)$ is the baseline pdf and $\bar{G}(x;\xi)=1-G(x;\xi)$. The main motivations for using the OGE-G family are to make the kurtosis more flexible (compared to the baseline model). In addition to construct heavy-tailed distributions that are not long-tailed for modeling real data. The class of OGE-G distributions shares an attractive physical interpretation of *X* when θ is an integer. Consider a system formed by θ independent components following the odd exponential-G class ([30]) given by

$$H(x;\lambda,\xi)=1-\mathrm{e}^{-\lambda\frac{G(x;\xi)}{\bar{G}(x;\xi)}}.$$

Suppose the system fails if all θ components fail and let *X* denote the lifetime of the entire system. Then, the cdf of *X* is $F(x;\lambda,\theta,\xi)=(H(x;\lambda,\xi))^{\theta}$, which is identical to (3).

To increase the flexibility for modeling purposes it will be useful to consider further alternatives to PF distribution. Our purpose is to provide a new four-parameter model, named as *odd generalized exponential power function* (OGEPF) using the OGE-G family. The suggsted model is quite flexible in terms of hazard rate could be increasing, decreasing, U and J-shaped. Also, we show its flexibility on the basis of three real life data.

This paper is organized as follows. The pdf, cdf, survival function, *hazard rate function* (hrf), reversed-hazard rate function and cumulative hazard rate function of the OGEPF are defined in Section 2. Mathematical properties including, expansions of its pdf and cdf, quantile function, moments, probability weighted moments, incomplete moments, entropy and order statistics are studied in Section 3. In Section 4, maximum likelihood and percentiles estimators are derived for the population parameters of the OGEPF distribution. A simulation study is established for evaluating parameter estimates in Section 5. Three real data sets are analyzed and compared with other fitted models in Section 6. At the end, concluding remarks are presented in Section 7.

2. THE OGEPF DISTRIBUTION

In this section, we introduce the odd generalized exponential power function distribution. The pdf, cdf, reliability function, hrf, reversed-hazard rate function and cumulative hazard rate function of the OGEPF distribution are derived.

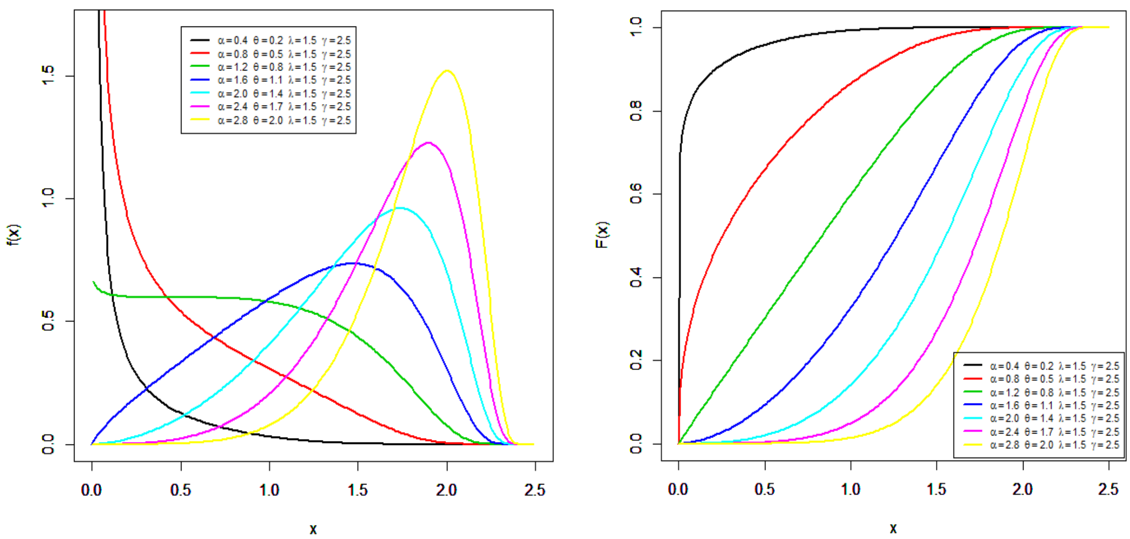
The cdf of OGEPF distribution, denoted by OGEPF$(\lambda,\alpha,\gamma,\theta)$, is obtained by inserting the pdf (1) and cdf (2) in cdf (3) as follows

$$F(x;\Phi)=\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{\theta};\quad 0<x<\gamma \quad ,\alpha,\theta>0, \tag{5}$$

where, $\Phi\equiv(\lambda,\alpha,\gamma,\theta)$. The pdf of OGEPF distribution is obtained by inserting the pdf (1) and cdf (2) into (4) as the following

$$f(x;\Phi)=\theta\lambda\alpha\gamma^{\alpha}x^{\alpha-1}\left(\gamma^{\alpha}-x^{\alpha}\right)^{-2}e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{\theta-1};0<x<\gamma. \tag{6}$$

For $\theta=1$, the pdf of OGEPF model reduces to the odd exponential- PF model. Figure 1 displays some plots of the pdf and cdf of OGEPF distribution for some selected parameter values. Figure 1 indicates that the densities of the OGEPF take different shapes.



(a) (b)

Figure 1. *Plots of (a) pdf and (b) cdf of OGEPF for some selected values of parameters*

Furthermore, the survival function, hrf, reversed-hazard rate function and cumulative hazard rate function of OGEPF distribution are respectively given by

$$\bar{F}(x;\Phi)=1-\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{\theta},$$

$$h(x;\Phi)=\frac{\theta\lambda\alpha\gamma^{\alpha}x^{\alpha-1}e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{\theta-1}}{\left(\gamma^{\alpha}-x^{\alpha}\right)^{2}\left\{1-\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{\theta}\right\}},$$

$$\vartheta(x;\Phi)=\theta\lambda\alpha\gamma^{\alpha}x^{\alpha-1}e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\left(\gamma^{\alpha}-x^{\alpha}\right)^{-2}\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{-1},$$

and

$$H(x;\Phi)=-\ln\left(1-\left[1-e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha}-x^{\alpha}}\right)}\right]^{\theta}\right).$$

Figure 2 indicates that OGEPF hrfs can have increasing, decreasing, J and U-shaped. This fact implies that the OGEPF can be very useful for fitting data sets with various shapes.

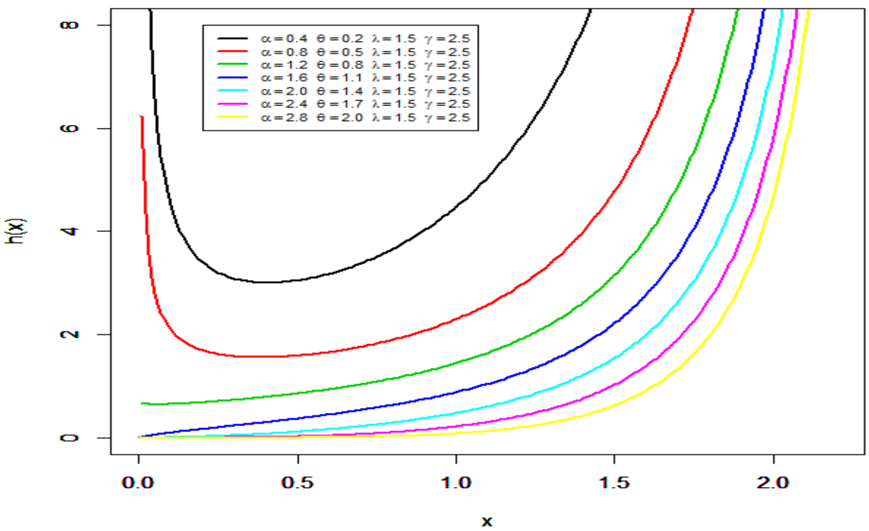


Figure 2. *Plots hrf of OGEPF for some selected values of parameters*

3. SOME MATHEMATICAL PROPERTIES

In this section, some mathematical properties of the OGEPF distribution, including, pdf and cdf expansions, quantile function, moments and incomplete moments, probability weighted moments, order statistics and entropy measure are derived.

3.1. Quantile Measures

The quantile function, say $x = Q(u) = F^{-1}(u)$ of X can be obtained by inverting (5) as follows

$$Q(u) = \gamma\left[\frac{-ln\left(1-(u)^{\frac{1}{\theta}}\right)}{\lambda - ln\left(1-(u)^{\frac{1}{\theta}}\right)}\right]^{\frac{1}{\alpha}}, \tag{7}$$

where, u is a uniform variate on the unit interval (0,1). In particular, the first quartile, median and third quartile are obtained by subsituting u=0.25,0.5 and 0.75 in (7).

The Bowley skewness (see [35]), based on quantiles, is given by

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Further, the Moors kurtosis (see [36]) is defined as

$$M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

where Q (.) denotes the quantile function. The graphs of *Bowley skewness* (*B*) and *Moors kurtosis* (*M*) are given below for different values of the parameters. Plots of the skewness and kurtosis for some choices of the parameter θ as function of α, and for some choices of the parameter α as function of θ are illustrated in Figures 3 and 4. These plots show that the skewness decreases when θ increases for fixed α and when α increases for fixed θ. Figures 4 reveal that there is great flexibility of kurtosis curves.

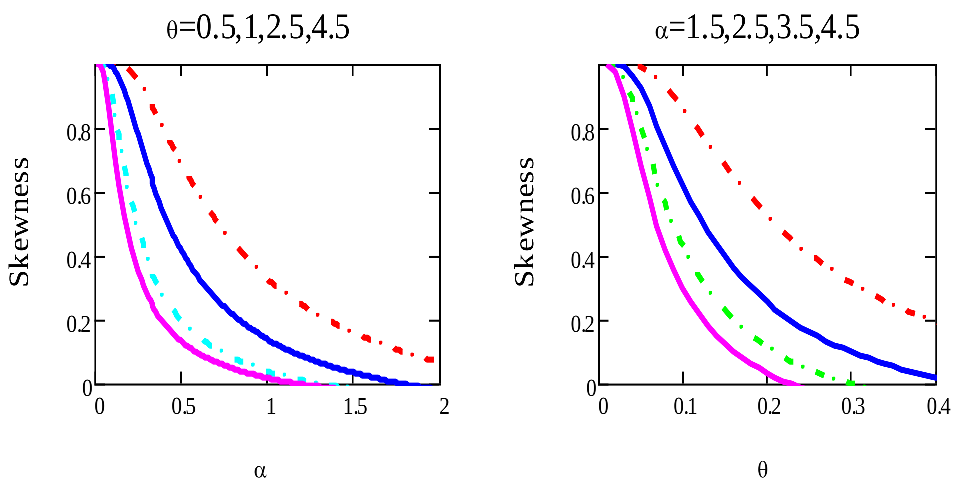


***Figure 3**. Skewnees of the OGEPF with different values of α and θ*

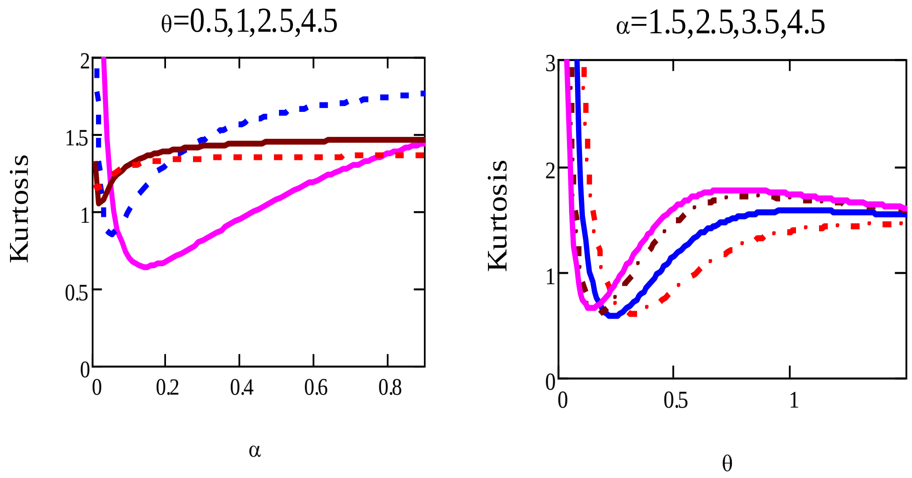


***Figure 4**. Kurtosis of the OGEPF with different values of α and θ*

3.2. Useful Expansion

Here, useful expansions are derived. Since, the pdf (6) can be rewritten as follows

$$f(x;\Phi)=\frac{\theta\lambda\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1}e^{-\lambda\left(\left(\frac{x}{\gamma}\right)^{\alpha}/1-\left(\frac{x}{\gamma}\right)^{\alpha}\right)}\left(1-\left(\frac{x}{\gamma}\right)^{\alpha}\right)^{-2}\left[1-e^{-\lambda\left(\left(\frac{x}{\gamma}\right)^{\alpha}/1-\left(\frac{x}{\gamma}\right)^{\alpha}\right)}\right]^{\theta-1}\quad ;0<x<\gamma. \tag{8}$$

By using the binomial expansion for the last term in (8) and further the exponential expansion, then the pdf (8) can be expressed as follows

$$f(x;\Phi)=\sum_{j,k=0}^{\infty}(-1)^{j+k}\binom{\theta-1}{j}\frac{\theta\lambda^{k+1}\alpha(j+1)^{k}}{\gamma k!}\left(\frac{x}{\gamma}\right)^{\alpha k+\alpha-1}\left(1-\left(\frac{x}{\gamma}\right)^{\alpha}\right)^{-k-2}. \tag{9}$$

Using the following series expansion

$$(1-z)^{-k}=\sum_{i=0}^{\infty}\frac{\Gamma(k+i)z^{i}}{\Gamma(k)i!},\quad |z|<1,k>0. \tag{10}$$

Then the pdf (9) takes the following form

$$\left.\begin{aligned}&f(x;\Phi)=\sum_{j,k,i=0}^{\infty}w_{j,k,i}\,g_{\alpha(k+i+1)}(x),\\&w_{j,k,i}=(-1)^{j+k}\binom{\theta-1}{j}\theta\lambda^{k+1}(j+1)^{k}\,\frac{\Gamma(k+1+i)}{k!\Gamma(k+2)i!},\ \sum_{j,k,i=0}^{\infty}w_{j,k,i}=1\end{aligned}\right\} \tag{11}$$

and $g_{\alpha(k+i+1)}$ denotes the pdf of the PF distribution with parameters $\alpha(k+i+1)$ and γ.

Further, an expansion for $F(x;\Phi)^{s}$, where s is an integer and θ is a real non integer, takes the following form

$$\left.\begin{aligned} F(x;\Phi)^s &= \sum_{m,l,p=0}^{\infty} \eta_{m,l,p} G_{\alpha(l+p)}(x), \\ \eta_{m,l,p} &= \sum_{m,l,p=0}^{\infty} (-1)^{m+l} \binom{\theta s}{m} \frac{\Gamma(p+l)}{\Gamma(l)\, p!} \frac{(\lambda m)^l}{l!}, \end{aligned}\right\} \tag{12}$$

and $G_{\alpha(l+p)}(x)$ is the cdf of PF with parameters $\alpha(l+p)$ and γ.

3.3. Probability Weighted Moments

The *probability weighted moments* (PWMs) can be used to derive estimators of the parameters and quantiles of generalized distributions. The PWM of *X* is defined by

$$\tau_{r,s} = E\left\{X^r \left[F(x)\right]^s\right\} = \int_{-\infty}^{\infty} x^r \left[F(x)\right]^s f(x)\,dx, \tag{13}$$

where, s and r are positive integers. Inserting pdf (11) and cdf (12) in (13), then the PWM of the OGEPF distribution is obtained as follows

$$\tau_{r,s} = \sum_{m,l,p=0}^{\infty} \sum_{j,k,i=0}^{\infty} \eta_{m,l,p} w_{j,k,i} \int_0^{\gamma} x^r \left(\frac{x}{\gamma}\right)^{\alpha(l+p)} \frac{\alpha(k+i+1)}{\gamma} \left(\frac{x}{\gamma}\right)^{\alpha k+\alpha+\alpha i-1} dx.$$

Let $z = x/\gamma \Rightarrow dz = dx/\gamma$, then $\tau_{r,s}$ is written as follows

$$\tau_{r,s} = \gamma^r \sum_{m,l,p=0}^{\infty} \sum_{j,k,i=0}^{\infty} \alpha(k+i+1) \eta_{m,l,p} w_{j,k,i} \int_0^1 z^{r+\alpha(l+k+i+p+1)-1} dz.$$

Therefore, the PWM of OGEPF distribution is given by

$$\tau_{r,s} = \sum_{m,l,p=0}^{\infty} \sum_{j,k,i=0}^{\infty} \eta_{m,l,p} w_{j,k,i} \frac{\gamma^r \alpha(k+i+1)}{\left(r+\alpha(l+k+i+p+1)\right)}.$$

3.4. Moments

Moments are necessary and important in any statistical analysis especially in applications. It can used to study the most important characteristics and features of distribution (e.g, dispersion, skewness, kurtosis and tendency). The *rth* moment of OGEPF is derived by using pdf (11) as follows

$$\mu_r' = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \int_0^{\gamma} x^r g_{\alpha(k+i+1)}(x)\,dx = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \int_0^{\gamma} x^r \frac{\alpha(k+i+1)}{\gamma} \left(\frac{x}{\gamma}\right)^{\alpha k+\alpha+\alpha i-1} dx.$$

Let $z = x/\gamma \Rightarrow dz = dx/\gamma$, then the previous equation takes the following form

$$\mu_r' = \sum_{j,k,i=0}^{\infty} w_{j,k,i}\, \alpha(k+i+1) \int_0^1 (z\gamma)^r z^{\alpha k+\alpha+\alpha i-1} dz.$$

After simplification, the *rth* moment of OGEPF is obtained as follows

$$\mu_r' = \sum_{i,j,k=0}^{\infty} w_{j,k,i} \frac{\gamma^r \alpha(k+i+1)}{\left(\alpha(k+i+1)+r\right)}; \quad r=1,2,3,...$$

In particular, the mean and variance of the OGEPF distribution are given by

$$E(X) = \sum_{i,j,k=0}^{\infty} w_{j,k,i} \frac{\gamma\alpha(k+i+1)}{\left(\alpha(k+i+1)+1\right)}, \tag{14}$$

and

$$\text{var}(X) = \sum_{i,j,k=0}^{\infty} w_{j,k,i} \frac{\gamma^2\alpha(k+i+1)}{\left(\alpha(k+i+1)+2\right)} - \left[\sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{\gamma\alpha(k+i+1)}{\left(\alpha(k+i+1)+1\right)}\right]^2.$$

Furthermore, the moment generating function of the OGEPF distribution is obtained as follows

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right) = \sum_{r=0}^{\infty} \sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{(t\gamma)^r \alpha(k+i+1)}{r!\left(\alpha(k+i+1)+r\right)}.$$

3.5. Incomplete Moments

The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. The *sth* incomplete moment, say $£_s(t;\Phi)$, is defined by

$$£_s(t;\Phi) = \int_{-\infty}^{t} x^s f(x)dx. \tag{15}$$

Hence, the *sth* incomplete moment of OGEPF is derived by using pdf (11) as follows

$$£_s(t;\Phi) = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \int_0^t x^s \frac{\alpha(k+i+1)}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha(k+i+1)-1} dx,$$

which leads to

$$£_1(t;\Phi) = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{t^{\alpha(k+i+1)+s}}{\gamma^{\alpha(k+i+1)}} \frac{\alpha(k+i+1)}{\alpha(k+i+1)+s}. \tag{16}$$

In particular, the first incomplete moment of the OGEPF distribution can be obtained by putting $s=1$ in (16), as follows

$$£_1(t;\Phi) = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{t^{\alpha(k+i+1)+1}}{\gamma^{\alpha(k+i+1)}} \frac{\alpha(k+i+1)}{\alpha(k+i+1)+1}. \tag{17}$$

The mean deviations provide useful information about the characteristics of a population and it can be calculated from the first incomplete moment. Indeed, the amount of dispersion in a population may be measured to some extent by the totality of the deviations from the mean and median. The mean deviations of *X* about the *means*can be calculated from the following relation) m(*median* and about the (μ)

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \quad \text{and} \quad \delta_2(X) = \mu - 2T(m),$$

where, *m* is obtained from (7) by setting $u = 0.5$, μ is defined in (14), $T(q) = \int_0^q xf(x)dx$ which is the first incomplete moment, then from (16);swollof sa ,ylevitcepser ,deniatbo era $T(m)$ dna $T(\mu)$

$$T(\mu) = \int_0^{\mu} xf(x)dx = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{\mu^{\alpha(k+i+1)+1}}{\gamma^{\alpha(k+i+1)}} \frac{\alpha(k+i+1)}{\alpha(k+i+1)+1},$$

and

$$T(m) = \int_0^{m} xf(x)dx = \sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{m^{\alpha(k+i+1)+1}}{\gamma^{\alpha(k+i+1)}} \frac{\alpha(k+i+1)}{\alpha(k+i+1)+1}.$$

Another application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The Lorenz and Bonferroni curves are obtained, respectively, as follows

$$L_F(x) = \frac{1}{E(X)} \int_0^x tf(t)dt = \frac{\sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{x^{\alpha(k+i+1)+1}}{\gamma^{\alpha(k+i+1)}} \frac{\alpha(k+i+1)}{\alpha(k+i+1)+1}}{\sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{\gamma\alpha(k+i+1)}{(\alpha(k+i+1)+1)}},$$

and

$$B_F(x) = \frac{L_F(x)}{F(x)} = \frac{\sum_{j,k,i=0}^{\infty} w_{j,k,i} \frac{x^{\alpha(k+i+1)+1}}{\gamma^{\alpha(k+i+1)}} \frac{\alpha(k+i+1)}{\alpha(k+i+1)+1}}{\left[1 - e^{-\lambda\left(\frac{x^{\alpha}}{\gamma^{\alpha} - x^{\alpha}}\right)}\right]^{\theta} \sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{\gamma\alpha(k+i+1)}{(\alpha(k+i+1)+1)}}.$$

3.6. Rényi Entropy

The entropy of a random variable X with density function $f(x)$ is a measure of the uncertainty variation. The Rényi entropy is defined as

$$I_R(\beta) = \frac{1}{1-\beta} ln\left\{\int_{-\infty}^{\infty} f^{\beta}(x)dx\right\}, \tag{18}$$

where $\beta > 0$ and $\beta \neq 1$. Using the binomial theory and exponential expansion, then the pdf $f(x;\Phi)^{\beta}$ can be expressed as follows

$$f(x;\Phi)^{\beta} = \sum_{j,k=0}^{\infty} \binom{(\theta-1)\beta}{j} (-1)^{j+k} \left(\frac{\theta\lambda\alpha}{\gamma}\right)^{\beta} \frac{(\beta\lambda+\lambda j)^{k}}{k!} \left(\frac{x}{\gamma}\right)^{\beta(\alpha-1)+\alpha k} \left(1-\frac{x}{\gamma}\right)^{-\alpha k-2\beta} . \tag{19}$$

Applying the binomial expansion (10) in (19), then $f(x;\Phi)^{\beta}$ can be written as follows

$$f(x;\Phi)^{\beta} = \sum_{j,k=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(\alpha k+2\beta+i)}{\Gamma(\alpha k+2\beta) i!} \binom{(\theta-1)\beta}{j} (-1)^{j+k} \left(\frac{\theta\lambda\alpha}{\gamma}\right)^{\beta} \frac{(\beta\lambda+\lambda j)^{k}}{k!} \left(\frac{x}{\gamma}\right)^{\beta(\alpha-1)+\alpha k+\alpha i} . \tag{20}$$

Hence, the Rényi entropy of the OGEPF model is obtained as follows

$$I_R(\beta) = \frac{1}{1-\beta} ln \left\{ \sum_{j,k,i=0}^{\infty} \binom{(\theta-1)\beta}{j} (-1)^{j+k} \frac{(\theta\lambda\alpha)^{\beta} (\beta\lambda+\lambda j)^{k}}{\gamma^{\beta-1}(\beta(\alpha-1)+\alpha k+i+1)k!} \frac{\Gamma(\alpha k+2\beta+i)}{\Gamma(\alpha k+2\beta) i!} \right\}.$$

3.7. Order Statistics

Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the order statistics for a random sample $X_1, X_2,.., X_n$ from OGEPF distribution with cdf (12) and pdf (11). The pdf of *rth* order statistics is given by

$$f_{(r)}(x;\Phi) = \frac{1}{B(r,n-r+1)} \sum_{\upsilon=0}^{n-r} \binom{n-r}{\upsilon} (-1)^{\upsilon} \left[F(x;\Phi)\right]^{\upsilon+r-1} f(x;\Phi). \tag{21}$$

Again, by using binomial expansion for $\left[F(x;\Phi)\right]^{\upsilon+r-1}$ and replacing s in (12) with $\upsilon+r-1$. Hence the pdf (21) takes the following form

$$f_{(r)}(x;\Phi) = \frac{1}{B(r,n-r+1)} \sum_{\upsilon=0}^{n-r} \sum_{j,k,i=0}^{\infty} (-1)^{\upsilon+j+k} \binom{n-r}{\upsilon} \binom{\theta-1}{j} \frac{\theta\lambda^{k+1}\alpha(j+1)^{k}\,\Gamma(k+2+i)}{\gamma k!\Gamma(k+2) i!} \left(\frac{x}{\gamma}\right)^{\alpha k+\alpha+\alpha i-1}$$
$$\times \sum_{m,l,p=0}^{\infty} (-1)^{m+l} \binom{\theta(\upsilon+r-1)}{m} \frac{(\lambda m)^{l}}{l!} \frac{\Gamma(p+l)}{\Gamma(l) p!} \left(\frac{x}{\gamma}\right)^{\alpha(l+p)} .$$

Hence,

$$f_{(r)}(x;\Phi) = \frac{1}{B(r,n-r+1)} \sum_{\upsilon=0}^{n-r} \sum_{j,k,i,m,l,p=0}^{\infty} \varpi_{\upsilon,j,k,i,m,l,p}\, g_{\alpha(k+1+i+l+p)}(x), \tag{22}$$

where

$$\varpi_{\upsilon,j,k,i,m,l,p} = (-1)^{\upsilon+j+k+m+l} \binom{\theta-1}{j} \binom{\theta(\upsilon+r-1)}{m} \frac{\theta\lambda^{k+1}\alpha(j+1)^{k}\,\Gamma(k+2+i)(\lambda m)^{l}\,\Gamma(p+l)}{\Gamma(k+2)\Gamma(l) k! l! p! i! (k+i+l+p+1)},$$

and $g_{\alpha(k+i+l+p+1)}(x)$ is the pdf of the PF distribution with parameters $(\alpha(k+i+l+p+1),\gamma)$.

In particular, the pdf of the smallest order statistics is obtained by substituting $r=1$ in (22) as follows

$$f_{(1)}(x;\Phi)=n\sum_{\upsilon=0}^{n-1}\sum_{j,k,i,m,l,p=0}^{\infty}\varsigma_{\upsilon,j,k,i,m,l,p}\,g_{\alpha(k+i+l+p+1)}(x),$$

where

$$\varsigma_{\upsilon,j,k,i,m,l,p}=(-1)^{\upsilon+j+k+m+l}\binom{\theta-1}{j}\binom{\theta\upsilon}{m}\theta\lambda^{k+1}\alpha(j+1)^{k}\frac{\Gamma(k+2+i)(\lambda m)^{l}\,\Gamma(p+l)}{\Gamma(k+2)\Gamma(l)k!l!p!i!(k+i+l+p+1)}.$$

Also, the pdf of largest order statistics is obtained by substituting $r=n$ in (22) as follows

$$f_{(n)}(x;\Phi)=n\sum_{j,k,i,m,l,p=0}^{\infty}\delta_{j,k,i,m,l,p}\,g_{\alpha(k+i+l+p+1)}(x),$$

where

$$\delta_{j,k,i,m,l,p}=(-1)^{j+k+m+l}\binom{\theta-1}{j}\binom{\theta(\upsilon+n-1)}{m}\theta\lambda^{k+1}\alpha(j+1)^{k}\frac{\Gamma(k+2+i)(\lambda m)^{l}\,\Gamma(p+l)}{\Gamma(k+2)\Gamma(l)k!l!p!i!(k+i+l+p+1)}.$$

4. PARAMETER ESTIMATION

In this section, the estimators of the OGEPF model parameters are obtained based on *maximum likelihood* (ML), and percentiles methods.

4.1. Maximum Likelihood Estimators

In this subsection, we consider the estimation of the unknown parameters of the OGEPF distribution using the ML method. Let $X_1,..,X_n$ be the observed values from the OGEPF distribution with set of parameters $\Phi=(\lambda,\alpha,\gamma,\theta)^{T}$. The total log-likelihood function, denoted by *LogL*, based on complete sample for the vector of parameters Φ can be expressed as

$$LogL=n\ln\theta+n\ln\lambda+n\ln\alpha+n\alpha\ln\gamma+(\alpha-1)\sum_{i=1}^{n}\ln x_i-2\sum_{i=1}^{n}\ln\left(\gamma^{\alpha}-x^{\alpha}\right)-\lambda\sum_{i=1}^{n}D_i$$
$$+(\theta-1)\sum_{i=1}^{n}\ln\left(1-e^{-\lambda D_i}\right),$$

where, $D_i=[\frac{x^{\alpha}}{\left(\gamma^{\alpha}-x^{\alpha}\right)}]$. It is known that, the estimate of γ is the sample maxima, i.e $\hat{\gamma}=X_{(n)}$. The partial derivatives of the log-likelihood function with respect to θ, λ, and α components of the score vector $U_L=(U_\theta,U_\lambda,U_\alpha)^{T}$ can be obtained as follows

$$U_\theta=\frac{\partial LogL}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\ln\left(1-e^{-\lambda D_i}\right),$$

$$U_{\lambda}=\frac{\partial LogL}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}D_i+(\theta-1)\sum_{i=1}^{n}\frac{D_i}{\left(e^{\lambda D_i}-1\right)},$$

and

$$U_{\alpha}=\frac{\partial LogL}{\partial\alpha}=\frac{n}{\alpha}+\text{n}\ln\gamma+\sum_{i=1}^{n}\ln x_i-2\sum_{i=1}^{n}\frac{\gamma^{\alpha}\ln\gamma-x^{\alpha}\ln x}{(\gamma^{\alpha}-x^{\alpha})}-\lambda\sum_{i=1}^{n}\frac{\partial D_i}{\partial\alpha}+(\theta-1)\lambda\sum_{i=1}^{n}\frac{1}{\left(e^{\lambda D_i}-1\right)}\frac{\partial D_i}{\partial\alpha},$$

where, $\dfrac{\partial D_i}{\partial\alpha}=\dfrac{\gamma^{\alpha}x^{\alpha}(\ln x-\ln\gamma)}{\left(\gamma^{\alpha}-x^{\alpha}\right)^2}$.

Then the ML estimators of the parameters, θ, λ, and α are obtained by setting U_{θ}, U_{λ}, and U_{α} to be zeros and solving them numerically.

4.2. Percentiles Estimator

Let $X_1, X_2,.., X_n$ be a random sample from the OGEPF distribution and $X_{(i)}$ denotes the *ith* order statistic, i.e, $X_{(1)} < X_{(2)} <...< X_{(n)}$. If p_i denotes some estimators of $F\left(x_{(i)};\Phi\right)$, then the estimator of Φ can be obtained by minimizing the following equation with respect to the unknown parameters

$$\sum_{i=1}^{n}\left[\ln\left(p_i\right)-\ln\left[1-e^{-\lambda\left(\frac{x_{(i)}^{\alpha}}{\gamma^{\alpha}-x_{(i)}^{\alpha}}\right)}\right]^{\theta}\right]^2 .$$

In *percentiles method* (PM) of estimate, p_i takes a several possible choice as estimates for $F\left(x;\Phi\right)$, in this study, we use the formula $p_i=\dfrac{i}{n+1}$.

5. NUMERICAL STUDY

In this section, we perform simulation study to evaluate and compare the performance of the estimates with respect to their biases, *standard errors* (SEs) and *mean square errors* (MSEs) for different sample sizes and for different parameter values. The numerical procedures are described through the following algorithm.

Step(1): A random sample $X_1,.., X_n$ of sizes n =10,20,30,50 and 100 are selected, these random samples are generated from the OGEPF distribution.

Step(2): Assume that the scale parameter γ is known and equal one throughout the experiment. Eight different set values of the parameters are selected as, $Set1\equiv\left(\alpha=0.2,\ \lambda=0.5,\ \theta=0.5\right)$, $Set\,2\equiv\left(\alpha=0.2,\ \lambda=0.5,\ \theta=1\right)$, $Set\,3\equiv\left(\alpha=0.2,\ \lambda=0.5,\ \theta=1.5\right)$, $Set\,4\equiv\left(\alpha=0.2,\ \lambda=0.5,\ \theta=2\right)$, $Set\,5\equiv\left(\alpha=0.7,\ \lambda=0.5,\ \theta=1\right)$, $Set\,6\equiv\left(\alpha=1.2,\ \lambda=0.5,\ \theta=1\right)$, $Set\,7\equiv\left(\alpha=1.7,\ \lambda=0.5,\ \theta=1\right)$ and $Set\,8\equiv\left(\alpha=2.2,\ \lambda=0.5,\ \theta=1\right)$.

Step(3): For each model parameters and for each sample size, the ML and percentiles estimates of λ, α and θ are computed.

Step(4): Steps from 1 to 3 are repeated 1000 times for each sample size and for selected sets of parameters. Then, the biases, SEs and MSEs of the estimates of the unknown parameters are computed.

Numerical results are listed in Tables 1 and 2.

Table 1. *Biases, SEs and MSEs for Set1, Set 2, Set 3 and Set 4 of parameters*

Set of parameters														
			Set1≡(α=0.2, λ=0.5, θ=0.5)			Set2≡(α=0.2, λ=0.5, θ=1)			Set3≡(α=0.2, λ=0.5, θ=1.5)			Set4≡(α=0.2, λ=0.5, θ=2)		
N	Method	Properties	*α*	*λ*	*Θ*	*α*	*Λ*	*θ*	*α*	*λ*	*θ*	*α*	*λ*	*θ*
10	ML	MSE	1.995	0.246	0.215	1.255	0.327	0.719	0.235	1.837	0.660	0.209	1.504	2.866
		Bias	1.204	-0.435	-0.459	0.463	0.242	0.036	0.348	0.968	-0.244	0.277	0.773	0.164
		SE	0.074	0.024	0.007	0.102	0.052	0.085	0.034	0.095	0.077	0.036	0.095	0.168
	PM	MSE	1.579	0.245	0.236	0.665	0.314	0.893	1.442	0.277	1.922	0.944	0.275	3.621
		Bias	1.069	-0.494	-0.049	0.071	-0.530	-0.815	0.317	-0.501	1.335	0.244	-0.475	-1.796
		SE	0.066	0.001	0.001	0.081	0.018	0.048	0.116	0.016	0.038	0.094	0.022	0.063
20	ML	MSE	0.918	0.229	0.209	0.318	0.199	0.287	0.163	1.026	0.404	0.140	0.846	1.184
		Bias	0.851	-0.421	-0.453	0.215	0.186	-0.008	0.299	0.747	-0.378	0.221	0.558	-0.091
		SE	0.022	0.011	0.003	0.026	0.020	0.027	0.014	0.034	0.026	0.015	0.037	0.054
	PM	MSE	0.774	0.244	0.239	0.219	0.291	0.859	0.342	0.261	1.937	0.348	0.246	3.522
		Bias	0.822	-0.494	-0.489	-0.067	-0.521	-0.848	0.110	-0.503	-1.372	0.118	-0.492	-1.871
		SE	0.016	0.000	0.000	0.023	0.007	0.019	0.029	0.004	0.012	0.029	0.003	0.007
30	ML	MSE	0.642	0.242	0.195	0.164	0.233	0.175	0.129	0.725	0.383	0.118	0.648	0.847
		Bias	0.720	-0.387	-0.435	0.142	0.158	-0.028	0.279	0.657	-0.422	0.195	0.466	-0.164
		SE	0.012	0.010	0.003	0.013	0.015	0.014	0.008	0.018	0.015	0.009	0.022	0.030
	PM	MSE	0.613	0.243	0.240	0.200	0.273	0.829	0.246	0.266	1.959	0.209	0.251	3.561
		Bias	0.743	-0.493	-0.490	-0.105	-0.513	-0.878	0.059	-0.506	-1.376	0.074	-0.498	-1.884
		SE	0.008	0.000	0.000	0.014	0.003	0.008	0.016	0.003	0.009	0.015	0.002	0.004
50	ML	MSE	0.416	0.215	0.195	0.046	0.062	0.105	0.102	0.548	0.328	0.087	0.478	0.612
		Bias	0.594	-0.388	-0.433	0.056	0.078	0.002	0.253	0.574	-0.449	0.156	0.362	-0.181
		SE	0.005	0.005	0.002	0.004	0.005	0.006	0.004	0.009	0.007	0.005	0.012	0.015
	PM	MSE	0.446	0.242	0.241	0.133	0.269	0.827	0.080	0.253	1.959	0.066	0.256	3.595
		Bias	0.634	-0.492	-0.491	-0.162	-0.511	-0.885	0.001	-0.502	-1.397	0.018	-0.502	-1.887
		SE	0.004	0.000	0.000	0.007	0.002	0.004	0.006	0.001	0.002	0.005	0.001	0.004
100	ML	MSE	0.272	0.213	0.205	0.007	0.031	0.047	0.086	0.449	0.303	0.073	0.371	0.455
		Bias	0.940	-0.041	-0.446	0.017	0.037	0.014	0.241	0.539	-0.473	0.123	0.277	-0.155
		SE	0.002	0.002	0.001	0.001	0.002	0.002	0.002	0.004	0.003	0.002	0.005	0.007
	PM	MSE	0.300	0.240	0.243	0.082	0.255	0.825	0.043	0.251	1.970	0.013	0.250	3.612
		Bias	0.522	-0.490	-0.493	-0.174	-0.503	-0.903	-0.020	-0.500	-1.403	-0.004	-0.500	-1.900
		SE	0.002	0.000	0.000	0.002	0.000	0.001	0.002	0.000	0.000	0.001	0.000	0.000

Table 2. *Biases, SEs and MSEs for Set 5, Set 6, Set 7 and Set 8 of parameters*

Set of parameters														
			Set5≡(α=0.7, λ=0.5, θ=1)			Set6≡(α=1.2, λ=0.5, θ=1)			Set7≡(α=1.7, λ=0.5, θ=1)			Set8≡(α=2.2, λ=0.5, θ=1)		
n	Method	Properties	*α*	*λ*	*Θ*	*α*	*λ*	*Θ*	*α*	*λ*	*θ*	*α*	*λ*	*θ*
10	ML	MSE	1.603	0.664	1.856	1.421	0.250	1.215	2.889	0.250	0.816	4.839	0.250	1.001
		Bias	0.423	0.299	0.498	-1.186	-0.493	-0.817	-1.700	-0.500	-0.764	-2.200	-0.500	-0.978
		SE	0.119	0.076	0.127	0.012	0.008	0.074	0.000	0.000	0.048	0.000	0.000	0.021
	PM	MSE	0.460	0.250	0.900	1.436	0.246	0.998	2.890	0.240	0.960	4.840	0.239	0.996
		Bias	-0.678	-0.500	-0.948	-1.198	-0.496	-0.999	-1.700	-0.490	-0.980	-2.200	-0.489	-0.998
		SE	0.000	0.000	0.001	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	ML	MSE	1.035	0.373	0.929	1.421	0.250	1.523	2.889	0.250	0.815	4.839	0.250	0.982
		Bias	0.236	0.132	0.335	-1.186	-0.494	-0.804	-1.700	-0.500	-0.782	-2.200	-0.500	-0.976
		SE	0.049	0.030	0.045	0.006	0.004	0.047	0.000	0.000	0.023	0.000	0.000	0.009
	PM	MSE	0.460	0.250	0.899	1.435	0.248	0.998	2.890	0.240	0.960	4.840	0.239	0.996
		Bias	-0.678	-0.500	-0.948	-1.198	-0.498	-0.999	-1.700	-0.490	-0.980	-2.200	-0.489	-0.998
		SE	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	ML	MSE	0.860	0.295	0.531	1.404	0.244	1.543	2.889	0.250	0.803	4.839	0.250	0.977
		Bias	0.209	0.104	0.234	-1.182	-0.491	-0.811	-1.700	-0.500	-0.795	-2.200	-0.500	-0.981
		SE	0.030	0.018	0.023	0.003	0.002	0.031	0.000	0.000	0.014	0.000	0.000	0.004
	PM	MSE	0.460	0.250	0.899	1.435	0.248	0.998	2.890	0.240	0.960	4.840	0.239	0.996
		Bias	-0.678	-0.500	-0.948	-1.198	-0.498	-0.999	-1.700	-0.490	-0.980	-2.200	-0.489	-0.998
		SE	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	ML	MSE	0.558	0.188	0.385	1.421	0.251	1.294	2.889	0.250	0.822	4.839	0.250	0.976
		Bias	0.056	0.006	0.231	-1.187	-0.494	-0.822	-1.700	-0.500	-0.835	-2.200	-0.500	-0.959
		SE	0.015	0.009	0.012	0.002	0.002	0.016	0.000	0.000	0.007	0.000	0.000	0.005
	PM	MSE	0.460	0.250	0.898	1.435	0.249	0.998	2.890	0.240	0.960	4.840	0.239	0.996
		Bias	-0.678	-0.500	-0.948	-1.198	-0.499	-0.999	-1.700	-0.490	-0.980	-2.200	-0.489	-0.998
		SE	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
100	ML	MSE	0.275	0.105	0.244	1.402	0.243	1.206	2.889	0.250	0.877	4.839	0.250	0.960
		Bias	-0.080	-0.067	0.224	-1.181	-0.491	-0.809	-1.700	-0.500	-0.890	-2.200	-0.500	-0.956
		SE	0.005	0.003	0.004	0.001	0.000	0.007	0.000	0.000	0.003	0.000	0.000	0.002
	PM	MSE	0.460	0.250	0.898	1.435	0.250	0.997	2.890	0.240	0.960	4.840	0.239	0.996
		Bias	-0.678	-0.500	-0.948	-1.198	-0.500	-0.999	-1.700	-0.490	-0.980	-2.200	-0.489	-0.998
		SE	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

From the above tables, the following conclusions can be observed on the properties of estimated parameters of OGEPF distribution.

1. For the two methods of estimation, it is clear that biases and MSEs decrease as sample sizes increase (see Tables 1 and 2).
2. For fixed values of λ, α and as the values of θ increase, the biases and MSEs are decreasing, in approximately most of situations (see Table 1). As the values of α increase and for fixed values of λ and θ, the biases and MSEs decrease in approximately, most sample sizes (see Tables 1 and 2).

6. APPLICATIONS

In this section, three real data sets are considered to illustrate that the OGEPF model can be a good lifetime distribution comparing with main five models; McPF, KwPF, EKwPF, TPF and PF. In each real data set, the ML estimate and their corresponding SEs (in parentheses) of the model parameters are obtained. The model selection is carried out using *-2 log-likelihood* (-2Log*L*), *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC), the *correct Akaike information criterion* (CAIC) and *Hannan-Quinn* *information criterion* (HQIC). However, the better distribution corresponds to the smaller values of -2Log*L*, AIC, BIC, CAIC, and HQIC criteria. Furthermore, we plot the histogram for each data set and the estimated pdf for the six models. Moreover, the plots of empirical cdf of the data sets and estimated cdf for the six models are displayed.

Data set 1: The first data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by [37]. The data are:
0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55

Data set 2: The second data represent the time to failure (103h) of turbocharger of one type of engine [38]. The data are:
1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

Data set 3: The third data have been used in [39].The data represent the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data are:
0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Tables 3, 5 and 7 give the ML estimates of the model parameters and their SEs (in the parentheses) for the three real data sets. The results in Tables 4, 6 and 8 indicate that the OGEPF model is suitable for these data set based on the selected criteria. The OGEPF model has the smallest; -2 Log*L*, AIC, BIC, CAIC and HQIC. It is also clear from Figures 5, 6 and 7 that the OGEPF distribution provides a better fit and therefore be one of the best models for these data sets.

Table 3. *ML* estimates *of the model parameters and the corresponding SEs (in parentheses) for the first data se*t

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
OGEPF	2.787 (0.234)	1.210 (0.293)	1.219 (0.098)	5.55 -	- -	- -	- -	- -
McPF	- -	0.698 (0.549)	- -	- -	5.55 -	0.95 (0.740)	1 0.000	2.140 (1.681)
KwPF	5.55 -	0.760 (0.324)	- -	- -	- -	1.52 (0.652)	2.222 (0.380)	- -
EkwPF	- -	11.216 (6.769)	0.434 (0.373)	- -	5.55 -	0.488 (0.419)	1.847 (0.222)	- -
PF	5.55 -	- -	0.663 (0.0781)	- -	- -	- -	- -	- -
TPF	5.55 -	1 (0.095)	1.089 (0.0954)	- -	- -	- -	- -	- -

Table 4. *The statistics, -2LogL, AIC, BIC, CAIC and HQIC for the first data set*

Distribution	-2Log*L*	AIC	BIC	CAIC	HQIC
OGEPF	211.014	217.014	223.844	217.367	219.733
McPF	949.170	957.170	966.276	957.767	960.795
KwPF	220.317	226.317	233.147	226.770	229.036
EKwPF	216.355	224.355	233.462	224.952	227.980
PF	256.590	260.590	265.143	260.764	262.403
TPF	227.836	231.836	236.389	232.010	233.649

Table 5. *ML* estimates *of the model parameters and the corre sponding SEs (in parentheses) for the second data set*

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
OGEPF	2.93 (0.542)	0.397 (0.115)	6.555 (1.134)	9 -	- -	- -	- -	- -
McPF	- -	1.145 (4.061)	- -	- -	9 -	1.631 (5.786)	1 (0.00087)	2.808 (14.851)
KwPF	9 -	0.182 (0.162)	- -	- -	- -	17.308 (15.487)	2.492 (0.642)	- -
EKwPF	- -	62.376 (0.00227)	0.138 (13.419)	- -	9 -	0.823 (2.269)	1.519 (0.191)	- -
PF	9 -	- -	1.867 (0.295)	- -	- -	- -	- -	- -
TPF	9 -	152.128 (35.602)	0.145 (0.0084)	- -	- -	- -	- -	- -

Table 6. *The statistics,-2LogL, AIC, BIC, CAIC and HQIC for the second data set*

Distribution	-2Log*L*	AIC	BIC	CAIC	HQIC
OGEPF	156.326	164.326	162.734	165.468	157.963
McPF	555.230	563.230	561.639	564.373	556.868
KwPF	159.526	165.526	160.754	166.192	160.754
EKwPF	162.493	170.493	163.311	171.636	164.130
PF	171.419	175.419	174.623	175.744	172.238
TPF	324.886	328.886	328.090	329.211	325.705

Table 7. *ML* estimates *of the model parameters and the corresponding SEs (in parentheses) for the third data set*

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
OGEPF	1.913 (0.164)	1.414 (0.344)	2.774 (0.254)	2.24 -	- -	- -	- -	- -
McPF	- -	0.865 (2.218)	- -	- -	2.24 -	1.456 (3.735)	1 (0.00058)	3.803 (5.163)
KwPF	2.24 -	1.396 (1.5)	- -	- -	- -	2.76 (2.9549)	2.434 (0.461)	- -
EKwPF	- -	142.385 (0.0063)	0.119 (21.122)	- -	2.24 -	2.988 (1.354)	1.519 (0.398)	- -

PF	2.24 -	- -	1.259 (0.1586)	- -	- -	- -	- -	- -	
TPF	2.24 -	53.147 (5.131)	0.071 (0.0015)	- -	- -	- -	- -	- -	

Table 8. *The statistics, -2LogL, AIC, BIC, CAIC and HQIC for the third data set*

Distribution	-2Log*L*	AIC	BIC	CAIC	HQIC
OGEPF	42.993	50.993	50.191	51.683	45.034
McPF	225.420	233.420	232.617	234.109	227.461
KwPF	46.937	52.937	52.335	53.344	48.468
EKwPF	53.501	61.501	54.521	62.190	55.542
PF	145.030	149.030	148.629	149.230	146.051
TPF	610.580	614.580	614.179	614.780	611.601

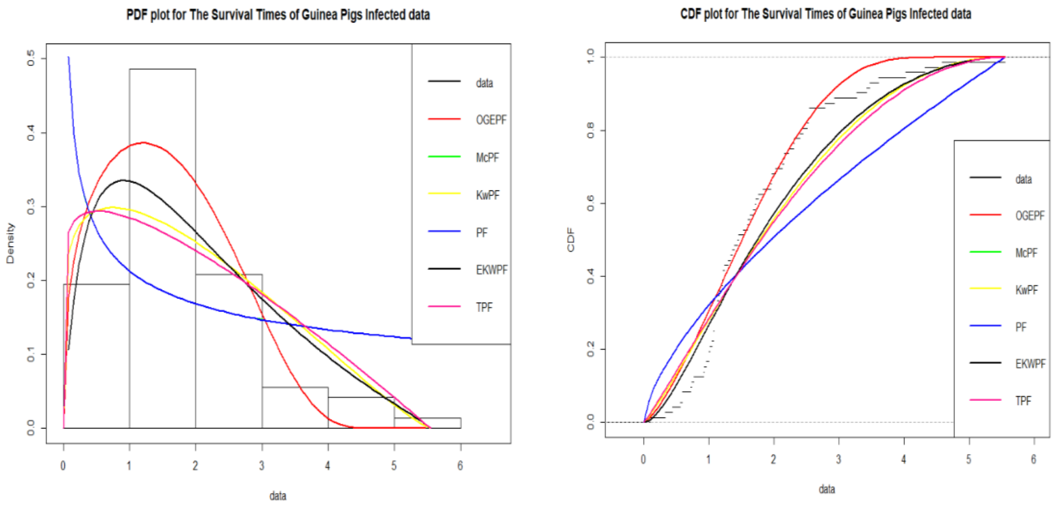


Figure 5*. Estimated densities and estimated distributions of models for the first data*

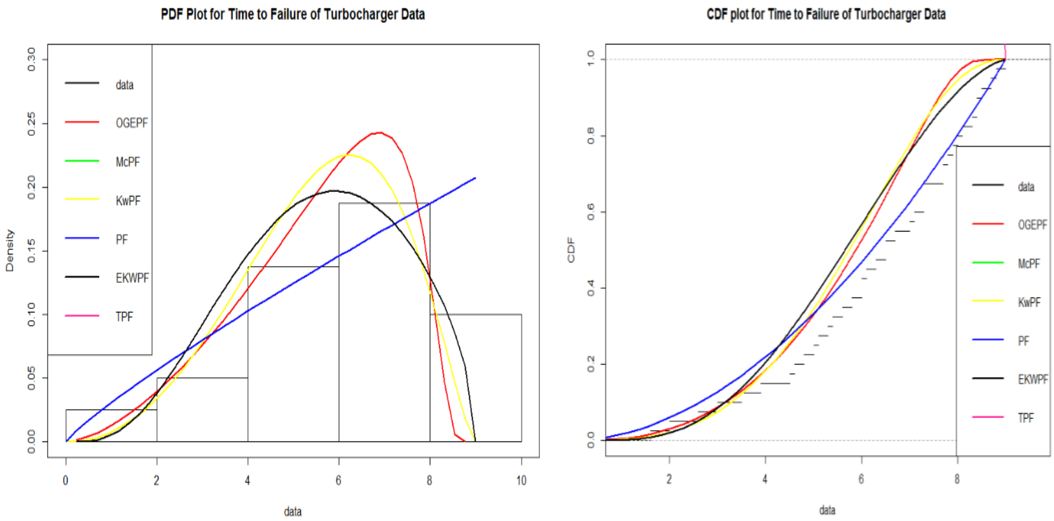


Figure 6*. Estimated densities and estimated distributions of models for the second data*

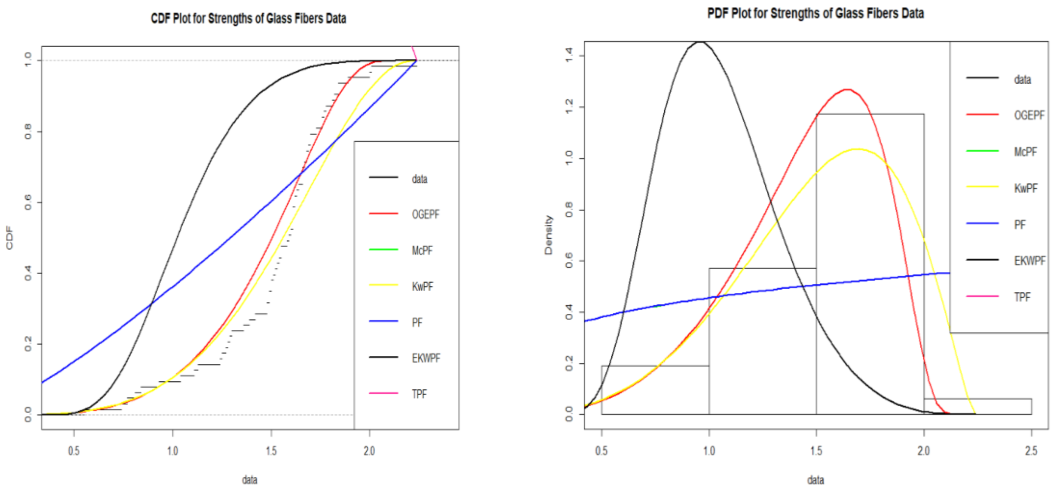


Figure 7*. Estimated densities and estimated distributions of models for the third data*

7. CONCLUDING REMARKS

In this paper, we introduce a new probability distribution called the odd generalized exponential power function distribution. The structural properties of this distribution are studied. The estimation of the model parameters is approached by maximum likelihood and percentiles methods. Simulation study is conducted in order to compare the performance of ML estimates with percentiles estimates for different sample sizes. It can be conclude that the behavior of the percentiles estimates is better than the corresponding ML estimates. An application of the OGEPF to three real data shows that the new distribution can be used quite effectively to provide better fits than, McPF, KwPF, EKwPF, TPF and PF distributions. We expect that the proposed model may be an interesting alternative model for a wider range of statistical research.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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