Uludağ University Journal of The Faculty of Engineering, Vol. 24, No.1, 2019 RESEARCH

DOI: 10.17482/uumfd.480661

DETERMINATION OF BUCKLING TEMPERATURES FOR ELLIPTICAL FGM PLATES WITH VARIABLE THICKNESSES

Işıl SANRI KARAPINAR *

Received: 08.11.2018; revised: 06.02.2019; accepted: 07.02.2019

Abstract: In this paper, the thermal buckling loads of elliptical thin plates made of functionally graded material (FGM) with thicknesses that vary parabolically are examined. The aim is to study the effect of parabolic thickness variations in the directions of both axes on the thermal buckling of FGM plates. In the analyses, the boundaries are assumed to be simply supported and clamped. Rayleigh-Ritz method is applied to solve the partial differential equations. The Poisson's ratios of the plates are kept constant, but their moduli of elasticity and thermal expansion coefficients are assumed to vary functionally in the thickness direction due to the material characteristics of FGMs. The study is carried out for several plate aspect ratios. Thermal buckling results of elliptical FGM plates with parabolically varying thicknesses are determined and the critical temperatures of those plates are obtained.

Keywords: Elliptical plate, Thermal buckling, Variable thickness, Temperature, FGM, Rayleigh-Ritz method.

Değişken Kalınlıklı Eliptik FGM Plakların Burkulma Sıcaklıklarının Belirlenmesi

Öz: Bu çalışmada, bir fonksiyona bağlı olarak değişen malzemeden (FGM) yapılmış, kalınlıkları parabolik olarak değişen, eliptik ince plakların termal burkulma yükleri incelenmiştir. Amaç, her iki eksendeki parabolik kalınlık değişimlerinin FGM plakların termal burkulması üzerindeki etkisini incelemektir. Analizlerde, sınır koşulları basit mesnetli ve ankastre olarak varsayılmaktadır. Kısmi diferansiyel denklemleri çözmek için Rayleigh-Ritz yöntemi kullanılmıştır. Plakların Poisson oranlarının sabit olduğu, ancak elastisite modülü ve ısı genleşme katsayılarının, FGM'lerin malzeme özellikleri nedeniyle kalınlık yönünde fonksiyonel olarak değiştiği varsayılır. Çalışma çeşitli plak en/boy oranları için gerçekleştirilmiştir. Parabolik olarak değişen kalınlıklara sahip eliptik FGM plaklarının termal burkulma sonuçları belirlenip, bu plakların kritik sıcaklıkları elde edilmiştir.

Anahtar Kelimeler: Eliptik plak, Termal burkulma, Değişken kalınlık, Sıcaklık, FGM, Rayleigh-Ritz yöntemi.

1. INTRODUCTION

Because of their extensive application areas, plates and plate-type structures have gained special importance and the problems related to their mechanical behavior have become more complex. Therefore, the demand for the solutions of these problems has increased. Thermal buckling of plates is one of these problems which has received widespread attention by many researchers. Since the investigations on the thermal effect on plates show that the traditional composite materials do not have high-temperature resistance property, the need for the use of composite materials which provide high-temperature resistance has arised.

* Maltepe Üniversitesi, Mühendislik ve Doğa Bilimleri Fakültesi, İnşaat Mühendisliği Bölümü, 34857, Maltepe, İstanbul

Correspondence Author: Işıl Sanrı Karapınar (isilkarapinar@maltepe.edu.tr)

Functionally graded materials (FGMs) are one of these composite materials that perform well in many engineering applications in high-temperature environments. FGMs have a smooth and continuous variation of material properties from one surface to another. Generally, they are composed of ceramic and metal or a combination of different materials. Since they are mainly designed to resist high-temperatures, their thermal behavior has been a big concern for researchers.

When FGMs are under high-temperatures, thermal loads induce instability. At this point, the temperature at which the structure becomes unstable is called buckling temperature. Exceeding this temperature causes big increments in deflections and stresses which result in the failure of the structure. As a consequence, in recent years most researchers have been studying on the buckling and post-buckling of FGM plates subjected to thermal loading. Among those studies, thermal buckling of simply supported FGM rectangular plates was investigated by using the classical and higher order shear deformation plate theories (Javaheri and Eslami, 2002). Buckling temperature of FGM plates with fully clamped edges was calculated by using solid finite elements (Na and Kim, 2004, 2006). The authors investigated the thermal buckling and post-buckling behaviors due to uniform and non-uniform temperature rise. Axisymmetric buckling of simply supported and clamped circular FGM plates was studied under a uniform temperature rise or a radial compression using the higher order shear deformation theory (Najafizadeh and Heydari, 2004, 2008). As a result, the buckling loads of a circular FGM plate obtained by the first-order shear deformation and classical plate theories were compared. Buckling analysis of simply supported FGM skew plates was carried out based on the first-order shear deformation theory in conjunction with the finite element approach (Ganapathi and Prakash, 2006). This study was extended and nonlinear post-buckling response of the same plates were investigated under different boundary conditions by using Mori-Tanaka scheme (Prakash et al., 2008). In another study, in order to understand the effects of geometric deficiencies on the buckling response of FGM plates, Galerkin method was employed (Tung and Duc, 2010). This work was continued with the postbuckling analysis of shear deformable FGM plates on elastic foundations and the consequences of geometric parameters and boundary conditions were also revealed (Duc and Tung, 2011). There are also few studies about thermal buckling analysis of FGM shells. In one of these studies, thermal buckling loads of functionally graded truncated conical shells were investigated by using Love's shell theory (Sofiyev, 2007). In this study, three types of thermal loading were applied on the simply supported truncated conical shells and the critical temperatures were determined.

Furthermore, uniform temperature increment and gradient through the thickness effects on the thermal buckling of simply supported FGM rectangular plates were studied by using the first order shear deformation plate theory (Wu, 2004). Thermal buckling of imperfect FGM rectangular plates was examined under uniform temperature rise, the non-linear temperature rise through the thickness and axial temperature rise (Shariyat and Eslami, 2005, 2006). The analyses were carried out based on the first order shear deformation and the classical thin plate theories. The results were compared with the results of perfect FGM and imperfect isotropic plates. Also, buckling analysis of rectangular thick FGM plates under mechanical and thermal loads was investigated (Shariyat and Eslami, 2007). The equilibrium and stability equations were determined using the third order shear deformation plate theory. Mechanical and thermal loads were applied on sandwich plates with FGM face sheets and post-buckling analysis was performed (Shen and Li, 2008). Moreover, for simply supported, midplane symmetric FGM plates under in-plane non-uniform parabolic temperature distribution, thermal post-buckling analysis, and heat conduction were examined (Shen, 2007). Moderately thick simply supported FGM plates were studied under a uniform temperature rise and a steady state temperature across the plate thickness by using the first order shear deformation theory (Lanhe, 2004).

It is observed that there is a very limited work about the thermal buckling of plates with variable thicknesses. However, there are quite a number of researches about the vibration analysis of tapered plates. Recently, free vibration analysis was performed for some composite doubly-curved shells, singly-curved shells, and plates having a continuous thickness change (Bacchiochi et al., 2016). Also, natural frequencies of several doubly-curved shells with variable thickness were evaluated by using Generalized Differential Quadrature method (Tornabene et al., 2016).

Moreover, an elastic buckling analysis of linearly and parabolically tapered circular plates was presented (Wang et al., 1995). In another study, the power series method was developed in order to solve the buckling problem of uniaxially compressed rectangular plates with a linearly tapered thickness (Kobayashi and Sonoda, 1990). An analog equation solution to plates with variable thicknesses for the buckling analysis was applied and numerical results were given to illustrate the effectiveness of the proposed method (Nerantzaki and Katsikadelis, 1996).

In the present study, the buckling load which is highly dependent on the thickness variations and the thermal buckling behavior of elliptical FGM plates with variable thicknesses are analyzed. In the analysis, the boundaries are considered as simply supported and clamped. Rayleigh-Ritz method is used to solve the partial differential equations. The study examines the effects of the parabolic thickness variations, volume fraction indices, aspect ratios, and boundary conditions on the thermal buckling behavior of FGM plates and presents the obtained buckling temperatures which can lead to a failure in the plates.

2. METHODOLOGY

2.1. Material Properties of FGMs

FGMs are composed of two or more constituent phases of materials in which not only volume fraction distribution of constituent materials from one to another changes gradually, but also the thermal and mechanical properties change consequently. Most of the FGM plates are generally made of a controlled mixture of ceramics and metals. Since the ceramic constituent of the material has a low thermal conductivity, it provides high-temperature resistance. In addition to that, the ductile metal constituent prevents failure caused by stresses due to the high-temperature gradient in a very short period of time. In the present study, the material in the upper half of the plate ($0 \leq z \leq t/2$) is ceramic rich, whereas that in the lower half ($-t/2 \leq z \leq 0$) is metal-rich as shown in Fig. 1. For the validity of the classical thin plate theory, the transverse deflections are assumed to be small compared to plate dimensions.

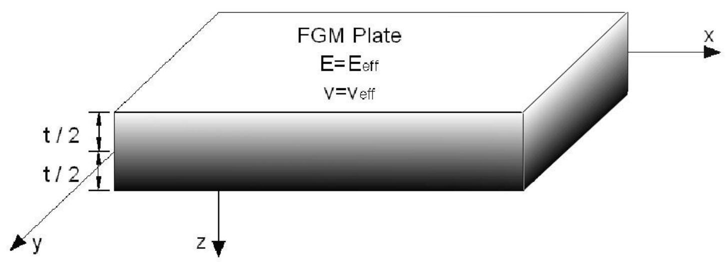


Figure 1:
Representative grading of an FGM plate in the thickness direction z

Since FGM plates are composed of more than one material, the plate behavior is governed by the effective material properties of the mixture. Through-the-thickness composition of the material is assumed to be governed by a volume fraction rule. The volume fractions of ceramic, V_c, and metal, V_m, are expressed as follows:

$$V_c(z) = \left(\frac{2z+t}{2t}\right)^n ; \qquad V_c = \int_{-t/2}^{t/2} V_c(z)\,dz \tag{1}$$

$$V_m = 1 - V_c$$

where z is the thickness coordinate variable and $-t/2 \le z \le t/2$, t is the plate thickness and n is the power law index ($0 \le n \le \infty$). When n=1, the change in the composition of ceramic and metal is linear and as it is seen from Eqs. (1), the value of $n=0$ represents a fully ceramic plate. Also, for high values of n, the dominant constituent material first exhibits changes in small increments and then changes rapidly on the opposite side. On the other hand, for low values of n, the material properties change rapidly near the surface. The variations of the volume fraction of ceramic through the thickness are illustrated in Fig. 2.

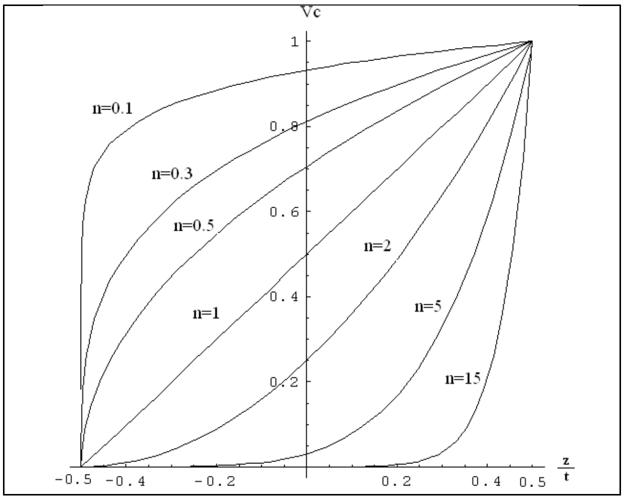


Figure 2:
Ceramic's volume fraction distribution through the thickness

It is accepted that the non-homogenous material properties such as modulus of elasticity, E, and coefficient of thermal expansion, α, change in the thickness direction, z, whereas Poisson's ratio, ν, is constant. Their expressions are

$$\begin{aligned} E_{eff} &= E_c V_c + E_m (1 - V_c) \\ \alpha_{eff} &= \alpha_c V_c + \alpha_m (1 - V_c) \\ \nu_{eff} &= \nu_0 \end{aligned} \tag{2}$$

where subscripts *m* and *c* represent the metal and ceramic constituents, respectively. When Eq. (1) is substituted into Eq. (2), the material properties of the FGM plate are determined. Thus,

$$\begin{aligned} E(z) &= (E_c - E_m)\left(\frac{2z+t}{2t}\right)^n + E_m \\ \alpha(z) &= (\alpha_c - \alpha_m)\left(\frac{2z+t}{2t}\right)^n + \alpha_m \\ \nu(z) &= \nu_0 \end{aligned} \tag{3}$$

The effective properties obtained by using Eq. (4) are position dependent in the thickness direction. The effective bending rigidity, D_{eff}, of the plate can be obtained by integration over the thickness as

$$D_{eff} = \int_{-t/2}^{t/2} \frac{E_{eff}(z)}{1-\nu_{eff}{}^{2}(z)}\, z^2\, dz \tag{4}$$

where ν_{eff} is the effective Poisson's ratio of the FGM plate. In many studies on FGM, Poisson's ratio effect on the deformations is minor compared to that of the modulus of elasticity as given in Delale and Erdoğan (1983). Therefore, Poisson's ratio is kept constant and modulus of elasticity is assumed to vary in the direction of thickness.

2.2. Constitutive Relations

The stress-strain relationship for the plates subjected to temperature changes is expressed as

$$\begin{aligned} \sigma_x &= \frac{E_{eff}}{1-\nu_{eff}{}^2}\left[\varepsilon_x + \nu_{eff}\varepsilon_y - (1+\nu_{eff})\alpha_{eff}\left(\Delta T\right)\right] \\ \sigma_y &= \frac{E_{eff}}{1-\nu_{eff}{}^2}\left[\varepsilon_y + \nu_{eff}\varepsilon_x - (1+\nu_{eff})\alpha_{eff}\left(\Delta T\right)\right] \\ \lambda_{xy} &= G\gamma_{xy} = \frac{E_{eff}}{2(1+\nu_{eff})}\gamma_{xy} \end{aligned} \tag{5}$$

where G is the shear modulus and ΔT is the temperature change. When the stress resultants are given in terms of displacements, by the help of the well-known strain-displacement relations, Eq. (6) can be determined.

$$\begin{aligned} N_x &= \frac{E_{eff}t}{(1-\nu_{eff}{}^2)}\left(\frac{\partial u}{\partial x} + \nu_{eff}\frac{\partial v}{\partial y}\right) - \frac{N^*}{1-\nu_{eff}} & M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu_{eff}\frac{\partial^2 w}{\partial y^2}\right) - \frac{M^*}{1-\nu_{eff}} \\ N_y &= \frac{E_{eff}t}{(1-\nu_{eff}{}^2)}\left(\frac{\partial v}{\partial y} + \nu_{eff}\frac{\partial u}{\partial x}\right) - \frac{N^*}{1-\nu_{eff}} & M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu_{eff}\frac{\partial^2 w}{\partial x^2}\right) - \frac{M^*}{1-\nu_{eff}} \\ N_{xy} &= \frac{E_{eff}t}{2(1+\nu_{eff})}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & M_{xy} &= -D(1-\nu_{eff})\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{6}$$

In Eqs. (6), N* and M* are the thermal stress resultants which can be expressed as

$$\begin{aligned} N^* &= \alpha_{eff}E_{eff}\int_{-t/2}^{t/2} \Delta T\, dz \\ M^* &= \alpha_{eff}E_{eff}\int_{-t/2}^{t/2} \Delta T\, z\, dz \end{aligned} \tag{7}$$

and the flexural rigidity, D, of the plate per unit length is

$$D = \frac{E_{eff} t^3}{12(1-\nu_{eff}{}^2)} \tag{8}$$

2.3. Solution by Rayleigh-Ritz Method

The total energy function, F, of a system can be given as

$$F = U - \gamma_1 T + \gamma_2 V \tag{9}$$

where strain energy, kinetic energy, and potential energy are represented by U, T and V, respectively. The coefficients, γ_1 and γ_2 are chosen according to the type of the plate problem. For a vibration problem, γ_1=1 and γ_2=0 whereas for a buckling problem, γ_1=0 and γ_2=1. U, T, and V can be expressed as

$$\begin{aligned} U &= \iint_A \frac{D}{2}\left\{\left(\nabla^2 w\right)^2 + 2(1-\nu_{eff})\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dx\, dy \\ T &= \frac{1}{2}\rho t w^2 \iint_A w^2 dx\, dy \\ V &= -\frac{1}{2}\iint_A N\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dx\, dy \end{aligned} \tag{10}$$

where ρ is the mass density, N is the in-plane normal force per unit length and ∇^2 is the second order Laplace operator which is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{11}$$

In the solution of a plate buckling problem, after the substitution of the strain and the potential energy expressions, the total energy expression takes the following form:

$$F = \iint_A \frac{D}{2}\left\{\left(\nabla^2 w\right)^2 + 2(1-\nu_{eff})\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dx\, dy - \frac{1}{2}\iint_A \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy \tag{12}$$

In applying the Rayleigh-Ritz method, which is based on the energy principle, first, an appropriate deflection shape must be assumed for the system. In a plate problem, the deflected middle surface may be presented as series:

$$w(x,y) = \alpha_1 f_1(x,y) + \alpha_2 f_2(x,y) + \ldots + \alpha_n f_n(x,y) = \sum_{i=1}^{n} \alpha_i f_i(x,y) \tag{13}$$

where $f_i(x,y)$ are continuous functions which satisfy at least the geometrical boundary conditions and represents the deflected plate surface. By using the minimum potential energy principle, the unknown constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, are obtained. With this minimization procedure, n simultaneous algebraic equations will be determined in terms of the n unknown coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$.

2.4. Plates with Variable Thicknesses

In many engineering applications where the weight reduction of the structure is exceptionally important, the use of plates with variable thickness can be one of the best choices.

In a previous study which is done by Bayer et al. (2002), the plate thickness variation is considered as the following relation:

$$t(x,y) = c\,t_0\left[\alpha \mp \beta(x^2+y^2)\right] \tag{14}$$

where t_0 is the thickness of a plate with a constant thickness, α is a parameter defining the constant part of the thickness, β is the taper parameter controlling the thickness variation and c is a parameter controlling the volume of the plate and defined as

$$c = \frac{2}{2\alpha \mp \beta} \tag{15}$$

It should be noted that plus and minus signs in Eqs. (14) and (15) denote the clamped and simply supported cases, respectively. On account of thickness variety, since the flexural rigidity also changes, it should be rewritten as:

$$D(x,y) = D_0 c^3 H \tag{16}$$

where D_0 is the flexural rigidity of the plate with constant thickness and H is the function representing the variable plate thickness which are defined as follows:

$$D_0 = E{t_0}^3 / \left[12(1-\nu^2)\right] \tag{17}$$

$$H = \left[\alpha + \beta(x^2+y^2)\right]^3 \tag{18}$$

2.5. Basic Assumptions and Equations

Since the problem in hand considers elliptical boundaries, to solve the problem by Rayleigh-Ritz method, firstly, continuous functions representing the plate deflections are selected as follows:

$$w(x,y) = \sum_i^r \sum_j^r \alpha_{ij}\phi(x,y)\,x^i y^j \tag{19}$$

where *r* is the degree of the polynomial trial function with a constraint of $i+j \le r$ and α_{ij} are the coefficients to be determined. In this study, in order to obtain accurate results, an appropriate shape function at the order of *r*=4 is selected. In Eq. (19), $\phi(x,y)$ is the boundary shape equation which is defined for an elliptical plate as follows:

$$\phi(x,y) = \left[\frac{x^2}{(a/2)^2} + \frac{y^2}{(b/2)^2} - 1\right]^{\Omega} \tag{20}$$

In the equation, $\phi(x,y)$ assures that boundary conditions given in the problem are satisfied by each element of these trial functions. In order to obtain an even function for the deflection function of the chosen system, the trial function's elements having odd powers of x and y are cancelled. It should be noted that the selection of the trial functions is very important to have better approximations and consume time.

If the edges are simply supported, then in Eq. (20), Ω=1. If the edges are clamped, then, Ω=2. In this investigation, the plate thickness is expected to differ parabolically. Thus, by using Eqs. (15), (16) and (18), the total energy expression for the buckling problem of an elliptical plate takes the following form:

$$F=\iint_A \frac{D}{2}\left\{c^3\left(\alpha+\beta(x^2/a^2+y^2/b^2)\right)^3\left(\nabla^2 w\right)^2+2(1-\nu_{eff})\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}dx\,dy-\frac{1}{2}\iint_A\left[N_x\left(\frac{\partial w}{\partial x}\right)^2+N_y\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy \tag{21}$$

During the solution of the differential equations, there are difficulties because of the integration over the elliptical area. Therefore, the integration over the elliptical region should be taken in accordance with the elliptical boundary equation which defines the ranges of the variables as

$$\begin{aligned} x&=\mp a/2\\ y&=\mp(b/2)\sqrt{1-(x^2/(a^2/4)} \end{aligned} \tag{22}$$

In the application of the Rayleigh-Ritz method, the thickness variation should be taken into account. This effect introduces the parameters α and β. During the numerical applications one dimension of the plate is kept constant (a=1). The boundary conditions are assumed as fully clamped and simply supported.

For the solution of FGM plates, effective material properties are obtained by using Eqs. (1) and (2) in accordance with the variation of the power law index, n. Again, it is assumed that plate layers are composed of ceramics and metals with the mixing ratio changing continuously in the thickness direction. The effective modulus of elasticity and the thermal expansion coefficient of the FGM layers are temperature-dependent, whereas Poisson's ratio depends weakly on the temperature change. Therefore, in this study, it is assumed to be constant and taken as 0.3. As a result of the analyses, the thermal buckling load factor, λ, changes due to the change in the material properties and the variation of the α and β parameters. According to this, λ is calculated for the different values of a/b ratios and used in the calculation of the buckling temperature of the elliptical FGM plate with variable thickness. After having found the buckling factors, buckling temperatures are calculated by substituting Eq. (7) into Eq. (6) where ΔT denotes the buckling temperature, T_{cr}. Hence, the formulation for T_{cr} can be written as;

$$N=\frac{E_{eff}\,t}{(1-\nu_{eff}{}^2)}\left(\frac{\partial u}{\partial x}+\nu_{eff}\frac{\partial v}{\partial y}\right)-\frac{\alpha_{eff}E_{eff}\int_{-t/2}^{t/2}T_{cr}\,dz}{1-\nu_{eff}} \tag{23}$$

3. NUMERICAL RESULTS AND DISCUSSION

The analyses for the buckling temperatures of elliptical FGM plates with parabolic thickness variation start with the calculation of the buckling parameters. Before the thermal analyses, the buckling loads of homogenous, isotropic elliptical plates are obtained by using the Rayleigh-Ritz method with clamped and simply supported boundaries. The results are compared in Table 1with those obtained in the study done by Wang et al. (1994). So, for the comparison of the results, the same aspect ratios are used and all values are normalized by the parameter $\lambda = Na^2/D$.

Table 1. Comparison of the buckling load factors of homogenous elliptical plates

Buckling Load Factors for Simply Supported Elliptical Plates (λ=Na²/D)								
a/b	1	1.125	1.25	1.375	1.5	2	2.5	3
Wang et al. (1994)	16.78	19.13	21.99	25.36	29.19	48.76	74.15	104.65
Present study	16.79	19.15	22.02	25.39	29.23	48.82	74.25	104.8
Buckling Load Factors for Clamped Elliptical Plates (λ=Na²/D)								
a/b	1	1.125	1.25	1.375	1.5	2	2.5	3
Wang et al. (1994)	58.67	66.85	76.68	88.1	101.06	166.84	253.06	358.67
Present study	58.73	66.91	76.74	88.17	101.13	166.94	253.18	358.87

Table 2. Buckling parameters for the simply supported elliptical plate with variable thickness

		a/b									
		1	1.11	1.25	1.43	1.67	2	2.5	3.33	5	10
α	β	λ=Na²/D									
2.0	0.0	16.79	18.86	22.02	26.98	35.04	48.82	74.26	127.91	276.27	1060.45
2.0	0.2	19.11	21.47	25.07	30.72	39.92	55.67	84.80	146.37	316.92	1219.21
2.0	0.6	25.40	28.54	33.33	40.86	53.14	74.22	113.37	196.52	427.66	1652.79
2.0	1.0	35.16	39.50	46.14	56.60	73.67	103.06	157.86	274.81	601.12	2333.88
1.8	0.0	16.79	18.86	22.02	26.98	35.04	48.82	74.26	127.91	276.27	1060.45
1.8	0.2	19.40	21.79	25.45	31.18	40.52	56.51	86.10	148.64	321.94	1238.86
1.8	0.6	26.73	30.03	35.08	43.01	55.94	78.16	119.43	207.17	451.23	1745.22
1.8	1.0	38.82	43.61	50.95	62.50	81.37	113.89	174.58	304.26	666.52	2591.14
1.4	0.0	16.79	18.86	22.02	26.98	35.04	48.82	74.26	127.91	276.27	1060.45
1.4	0.2	20.25	22.75	26.57	32.56	42.32	59.03	89.97	155.44	336.93	1297.48
1.4	0.6	31.14	34.98	40.86	50.11	65.21	91.18	139.52	242.50	529.47	2052.30
1.4	1.0	52.82	59.35	69.35	85.09	110.86	155.35	238.65	417.36	918.10	3582.57
1.2	0.0	16.79	18.86	22.02	26.98	35.04	48.82	74.26	127.91	276.27	1060.45

Table 2. Buckling parameters for the simply supported elliptical plate with variable thickness (continued)

		a/b									
		1	1.11	1.25	1.43	1.67	2	2.5	3.33	5	10
α	β	λ=Na²/D									
1.2	0.2	20.93	23.52	27.46	33.65	43.74	61.03	93.05	160.83	348.83	1344.01
1.2	0.6	35.16	39.50	46.14	56.60	73.67	103.06	157.86	274.81	601.12	2333.88
1.2	1.0	68.51	76.98	89.95	110.41	143.91	201.84	310.56	544.50	1201.63	4702.40
1.0	0.0	16.79	18.86	22.02	26.98	35.04	48.82	74.26	127.91	276.27	1060.45
1.0	0.2	21.94	24.64	28.78	35.27	45.85	63.99	97.61	168.84	366.49	1413.14
1.0	0.6	42.15	47.36	55.32	67.87	88.38	123.75	189.80	331.11	726.18	2826.02
1.0	1.0	104.00	116.87	136.59	167.70	218.73	307.15	473.61	833.32	1847.41	7259.47

Table 3. Buckling parameters for the clamped elliptical plate with variable thickness

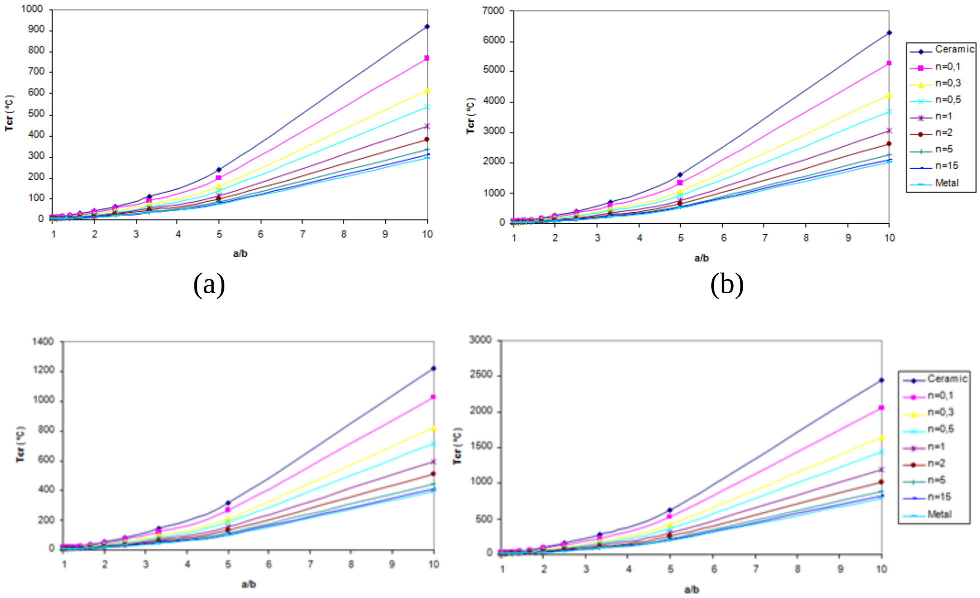
		a/b									
		1	1.11	1.25	1.43	1.67	2	2.5	3.33	5	10
α	β	λ=Na²/D									
1.0	0.0	58.73	65.92	76.74	93.55	120.67	166.98	253.47	442.55	995.53	4047.11
1.0	0.2	46.99	52.74	61.38	74.78	96.39	133.24	202.07	352.78	794.19	3228.95
1.0	0.6	32.04	35.96	41.83	50.92	65.53	90.42	136.96	239.20	539.27	2192.66
1.0	1.0	23.28	26.12	30.37	36.93	47.48	65.43	99.04	173.11	390.76	1588.66
0.8	0.0	58.73	65.92	76.74	93.55	120.67	166.98	253.47	442.55	995.53	4047.11
0.8	0.2	44.60	50.06	58.26	70.97	91.46	126.39	191.64	334.59	753.38	3063.11
0.8	0.6	28.24	31.69	36.86	44.85	57.70	79.58	120.49	210.50	474.80	1930.48
0.8	1.0	19.53	21.91	25.48	30.97	39.79	54.80	82.93	145.05	327.64	1331.88
0.4	0.0	58.73	65.92	76.74	93.55	120.67	166.98	253.47	442.55	995.53	4047.11
0.4	0.2	35.02	39.30	45.73	55.67	71.68	98.95	149.90	261.77	589.96	2398.76
0.4	0.6	16.64	18.67	21.70	26.37	33.86	46.62	70.55	123.48	279.07	1134.26
0.4	1.0	9.84	11.04	12.82	15.57	19.97	27.47	41.61	73.01	165.29	671.29
0.2	0.0	58.73	65.92	76.74	93.55	120.67	166.98	253.47	442.55	995.53	4047.11
0.2	0.2	23.28	26.12	30.37	36.93	47.48	65.43	99.04	173.11	390.76	1588.66
0.2	0.6	7.97	8.94	10.38	12.60	16.15	22.22	33.69	59.19	134.06	544.24
0.2	1.0	4.24	4.76	5.52	6.70	8.60	11.84	18.01	31.75	71.98	291.81

In the analysis of FGM plates, the volume fractions of ceramic, V_c, and metal, V_m, effective moduli of elasticity, E_{eff}, and effective thermal expansion coefficients, α_{eff}, are obtained in accordance with the variation of the power law index, n. Seven arbitrary values of the power law index, n= 0.1, 0.3, 0.5, 1, 2, 5, 15 are considered in the calculations. The case n=0 represents a fully ceramic plate and n=1 represents the linear variation of the composition of ceramics and metals. During the numerical applications, modulus of elasticity and coefficient of thermal expansion for metal are taken as E_m=70 GPa and α_m=23x10^{-6} 1/ºC and for the ceramic E_c=380 GPa and α_c=7.4x10^{-6} 1/ºC, respectively. Poisson's ratio is assumed to be 0.3. The results of these calculations are presented in Table 4.

Table 4. Effective material properties

n	V_c	V_m	E_{eff} *(GPa)*	α_{eff} *(1/ºC)*
0.1	0.909	0.091	351.79	8.82E-06
0.3	0.769	0.231	308.39	1.1E-05
0.5	0.667	0.333	276.77	1.26E-05
1	0.5	0.5	225	1.52E-05
2	0.333	0.667	173.23	1.78E-05
5	0.167	0.833	121.77	2.04E-05
15	0.062	0.938	89.22	2.2E-05

After obtaining the effective values of moduli of elasticity and the thermal expansion coefficients, it is possible to achieve the buckling load factors for elliptical FGM plates. Since the effect of the parabolic thickness variation is taken into account, the buckling load factors for different thickness variations and a/b ratios are calculated. In the calculations, the controlling parameter of the constant part of the thickness, α, is taken as 1 while the controlling parameter of the thickness variation, β, changes. When the buckling load factors for all cases are found for the selected α and β values, the buckling temperatures of elliptical FGM plates with variable thicknesses can be obtained. The critical temperature results under the uniform temperature rise for the simply supported and clamped elliptical FGM plates with variable thicknesses are presented in Figs. 3 and Figs. 4, respectively.



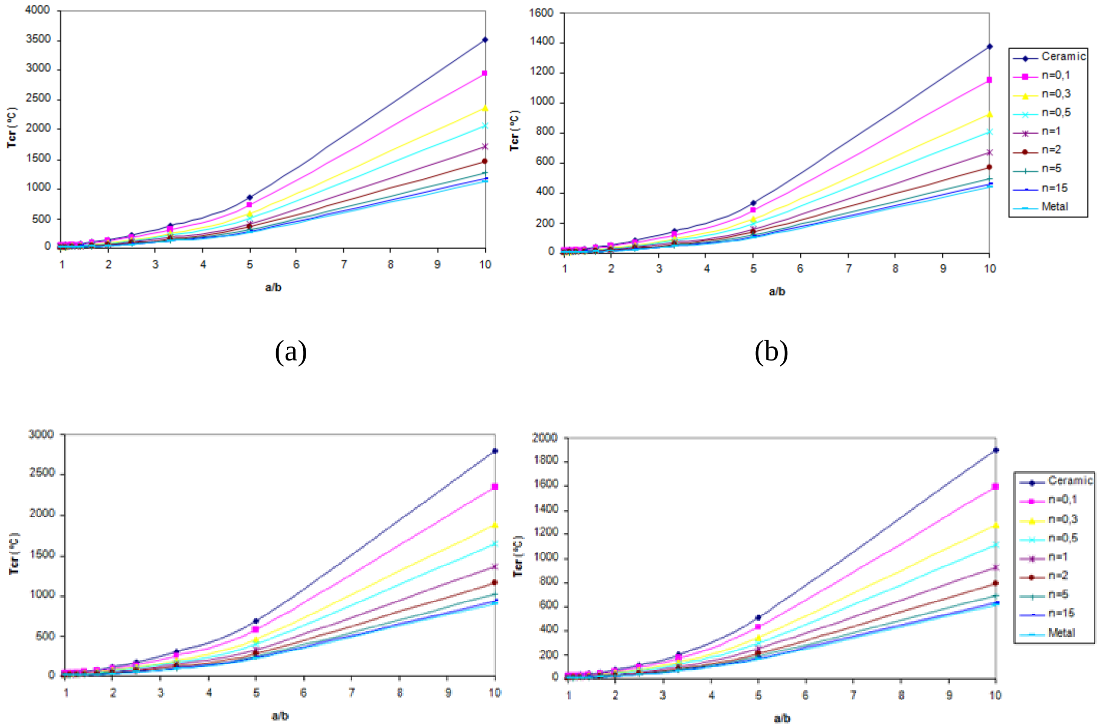
(a) (b)

(c) (d)

Figure 3:

Buckling temperatures of simply supported elliptical FGM plates

(a) *α=1, β=0,* ***(b)*** *α=1, β=1,* ***(c)*** *α=1, β=0.2,* ***(d)*** *α=1, β=0.6*



(a) (b)

(c) (d)

Figure 4:
Buckling temperatures of clamped elliptical FGM plates
(a) *for α=1, β=0,* ***(b)*** *for α=1, β=1, ,* ***(c)*** *α=1, β=0.2,* ***(d)*** *α=1, β=0.6*

4. CONCLUSIONS

In this study, thermal buckling analyses of elliptical FGM plates with variable thicknesses are studied. First, buckling analyses for the homogenous elliptical plates are employed. The plates are assumed as simply supported and fully clamped. The results in the literature for the same kind of plates are compared with the present study. In the studies encountered in the literature, the buckling load factors are defined in different ways. For the comparison, the results are converted into the same buckling load factor definition which is $\lambda=Na^2/D$. Buckling factors for elliptical plates with a wide range of aspect ratios are computed by using the Rayleigh-Ritz method. These buckling results for simply supported and clamped plates are given in Tables 1-2. In comparison, it is seen that very accurate buckling load factors with rapid convergence are obtained. The difference between the results of the present investigation and those acquired in the study done by Wang et al. (1994) is less than 1%.

After obtaining the buckling load factors for the homogenous plates, thermal buckling analyses of FGM tapered plates are employed. In the case of the parabolic variation of the plate thicknesses, for all the boundary conditions, controlling parameter of the constant part of the plate is kept constant (α=1) and the controlling parameter of the thickness variation is changed between 0 and 1 ($0\leq\beta\leq1$). It should be noted that the results of α=1, β=0 represents the plates with a constant thickness. While the controlling parameter of the thickness variation, β decreases, the buckling load factors and as a result of this, buckling temperatures decrease as well. The buckling temperatures for the elliptical plates increase by the increase of *a/b* and decrease by the increase of the power law index, *n*. It is also observed that for the higher values of *n*, there is no significant difference in the temperatures for the FGM plates with variable thicknesses. In the case of parabolic variation of the plate thickness, no matter what the boundary conditions are, the buckling load factor increases according to the constant thickness values.

REFERENCES

1. Bacciocchi, M., Eisenberger, M., Fantuzzi, N., Tornabene, F. and Viola, E. (2016) Vibration analysis of variable thickness plates and shells by the Generalized Differential Quadrature method, *Composite Structures*, 156, 218-237. doi: 10.1016/j.compstruct.2015.12.004
2. Delale, F. and Erdoğan, F. (1983) The crack problem for a non-homogenous plane, *ASME Journal of Applied Mechanics*, 50, 609-614. doi: 0.1115/1.3167098
3. Duc, N. D. and Van Tung H. (2011) Mechanical and thermal postbuckling of higher order shear deformable functionally graded plates on elastic foundations, *Composite Structures*, 93, 2874-2881. doi: 10.1016/j.compstruct.2011.05.017
4. Ganapathi, M. and Prakash, T. (2006) Thermal buckling of simply supported functionally graded skew plates, *Composite Structures,* 74, 247-250. doi: 10.1016/j.compstruct.2005.04.004
5. Javaheri, R. and Eslami, M. R. (2002) Thermal buckling of functionally graded plates based on higher order theory, *Journal of Thermal Stresses*, 25, 603-625. doi: 10.1080/01495730290074333
6. Kobayashi, H. and Sonada, K. (1990) Buckling of rectangular plates with tapered thickness, *Journal of Structural Engineering*, 116, 1278-1289. doi: 10.1061/(ASCE)0733-9445(1990)116:5(1278)
7. Lanhe, W. (2004) Thermal buckling of a simply supported moderately thick rectangular FGM plate, *Composite Structures*, 64, 211-218. doi: 10.1016/j.compstruct.2003.08.004
8. Na, K. S. and Kim, J. H. (2004) Three dimensional thermal buckling analysis of functionally graded materials, *Composites Part B*, 35, 429-437. doi: 10.1016/j.compositesb.2003.11.013
9. Na, K. S. and Kim, J. H. (2006) Thermal postbuckling investigations of functionally graded plates using 3-D finite element method, *Finite Elements in Analysis and Design*, 42, 749-756. doi: 10.1016/j.finel.2005.11.005
10. Najafizadeh, M. M. and Heydari, H. R. (2004) Thermal buckling of functionally graded circular plates based on higher order shear deformation plate theory, *European Journal of Mechanics A/Solids*, 23, 1085-1100. doi: 10.1016/j.euromechsol.2004.08.004
11. Najafizadeh, M. M. and Heydari, H. R. (2008) An exact solution for buckling of functionally graded circular plates based on higher order shear deformation plate theory under uniform radial compression, *International Journal of Mechanical Sciences*, 50, 603-612. doi: 10.1016/j.ijmecsci.2007.07.010
12. Nerantzaki, M. S. and Katsikadelis, J. T. (1996) Buckling of plates with variable thickness-an analog equation solution. *Engineering Analysis with Boundary Elements*, 18, 149-154. doi: 10.1016/S0955-7997(96)00045-8
13. Prakash, T., Singha, M. K. and Ganapathi, M. (2008). Thermal post-buckling analysis of FGM skew plates. *Engineering Structures*, 30, 22-32. doi: 10.1016/j.engstruct.2007.02.012
14. Shariyat, B. A. S. and Eslami, M. R. (2005). Effect of initial imperfections on thermal buckling of functionally graded plates, *Journal of Thermal Stresses*, 28, 1183-1198. doi: 10.1080/014957390967884

15. Shariyat, B. A. S. and Eslami, M. R. (2006) Thermal buckling of imperfect functionally graded plates, *International Journal of Solids and Structures*, 43, 4082-4096. doi: 10.1016/j.ijsolstr.2005.04.005

16. Shariyat, B. A. S. and Eslami, M. R. (2007) Buckling of thick functionally graded plates under mechanical and thermal loads, *Composite Structures,* 78, 433-439. doi: 10.1016/j.compstruct.2005.11.001

17. Shen, H. S. (2007) Thermal post-buckling behavior of shear deformable FGM plates with temperature-dependent properties, *International Journal of Mechanical Sciences*, 49, 466-478. doi: 10.1016/j.ijmecsci.2006.09.011

18. Shen, H. S. and Li, S. R. (2008) Postbuckling of sandwich plates with FGM face sheets and temperature dependent properties, *Composites Part B-Engineering*, 39, 332-344. doi: 10.1016/j.compositesb.2007.01.004

19. Sofiyev, A. H. (2007) Thermoelastic stability of functionally graded truncated conical shells, *Composite Structures,* 77, 56-65. doi: 10.1016/j.compstruct.2005.06.004

20. Tornabene, F. (2009) Free vibration analysis of functionally graded conical, cylindrical shell and annular plate structures with a four-parameter power-law distribution, *Computer Methods in Applied Mechanics and Engineering*, 198, 2911-2935. doi: 10.1016/j.cma.2009.04.011

21. Tornabene, F., Fantuzzi, N. and Bacciocchi, M. (2016) The local GDQ method for the natural frequencies of doubly-curved shells with variable thickness: A general formulation, *Composites Part B Engineering*, 92, 265-289. doi: 10.1016/j.compositesb.2016.02.010

22. Van Tung, H. and Duc N. D. (2010) Nonlinear analysis of stability for functionally graded plates under mechanical and thermal loads, *Composite Structures*, 92, 1184-1191. doi: 10.1016/j.compstruct.2009.10.015

23. Wang, C. M., Wang, L. and Liew, K. M. (1994) Vibration and buckling of super elliptical plates, *Journal of Sound and Vibration*, 171, 301-314. doi: 10.1006/jsvi.1994.1122

24. Wang, C. M., Hong, G. M. and Tan, T. J. (1995) Elastic buckling of tapered circular plates, *Computers and Structures*, 55, 1055-1061. doi: 10.1016/0045-7949(94)00508-Z

25. Wu, L. (2004) Thermal buckling of simply supported moderately thick rectangular FGM plate, *Composite Structures*, 64, 211-218. doi: 10.1016/j.compstruct.2003.08.004