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# EQUILIBRIUM AND STABILITY ANALYSIS OF TAKAGI-SUGENO FUZZY DELAYED COHEN-GROSSBERG NEURAL NETWORKS

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Abstract. This paper carries out an investigation into the problem of the global asymptotic stability of the class of Takagi-Sugeno (T-S) fuzzy delayed Cohen-Grossberg neural networks involving discrete time delays and employing the nondecreasing and slope-bounded activation functions. A new sufficient criterion for the uniqueness and global asymptotic stability of the equilibrium point for this class of fuzzy neural networks is proposed. The uniqueness of the equilibrium point is proved by using the contradiction method, and the stability of the equilibrium point is established by utilizing a novel fuzzy type Lyapunov functional. The obtained stability condition is independent of the time delay parameters and, it can be easily verified by exploiting some commonly used norm properties of matrices. A constructive numerical example is also given to demonstrate the applicability of the proposed stability condition.

## 1. Introduction

Stability and equilibrium properties of Cohen-Grossberg neural network model proposed by Cohen and Grossberg in [1] have been extensively studied due to their potential applications in a variety of fields such as pattern recognition, parallel computation, associative memory design, signal and image processing and optimization. Such types of applications require that the neural network employed for solving these specific problems must possess a unique and globally asymptotically stable equilibrium point. On the other hand, time delays unavoidable exist in the mathematical model of neural networks due to many different reasons. For instance, the finite switching speed of amplifiers in neural systems may cause time delays. The existence of time delays may change the dynamics of the system and cause undesired complex dynamical behaviors. Therefore, it is of crucial importance to consider the effects of time delays when analyzing the stability of neural networks.

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In the recent years, a variety of sufficient conditions for the global asymptotic stability of delayed Cohen-Grossberg neural networks have been proposed [2]-[16].

Fuzzy logic theory has been effectively adopted for modelling the various classes of nonlinear systems to provide a more efficient tool with stability analysis of these systems. In particular, fuzzy systems in the form of the Takagiĭ-Sugeno (T-S) model [17] have attracted rapidly growing attention in recent years. A T-S fuzzy system is a class of nonlinear systems defined by a set of IF-THEN rules [36]. It has been shown that the T-S model method can provide an effective way with representing complex nonlinear systems by using simple local linear dynamical systems with their linguistic description. Some classes of nonlinear dynamical systems can be approximated by the overall fuzzy linear T-S models for the purpose of stability analysis [18]-[19]. In [20], a sufficient condition for the stability of the T-S fuzzy systems has been proposed by constructing a suitable Lyapunov functional. The methods and techniques used in [19] have been an inspiration for many researchers to extend the T-S fuzzy models to describe different classes of delayed neural networks. Some original and useful results for global stability of various classes of T-S fuzzy delayed neural networks can be found in [21]-[38].

This paper will study the equilibrium and stability properties of the class of T-S fuzzy Cohen-Grossberg neural networks with discrete time delays. First, by using the contradiction method, the condition ensuring the uniqueness of the equilibrium point for this class of neural networks is established. Then, by constructing a suitable fuzzy Lyapunov functional, it will be shown that the condition proposed for the uniqueness of the equilibrium point also implies the global asymptotic stability of the equilibrium point.

## 2. System Description and Preliminaries

Consider the following general Cohen-Grossberg neural network model with discrete time delays:

$$\dot{x}_i(t) = d_i(x_i(t))[-c_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + u_i],$$

where $n$ is the number of the neurons in the network, $x_i$ denotes the state of the $i$th neuron, $d_i(x_i)$ represents an amplification function, and $c_i(x_i)$ is a behaved function. The constants $a_{ij}$ are the neuron interconnection parameters of the neurons within the network, the constants $b_{ij}$ are interconnection parameters of the neurons with time delay parameters $\tau_j$. The $f_i(\cdot)$ corresponds to the activation functions of neurons. The constants $u_i$ are some external inputs. In system (1), $\tau_j \geq 0$ represent the time delay parameters with $\tau = max(\tau_j)$ for $j = 1, 2, ..., n$. The neural system (1) is accompanied by an initial condition of the form: $x_i(t) = \phi_i(t) \in C([-\tau, 0], R)$, where $C([-\tau,0],R)$ denotes the set of all continuous functions from $[-\tau,0]$ to $R$.

The usual assumptions on the functions $d_i$, $c_i$ and $f_i$ are defined to be as follows :

$H_1$ : For the amplification functions $d_i(x)$, $(i=1,2,\ldots,n)$, there exist positive constants $\psi_i$ and $\phi_i$ such that $0<\psi_i\leq d_i(x)\leq\phi_i$, $\forall x\in R$.

$H_2$ : For the functions $c_i(x)$, $(i=1,2,\ldots,n)$, there exist constants $\gamma_i>0$ such that

$$\frac{c_i(x)-c_i(y)}{x-y}:=\frac{|c_i(x)-c_i(y)|}{|x-y|}:\geq\gamma_i>0,:i=1,2,\ldots,n,:::\forall x,y\in R,\ x\neq y.$$

$H_3$ : For the activation functions $f_i(x)$, $(i=1,2,\ldots,n)$, there exist some positive constants $k_i$ such that

$$0\leq\frac{f_i(x)-f_i(y)}{x-y}\leq k_i,::i=1,2,\cdots,n,:::\forall x,y\in R,:x\neq y.$$

Now, let $x^*$ be an equilibrium point of Cohen-Grossberg neural network model (1). The transformation $z(t)=x(t)-x^*$ will shift the equilibrium point $x^*$ of system (1) to the origin. The transformed Cohen-Grossberg neural network model is now represented by the following new sets of differential equations :

$$\dot{z}_i(t)=\alpha_i(z_i(t))[-\beta_i(z_i(t))+\sum_{j=1}^{n}a_{ij}g_j(z_j(t))+\sum_{j=1}^{n}b_{ij}g_j(z_j(t-\tau_j))]$$

in which the following can be stated

$$\alpha_i(z_i(t))=d_i(z_i(t)+x_i^*),\ i=1,2,\ldots,n$$

$$\beta_i(z_i(t))=c_i(z_i(t)+x_i^*)-c_i(x_i^*),\ i=1,2,\ldots,n$$

$$g_i(z_i(t))=f_i(z_i(t)+x_i^*)-f_i(x_i^*),\ i=1,2,\ldots,n.$$

An equivalent mathematical model of (2) can be stated as follows :

$$\dot{z}(t)=\alpha(z(t))[-\beta(z(t))+Ag(z(t))+Bg(z(t-\tau))],$$

where $z(t)=(z_1(t),z_2(t),\ldots,z_n(t))^T$,
$\alpha(z(t))=diag(\alpha_1(z_1(t)),\alpha_2(z_2(t)),\ldots,\alpha_n(z_n(t)))$, $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$,
$\beta(z(t))=(\beta_1(z_1(t)),\beta_2(z_2(t)),\ldots,\beta_n(z_n(t)))^T$,
$g(z(t-\tau))=(g_1(z_1(t-\tau_1)),g_2(z_2(t-\tau_2)),\ldots,g_n(z_n(t-\tau_n)))^T$.

When introducing the T-S fuzzy model concept into (3), the model of fuzzy Cohen-Grossberg neural network with discrete time delays is obtained as follows [36]:

**Plant Rule** $r$ :

**IF** $\{\theta_1(t)\ is\ M_{r1}\}$ and $\cdots$ and $\{\theta_p(t)\ is\ M_{rp}\}$.

**THEN**

$$\dot{z}(t) = \alpha_r(z(t))[-\beta_r(z(t)) + A_r g(z(t)) + B_r g(z(t-\tau))],$$

where $\theta_l(t)(l = 1, 2, \ldots, p)$ are the premise variables. $M_{rl}(r \in \{1, 2, \ldots, m\},\ \ l \in \{1, 2, \ldots, p\}$ are the fuzzy sets and $m$ is the number of **IF-THEN** rules.

By inferring from the fuzzy models, the final model of a fuzzy Cohen-Grossberg neural network takes the following form [36] :

$$\dot{z}(t) = \sum_{r=1}^{m} h_r(\theta(t))\{\alpha_r(z(t))[-\beta_r(z(t)) + A_r g(z(t)) + B_r g(z(t-\tau))]\},$$

where $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]^T$, $\omega_r(\theta(t)) = \prod_{l=1}^{p} M_{rl}(\theta_l(t))$ and $h_r(\theta(t)) = \frac{\omega_r(\theta(t))}{\sum_{r=1}^{m} \omega_r(\theta(t))}$ denote the weight and averaged weight of each fuzzy rule, respectively. The term $M_{rl}(\theta_l(t))$ is the grade membership of $\theta_l(t)$ in $M_{rl}$. We assume that $\omega_r(\theta(t)) \geq 0,\ \ r \in \{1, 2, \ldots, m\}$. Therefore, we have $\sum_{r=1}^{m} h_r(\theta(t)) = 1$ for all $t \geq 0$.

Note that $A_r = (a_{ij}^{(r)})_{n\times n}$, $B_r = (b_{ij}^{(r)})_{n\times n}$,
$\alpha_r(z(t)) = diag(\alpha_{r1}(z_1(t)), \alpha_{r2}(z_2(t)), \ldots, \alpha_{rn}(z_n(t)))$,
$\beta_r(z(t)) = (\beta_{r1}(z_1(t)), \beta_{r2}(z_2(t)), \ldots, \beta_{rn}(z_n(t)))^T$, $r = 1, 2, \ldots, m$.

We also note that, in system (5), the assumptions $H_1$, $H_2$ and $H_3$ can now be respectively adopted as follows :

$$A_1 : 0 < \psi_{ri} \leq \alpha_{ri}(z_i(t)) \leq \phi_{ri}, \quad i = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, m$$
$$A_2 : z_i(t)\beta_{ri}(z_i(t)) \geq \gamma_{ri} z_i^2(t) \geq 0, \quad i = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, m$$
$$A_3 : |g_i(z_i(t))| \leq k_i |z_i(t)|, \quad z_i(t) g_i(z_i(t)) \geq 0, \quad i = 1, 2, \ldots, n.$$

## 3. Main Result

In this section, we will present two main theorems. The first theorem proves the uniqueness of equilibrium point for system (5), which is stated as follows :

**Theorem 1.** *Under the assumptions $A_1$, $A_2$ and $A_3$, the origin $z = 0$ of the T-S fuzzy Cohen-Grossberg neural network model defined by (5) is the unique equilibrium point if there exist positive constants $\xi_r$, $r = 1, 2, \cdots, m$ such that the following condition holds :*

$$\Omega = 2\psi\gamma K^{-1} - \sum_{r=1}^{m} \phi_r(|A_r| + |A_r^T|) - \sum_{r=1}^{m} \frac{1}{\xi_r}\phi_r^2 ||B_r||_2^2 I - \sum_{r=1}^{m} \xi_r I > 0,$$

*where* $\psi = min\{\psi_r\}$ *with* $\psi_r = min\{\psi_{ri}\}$, $\gamma = min\{\gamma_r\}$ *with* $\gamma_r = min\{\gamma_{ri}\}$, $\phi_r = max\{\phi_{ri}\}$, $i = 1, 2, \cdots, n$, $r = 1, 2, \cdots, m$, $K = diag(k_1, k_2, \cdots, k_n)$ *and* $|A_r| = (|a_{ij}^{(r)}|)_{n\times n}$.

*Proof.* We will prove this theorem by using the contradiction method. Let $z \neq 0$ be an equilibrium point of system (5). Then, we have

$$\dot{z}(t) = \sum_{r=1}^{m} h_r(\theta(t))\{\alpha_r(z)[-\beta_r(z) + A_r g(z) + B_r g(z)]\} = 0$$

which can be written as

$$-\sum_{r=1}^{m} h_r(\theta(t))\alpha_r(z)\beta_r(z) + \sum_{r=1}^{m} h_r(\theta(t))\alpha_r(z)A_r g(z) + \sum_{r=1}^{m} h_r(\theta(t))\alpha_r(z)B_r g(z) = 0.$$

Let $z \neq 0$ and $g(z) = 0$. Then, one has

$$\sum_{r=1}^{m} h_r(\theta(t))\alpha_r(z)\beta_r(z) = 0$$

and from (8)

$$\sum_{r=1}^{m} h_r(\theta(t))z^T\alpha_r(z)\beta_r(z) = 0.$$

Thus, we obtain

$$\sum_{r=1}^{m} h_r(\theta(t))z^T\alpha_r(z)\beta_r(z) \geq \psi\gamma z^T z > 0, \forall z \neq 0.$$

It is clear that if $z \neq 0$, then (7) cannot be satisfied. Therefore, at the equilibrium point, when $g(z) = 0$, $z \neq 0$ cannot be a solution of (6).

Let $z \neq 0$ and $g(z) \neq 0$. Then, we can write

$$\begin{aligned} &-2\sum_{r=1}^{m} h_r(\theta(t))g^T(z)\alpha_r(z)\beta_r(z) + 2\sum_{r=1}^{m} h_r(\theta(t))g^T(z)\alpha_r(z)A_r g(z) \\ &+2\sum_{r=1}^{m} h_r(\theta(t))g^T(z)\alpha_r(z)B_r g(z) = 0. \end{aligned} \tag{1}$$

We note that

$$\begin{aligned} -2\sum_{r=1}^{m} h_r(\theta(t))g^T(z)\alpha_r(z)\beta_r(z) &= -2\sum_{r=1}^{m} h_r(\theta(t))\sum_{i=1}^{n}\alpha_{ri}(z_i)\beta_{ri}(z_i)g_i(z_i) \\ &\leq -2\sum_{r=1}^{m} h_r(\theta)\sum_{i=1}^{n}\psi_{ri}\gamma_{ri}z_i g_i(z_i) \\ &\leq -2\sum_{r=1}^{m} h_r(\theta(t))\sum_{i=1}^{n}\frac{1}{k_i}\psi_{ri}\gamma_{ri}g_i^2(z_i) \end{aligned}$$

$$\begin{aligned}
&\leq -2\sum_{r=1}^{m} h_r(\theta(t))g^T(z)\psi\gamma K^{-1}g(z) \\
&= -2\psi\gamma|g^T(z(t))|K^{-1}|g(z(t))|, \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{m} h_r(\theta(t))\{2g^T(z)\alpha_r(z)A_r g(z)\} &\leq \sum_{r=1}^{m} 2|g^T(z)|\alpha_r(z)|A_r||g(z)| \\
&\leq \sum_{r=1}^{m} 2|g^T(z)|\phi_r|A_r||g(z)| \\
&= \sum_{r=1}^{m} \phi_r|g^T(z)|(|A_r|+|A_r^T|)|g(z)|, \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{m} h_r(\theta(t))2g^T(z)\alpha_r(z)B_r g(z) &\leq \sum_{r=1}^{m} h_r(\theta(t))2||\alpha_r(z)||_2||B_r||_2||g(z)||_2||g(z)||_2 \\
&\leq \sum_{r=1}^{m} 2\phi_r||B_r||_2||g(z)||_2||g(z)||_2 \\
&\leq \sum_{r=1}^{m} \frac{1}{\xi_r}\phi_r^2||B_r||_2^2||g(z)||_2^2 + \sum_{r=1}^{m} \xi_r||g(z)||_2^2. \qquad (4)
\end{aligned}$$

Using (11)-(13) in (10) yields

$$\begin{aligned}
&-2\psi\gamma|g^T(z)|K^{-1}|g(z)| + \sum_{r=1}^{m} \phi_r|g^T(z)|(|A_r|+|A_r^T|)|g(z)| \\
&+\sum_{r=1}^{m} \frac{1}{\xi_r}\phi_r^2||B_r||_2^2||g(z)||_2^2 + \sum_{r=1}^{m} \xi_r||g(z)||_2^2 \geq 0
\end{aligned}$$

which is of the form

$$|g^T(z)|(-\Omega)|g(z)| \geq 0$$

or equivalently

$$|g^T(z)|\Omega|g(z)| \leq 0.$$

On the other hand, if $\Omega$ is a positive definite matrix, then, for all $g(z(t)) \neq 0$, we have

$$|g^T(z)|\Omega|g(z)| > 0.$$

Obviously, when $\Omega > 0$, (14) contradicts with (15), implying that under the condition of Theorem 1, the equilibrium equation of system (5) given by (6) cannot have a solution where $g(z) \neq 0$. Thus, we can conclude that Theorem 1 guarantees that the origin of system (5) is the unique equilibrium point. □

We will now present the following theorem that proves the stability of system (5).

**Theorem 2.** *Under the assumptions $A_1$, $A_2$ and $A_3$, the T-S fuzzy Cohen-Grossberg neural network model defined by (5) is globally asymptotically stable if there exist positive constants $\xi_r$, $r = 1, 2, \cdots, m$ such that the following condition holds :*

$$\Omega = 2\psi\gamma K^{-1} - \sum_{r=1}^{m} \phi_r(|A_r| + |A_r^T|) - \sum_{r=1}^{m} \frac{1}{\xi_r}\phi_r^2||B_r||_2^2 I - \sum_{r=1}^{m} \xi_r I > 0,$$

*where $\psi = min\{\psi_r\}$ with $\psi_r = min\{\psi_{ri}\}$, $\gamma = min\{\gamma_r\}$ with $\gamma_r = min\{\gamma_{ri}\}$, $\phi_r = max\{\phi_{ri}\}$, $i = 1, 2, \cdots, n$, $r = 1, 2, \cdots, m$, $K = diag(k_1, k_2, \cdots, k_n)$ and $|A_r| = (|a_{ij}^{(r)}|)_{n\times n}$.*

*Proof.* Consider the following positive definite Lyapunov functional :

$$\begin{aligned} V(z(t)) &= z^T(t)z(t) + 2\varepsilon \sum_{i=1}^{n} \int_0^{z_i(t)} g_i(s)ds \\ &\quad + \varepsilon \sum_{r=1}^{m} \xi_r \sum_{j=1}^{n} \int_{t-\tau_j}^{t} g_j^2(z_j(\zeta))d\zeta + \eta \sum_{j=1}^{n} \int_{t-\tau_j}^{t} g_j^2(z_j(\zeta))d\zeta, \end{aligned}$$

where $\varepsilon$ and $\eta$ are some positive constants to be determined later. We can calculate the time derivative of $V(z(t))$ along the trajectories of neural system (5) as follows :

$$\begin{aligned} \dot{V}(z(t)) &= 2z^T(t)\dot{z}(t) + 2\varepsilon \sum_{i=1}^{n} g_i(z_i(t))\dot{z}_i(t) = 2z^T(t)\dot{z}(t) + 2\varepsilon g^T(z(t))\dot{z}(t) \\ &\quad + \varepsilon \sum_{r=1}^{m} \xi_r \sum_{j=1}^{n} g_j^2(z_j(t)) - \varepsilon \sum_{r=1}^{m} \xi_r \sum_{j=1}^{n} g_j^2(z_j(t-\tau_j)) \\ &\quad + \eta \sum_{j=1}^{n} g_j^2(z_j(t)) - \eta \sum_{j=1}^{n} g_j^2(z_j(t-\tau_j)) \\ &= 2z^T(t) \sum_{r=1}^{m} h_r(\theta(t))\{\alpha_r(z(t))[-\beta_r(z(t)) + A_r g(z(t)) + B_r g(z(t-\tau))]\} \\ &\quad + 2\varepsilon g^T(z(t)) \sum_{r=1}^{m} h_r(\theta(t))\{\alpha_r(z(t))[-\beta_r(z(t)) + A_r g(z(t)) + B_r g(z(t-\tau))]\} \\ &\quad + \varepsilon \sum_{r=1}^{m} \xi_r ||g(z(t))||_2^2 - \varepsilon \sum_{r=1}^{m} \xi_r ||g(z(t-\tau))||_2^2 \\ &\quad + \eta ||g(z(t))||_2^2 - \eta ||g(z(t-\tau))||_2^2 \\ &= \sum_{r=1}^{m} h_r(\theta(t))\{-2z^T(t)\alpha_r(z(t))\beta_r(z(t))\} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{m} h_r(\theta(t))\{2z^T(t)\alpha_r(z(t))A_r g(z(t))\}\\
&+\sum_{r=1}^{m} h_r(\theta(t))\{2z^T(t)\alpha_r(z(t))B_r g(z(t-\tau))\}\\
&+\varepsilon\sum_{r=1}^{m} h_r(\theta(t))\{-2g^T(z(t))\alpha_r(z(t))\beta_r(z(t))\}\\
&+\varepsilon\sum_{r=1}^{m} h_r(\theta(t))\{2g^T(z(t))\alpha_r(z(t))A_r g(z(t))\}\\
&+\varepsilon\sum_{r=1}^{m} h_r(\theta(t))\{2g^T(z(t))\alpha_r(z(t))B_r g(z(t-\tau))\}\\
&+\varepsilon\sum_{r=1}^{m}\xi_r||g(z(t))||_2^2-\varepsilon\sum_{r=1}^{m}\xi_r||g(z(t-\tau))||_2^2\\
&+\eta||g(z(t))||_2^2-\eta||g(z(t-\tau))||_2^2. \qquad (5)
\end{aligned}
$$

We first note the following inequalities :

$$
\begin{aligned}
\sum_{r=1}^{m} h_r(\theta(t))\{-2z^T(t)\alpha_r(z(t))\beta_r(z(t))\} &= \sum_{r=1}^{m} h_r(\theta(t))\{-2\sum_{i=1}^{n}\alpha_{ri}(z_i(t))\beta_{ri}(z_i(t))z_i(t)\}\\
&\leq \sum_{r=1}^{m} h_r(\theta(t))\{-2\sum_{i=1}^{n}\psi_{ri}\gamma_{ri}z_i^2(t)\}\\
&\leq \sum_{r=1}^{m} h_r(\theta(t))\{-2\psi_r\gamma_r\sum_{i=1}^{n}z_i^2(t)\}\\
&\leq -2\psi\gamma\sum_{r=1}^{m} h_r(\theta(t))\{\sum_{i=1}^{n}z_i^2(t)\}\\
&= -2\psi\gamma||z(t)||_2^2\sum_{r=1}^{m} h_r(\theta(t))\\
&= -2\psi\gamma||z(t)||_2^2, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{r=1}^{m} h_r(\theta(t))\{2z^T(t)\alpha_r(z(t))A_r g(z(t))\}\\
=\;&\sum_{r=1}^{m} h_r(\theta(t))\{-(\sqrt{\psi\gamma}z(t)-\frac{1}{\sqrt{\psi\gamma}}\alpha_r(z(t))A_r g(z(t)))^T(\sqrt{\psi\gamma}z(t)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{\sqrt{\psi\gamma}}\alpha_r(z(t))A_r g(z(t)))\} + \sum_{r=1}^{m} h_r(\theta(t))\{\psi\gamma z^T(t)z(t) \\
& +\frac{1}{\psi\gamma}g^T(z(t))A_r^T\alpha_r^2(z(t))A_r g(z(t)))\} \\
\leq\ & \sum_{r=1}^{m} h_r(\theta(t))\{\leq \psi\gamma z^T(t)z(t) + \frac{1}{\psi\gamma}g^T(z(t))A_r^T\alpha_r^2(z(t))A_r g(z(t)))\} \\
\leq\ & \sum_{r=1}^{m} h_r(\theta(t))\psi\gamma z^T(t)z(t) + \sum_{r=1}^{m} h_r(\theta(t))\frac{1}{\psi\gamma}||\alpha_r^2(z(t))||_2||A_r||_2^2||g(z(t))||_2^2 \\
\leq\ & \sum_{r=1}^{m} h_r(\theta(t))\psi\gamma||z(t)||_2^2 + \sum_{r=1}^{m} h_r(\theta(t))\frac{\phi^2}{\psi\gamma}||A_r||_2^2||g(z(t))||_2^2 \\
\leq\ & \psi\gamma||z(t)||_2^2 + \sum_{r=1}^{m}\frac{\phi^2}{\psi\gamma}||A_r||_2^2||g(z(t))||_2^2, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{r=1}^{m} h_r(\theta(t))\{2z^T(t)\alpha_r(z(t))B_r g(z(t-\tau))\} \\
=\ & \sum_{r=1}^{m} h_r(\theta(t))\{-(\sqrt{\psi\gamma}z(t) - \frac{1}{\sqrt{\psi\gamma}}\alpha_r(z(t))B_r g(z(t-\tau)))^T \\
& \times(\sqrt{\psi\gamma}z(t) - \frac{1}{\sqrt{\psi\gamma}}\alpha_r(z(t))B_r g(z(t-\tau)))\} \\
& + \sum_{r=1}^{m} h_r(\theta(t))\{\psi\gamma z^T(t)z(t) + \frac{1}{\psi\gamma}g^T(z(t-\tau))B_r^T\alpha_r^2(z(t))B_r g(z(t-\tau)))\} \\
\leq\ & \sum_{r=1}^{m} h_r(\theta(t))\{\leq \psi\gamma z^T(t)z(t) + \frac{1}{\psi\gamma}g^T(z(t-\tau))B_r^T\alpha_r^2(z(t))B_r g(z(t-\tau)))\} \\
\leq\ & \sum_{r=1}^{m} h_r(\theta(t))\psi\gamma z^T(t)z(t) + \sum_{r=1}^{m} h_r(\theta(t))\frac{1}{\psi\gamma}||\alpha_r^2(z(t))||_2||B_r||_2^2||g(z(t-\tau))||_2^2 \\
\leq\ & \sum_{r=1}^{m} h_r(\theta(t))\psi\gamma||z(t)||_2^2 + \sum_{r=1}^{m} h_r(\theta(t))\frac{\phi^2}{\psi\gamma}||B_r||_2^2||g(z(t-\tau))||_2^2 \\
\leq\ & \psi\gamma||z(t)||_2^2 + \sum_{r=1}^{m}\frac{\phi^2}{\psi\gamma}||B_r||_2^2||g(z(t-\tau))||_2^2, \qquad (8)
\end{aligned}
$$

$$
\varepsilon\sum_{r=1}^{m} h_r(\theta(t))\{-2g^T(z(t))\alpha_r(z(t))\beta_r(z(t))\}
$$

$$
\begin{aligned}
&= \varepsilon \sum_{r=1}^{m} h_r(\theta(t))\{-2\sum_{i=1}^{n} \alpha_{ri}(z_i(t))\beta_{ri}(z_i(t))g_i(z_i(t))\} \\
&\leq \varepsilon \sum_{r=1}^{m} h_r(\theta(t))\{-2\sum_{i=1}^{n} \psi_{ri}\gamma_{ri} z_i(t) g_i(z_i(t))\} \\
&\leq \varepsilon \sum_{r=1}^{m} h_r(\theta(t))\{-2\sum_{i=1}^{n} \frac{1}{k_i}\psi_{ri}\gamma_{ri} g_i^2(z_i(t))\} \\
&\leq \varepsilon \sum_{r=1}^{m} h_r(\theta(t))\{-2g^T(z(t))\psi\gamma K^{-1} g(z(t))\} \\
&\leq -2\varepsilon\psi\gamma|g^T(z(t))|K^{-1}|g(z(t))|, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon \sum_{r=1}^{m} h_r(\theta(t))\{2g^T(z(t))\alpha_r(z(t))A_r g(z(t))\} \\
&\leq \varepsilon \sum_{r=1}^{m} 2|g^T(z(t))|\alpha_r(z(t))|A_r||g(z(t))| \\
&\leq \varepsilon \sum_{r=1}^{m} 2|g^T(z(t))|\phi_r|A_r||g(z(t))| \\
&= \varepsilon \sum_{r=1}^{m} \phi_r|g^T(z(t))|(|A_r| + |A_r^T|)|g(z(t))|, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon \sum_{r=1}^{m} h_r(\theta(t))\{-2g^T(z(t))\alpha_r(z(t))B_r g(z(t-\tau))\} \\
&\varepsilon \leq \sum_{r=1}^{m} h_r(\theta(t))\{2||\alpha_r(z(t))||_2||B_r||_2||g(z(t))||_2||g(z(t-\tau))||_2\} \\
&\varepsilon \leq \sum_{r=1}^{m} 2\phi_r||B_r||_2||g(z(t))||_2||g(z(t-\tau))||_2 \\
\leq\ &\varepsilon \sum_{r=1}^{m} \frac{1}{\xi_r}\phi_r^2||B_r||_2^2||g(z(t))||_2^2 + \varepsilon \sum_{r=1}^{m} \xi_r||g(z(t-\tau))||_2^2. \qquad (11)
\end{aligned}
$$

Using (17)-(22) in (16) leads to

$$
\begin{aligned}
\dot{V}(z(t)) \leq\ & -2\psi\gamma||z(t)||_2^2 + \psi\gamma||z(t)||_2^2 + \sum_{r=1}^{m} \frac{\phi^2}{\psi\gamma}||A_r||_2^2||g(z(t))||_2^2 \\
& +\psi\gamma||z(t)||_2^2 + \sum_{r=1}^{m} \frac{\phi^2}{\psi\gamma}||B_r||_2^2||g(z(t-\tau))||_2^2
\end{aligned}
$$

$$
\begin{aligned}
&-2\varepsilon\psi\gamma|g^T(z(t))|K^{-1}|g(z(t))| \\
&+\varepsilon\sum_{r=1}^{m}|g^T(z(t))|(\phi_r|A_r|+|A_r^T|\phi_r)|g(z(t))| \\
&+\varepsilon\sum_{r=1}^{m}\frac{1}{\xi_r}\phi_r^2||B_r||_2^2||g(z(t))||_2^2+\varepsilon\sum_{r=1}^{m}\xi_r||g(z(t-\tau))||_2^2 \\
&+\varepsilon\sum_{r=1}^{m}\xi_r||g(z(t))||_2^2-\varepsilon\sum_{r=1}^{m}\xi_r||g(z(t-\tau))||_2^2 \\
&+\eta||g(z(t))||_2^2-\eta||g(z(t-\tau))||_2^2. \qquad (12)
\end{aligned}
$$

Let $||A||_2 = max\{||A_r||_2\}$ and $||B||_2 = max\{||B_r||_2\}$, $r = 1, 2, \cdots, m$ and $\eta = \frac{m\phi^2}{\psi\gamma}$. Then, (23) takes the form

$$
\begin{aligned}
\dot{V}(z(t)) \leq\;& \frac{m\phi^2}{\psi\gamma}||A||_2^2||g(z(t))||_2^2+\frac{m\phi^2}{\psi\gamma}||B||_2^2||g(z(t))||_2^2 \\
&-2\varepsilon\psi\gamma|g^T(z(t))|K^{-1}|g(z(t))|+\varepsilon\sum_{r=1}^{m}|\phi_r g^T(z(t))|(|A_r|+|A_r^T|)|g(z(t))| \\
&+\varepsilon\sum_{r=1}^{m}\frac{1}{\xi_r}\phi_r^2||B_r||_2^2||g(z(t))||_2^2+\varepsilon\sum_{r=1}^{m}\xi_r||g(z(t))||_2^2 \\
=\;& \frac{m\phi^2}{\psi\gamma}(||A||_2^2+||B||_2^2)||g(z(t))||_2^2 \\
&-\varepsilon|g^T(z(t))|(2\psi\gamma K^{-1}-\sum_{r=1}^{m}(\phi_r(|A_r|+|A_r^T|) \\
&+\frac{1}{\xi_r}\phi_r^2||B_r||_2^2 I+\xi_r I))|g(z(t))| \\
=\;& \frac{m\phi^2}{\psi\gamma}(||A||_2^2+||B||_2^2)||g(z(t))||_2^2-\varepsilon|g^T(z(t))|\Omega|g(z(t))| \\
\leq\;& \frac{m\phi^2}{\psi\gamma}(||A||_2^2+||B||_2^2)||g(z(t))||_2^2-\varepsilon\lambda_m(\Omega)||g(z(t))||_2^2, \qquad (13)
\end{aligned}
$$

where $\lambda_m(\Omega) > 0$ is the minimum eigenvalue of the positive definite matrix $\Omega$. The choice

$$
\varepsilon > \frac{m\phi^2}{\psi\gamma\lambda_m(\Omega)}(||A||_2^2+||B||_2^2)
$$

ensures that $\dot{V}(z(t))$ expressed by (24) is negative definite for all $g(z(t)) \neq 0$, or equivalently $\dot{V}(z(t)) < 0$ for all $z(t) \neq 0$ as $g(z(t)) \neq 0$ implies that $z(t) \neq 0$. In the case where $z(t) = 0$, ($z(t) = 0$ implies that $g(z(t)) = 0$), $\dot{V}(z(t))$ directly takes the form

$$\begin{aligned}\dot{V}(z(t)) &= -\varepsilon\sum_{r=1}^{m}\xi_r\sum_{j=1}^{n}g_j^2(z_j(t-\tau_j)) - \eta\sum_{j=1}^{n}g_j^2(z_j(t-\tau_j))\\ &\leq -\eta\sum_{j=1}^{n}g_j^2(z_j(t-\tau_j)) = -\eta g^T(z(t-\tau))g(z(t-\tau)). \qquad (14)\end{aligned}$$

It follows from (25) that if $g(z(t-\tau)) \neq 0$, then $\dot{V}(z(t)) < 0$. Note that $\dot{V}(z(t)) = 0$ if and only if $z(t) = g(z(t)) = g(z(t-\tau)) = 0$. On the other hand, one can easily check that $V(z(t))$ is radially unbounded. Therefore, from the standard Lyapunov stability theorems, we can conclude that the condition given in Theorem 2 establishes the global asymptotic stability of the origin of neural system (5). □

**Remark 3.** *In [21]-[38], the stability of fuzzy neural system (5) have been established by using the LMI (linear matrix inequality approach) or the M-matrix based approach. It should be pointed out here that stability conditions expressed in the LMI forms need to be checked for negative definiteness of the high dimensional matrices formed by network parameters of neural systems. On the other hand, the M-matrix-based approach neglects the sign of entries of the interconnection matrices, which may result in the possible conservativeness in the stability criteria. However, the stability conditions proposed in Theorems 1 and 2 establish a simple and easily verifiable relationship between the network parameters of the system without using the LMI-based or M-matrix-based approaches. Therefore, the result proposed in the current paper can be considered as an alternative condition to the previously published results.*

Now, the following numerical example is given in order to demonstrate the applicability of the theoretical results obtained in Theorems 1 and 2.

**Example 4.** *Let Takagi-Sugeno fuzzy delayed neural network (5) be defined by the parameters* $\psi = 1$, $\gamma = 1$, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 1$, $k_1 = k_2 = k_3 = k_4 = 1$ *and by the system matrices :*

$$A_1 = \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ -a & -a & -a & -a \\ -a & a & a & -a \\ \end{bmatrix}, : A_2 = \begin{bmatrix} -a & -a & -a & -a \\ a & -a & -a & a \\ a & -a & -a & a \\ a & a & a & a \\ \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a & -a & a & a \\ -a & -a & a & a \\ -a & a & a & -a \\ -a & -a & -a & a \\ \end{bmatrix}, : A_4 = \begin{bmatrix} a & -a & -a & -a \\ -a & a & -a & -a \\ a & -a & a & -a \\ a & a & -a & -a \\ \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b & b & b & b \\ b & b & -b & -b \\ b & -b & b & -b \\ -b & b & b & -b \end{bmatrix}, : B_2 = \begin{bmatrix} -b & -b & -b & -b \\ -b & -b & b & b \\ -b & b & -b & b \\ b & -b & -b & b \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -b & -b & -b & -b \\ b & b & -b & -b \\ b & -b & b & -b \\ -b & b & b & -b \end{bmatrix}, : B_4 = \begin{bmatrix} b & b & b & b \\ -b & -b & b & b \\ -b & b & -b & b \\ b & -b & -b & b \end{bmatrix},$$

*where $a > 0$ and $b > 0$ are some positive real constants. From the above matrices, we obtain*

$$|A_1| + |A_1^T| = |A_2| + |A_2^T| = |A_3| + |A_3^T| = |A_4| + |A_4^T| = 2 \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{bmatrix}$$

*and*

$$||B_1||_2 = ||B_2||_2 = ||B_3||_2 = ||B_4||_2 = 2b.$$

*Let $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 2b$. Then, $\Omega$ in Theorem 2 is obtained as follows :*

$$\begin{aligned} \Omega &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 8a & 8a & 8a & 8a \\ 8a & 8a & 8a & 8a \\ 8a & 8a & 8a & 8a \\ 8a & 8a & 8a & 8a \end{bmatrix} - \begin{bmatrix} 8b & 0 & 0 & 0 \\ 0 & 8b & 0 & 0 \\ 0 & 0 & 8b & 0 \\ 0 & 0 & 0 & 8b \end{bmatrix} \\ &= 2 \begin{bmatrix} 1-4a-4b & -4a & -4a & -4a \\ -4a & 1-4a-4b & -4a & -4a \\ -4a & -4a & 1-4a-4b & -4a \\ -4a & -4a & -4a & 1-4a-4b \end{bmatrix}, \end{aligned}$$

*where $\Omega > 0$ if $1 - 16a - 4b > 0$. Thus, for the network parameters of this example, the stability condition for system (5) is derived to be $16a + 4b < 1$.*

## 4. Conclusion

This paper has presented a sufficient condition for the uniqueness and global asymptotic stability of the equilibrium point for the class of Takagi-Sugeno (T-S) fuzzy delayed Cohen-Grossberg neural networks with discrete time delays in the presence of the nondecreasing and slope-bounded activation functions. The uniqueness of the equilibrium point has been proved by using the contradiction method, and the stability of the equilibrium point has been established by employing a new fuzzy type Lyapunov functional. A numerical example has been presented to support the effectiveness of the proposed stability criterion. The advantage of the obtained condition over the previously published literature results has also been addressed.

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