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Research Article

Detour g-interior nodes and detour g-boundary nodes in bipolar fuzzy graph with applications

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Abstract

In this paper, we obtain a characterization of bipolar fuzzy detour g-eccentric node. The concepts of bipolar fuzzy detour g-boundary nodes and bipolar fuzzy detour g-interior nodes in a bipolar fuzzy graph are examined. Also we establish the relationship between bipolar fuzzy cut node and bipolar fuzzy detour g-boundary node. Some properties of bipolar fuzzy detour g-boundary nodes, bipolar fuzzy detour g-interior nodes and bipolar fuzzy complete nodes are discussed. Bipolar fuzzy detour g-interior node and bipolar fuzzy detour g-boundary node of a bipolar fuzzy tree are introduced using maximum bipolar fuzzy spanning tree. Applications of detour g-distance, detour g-boundary node, detour g-interior node are given.

Mathematics Subject Classification (2010). 05C72, 05C76, 03E72

Keywords. Bipolar fuzzy detour g-distance, bipolar fuzzy detour g-interior node, bipolar fuzzy detour g-boundary node

1. Introduction

In real life, the concepts of graph theory are immensely utilized in various field including computer science, network routing, operation research, electrical engineering, artificial intelligence, signal processing, robotics and medical science. In 1965 Zadeh [42] replace the classical set by fuzzy set which gives better exactness in both theory and application. In 1975, Rosenfeld [36] initiate the notion of fuzzy graph and in various field it has manifold application. The concepts of bipolar fuzzy sets was established by Zhang [44] in 1994. Mordeson [25] explained the operations on fuzzy graph. The bipolar fuzzy graph and its different types of operations are discussed by Akram [1, 3] and certain type of product of bipolar fuzzy graphs given by [15]. The idea of strong arc in fuzzy graph was given by Kiran R. Bhutani and Rosenfeld [4] and types of arc in fuzzy graph was given by Sunil Mathew and M. S. Sunitha [23]. The notion of bridge, trees, cycles, cut node, end node were introduced by Rosenfeld [36]. The concepts of strength of connectedness in bipolar fuzzy graph, bipolar fuzzy tree, bipolar fuzzy cut node are established by Akram [1–3]. Different types fuzzy graph with their operation and application are explained in the references [5–7, 11, 13, 14, 16, 17, 22, 26–33, 35, 37–39].

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Rosenfeld and Bhutani [4] establish the notion of g-distance in fuzzy graph. The notion of g-boundary node, g-interior node, g-eccentric node was founded by J. P. Linda, and M. S. Sunitha [18]. A characterization of g-self centrad fuzzy graph was given by K. Sameena and M. S. Sunitha [40]. The length of longest $x-y$ path in a connected fuzzy graph G is the detour distance between two nodes x and y explained in G. Chartrand, P. Zang [12]. Gary Chartrand [9] discussed the concepts of detour center of a graph. The notion of detour number, detour set, detour nodes, detour basis in a graph were established by Gary Chatrand, G. L. Johns and Ping Zhang [10]. Interior nodes and boundary nodes are discussed in G. Chatrand, D. Erwin, G. L. Johns, P. Zhang [8]. Fuzzy detour g-distance was given by J. P. Linda and M. S. Sunitha [20]. Fuzzy detour g-interior nodes and fuzzy detour g-boundary nodes of a fuzzy graph are discussed by J. P. Linda and M. S. Sunitha [19]. In this paper we introduced bipolar fuzzy detour g-distance, bipolar fuzzy detour g-interior node, bipolar fuzzy detour g-boundary node and explained their relations. Also some properties of these have been established.

The paper is structured as follows: Section 2 contains a brief background about bipolar fuzzy sets, bipolar fuzzy graphs and path, connectedness, trees in bipolar fuzzy graphs. Bipolar fuzzy detour g-distance, bipolar fuzzy geodesic distance are defined with examples in Section 3. Section 4 proposes the concept of detour g-eccentric node, detour g-periphery, detour g-eccentric subgraphs in bipolar fuzzy graphs and some of theorems are given. The idea of bipolar fuzzy detour g-boundary node with example, complete node and Some properties of are introduced in Section 5. Section 6 the notion of detour g-interior node with example and some theorems. Some theorems on detour g-interior node, detour g-boundary using maximum bipolar fuzzy spanning tree are explained in Section 7. Section 8 contains applications of bipolar fuzzy detour g-distance, detour$g$-interior node, detour g-boundary node and finally conclusions are given in Section 9.

2. Preliminaries

Definition 2.1. [21, 44] Let X be a non-empty set. We say $A = \{(x, \mu_A^P(x), \mu_A^N(x)) : x \in X\}$ is a bipolar fuzzy set in X, where $\mu_A^P : X \to [0,1]$ and $\mu_A^N : X \to [-1,0]$ are mappings.

Definition 2.2. [43] Let X be a non-empty set. The mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0,1] \times [-1,0]$ is called a bipolar fuzzy relation on X, where $\mu_A^P(x,y) \in [0,1]$ and $\mu_A^N(x,y) \in [-1,0]$.

Definition 2.3. [43] Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two bipolar fuzzy sets on X. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy relation on X, then $B = (\mu_B^P, \mu_B^N)$ is said to be a bipolar fuzzy relation on $A = (\mu_A^P, \mu_A^N)$, where $\mu_B^P(x,y) \leqslant min(\mu_A^P(x), \mu_A^P(y))$ and $\mu_B^N(x,y) \geqslant max(\mu_A^N(x), \mu_A^N(y))$, $\forall (x,y) \in X$.

If $\mu_B^P(x,y) = \mu_B^P(y,x)$ and $\mu_B^N(x,y) = \mu_B^N(y,x), \forall x,y \in X$, then $B = (\mu_B^P, \mu_B^N)$ is called a symmetric bipolar fuzzy relation on X.

Definition 2.4. [41] If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set on an underlying set V and $B = (\mu_B^P, \mu_B^N)$ be a bipolar fuzzy set in $\widetilde{V^2}$ such that $\mu_B^P(x,y) \leqslant min\{\mu_A^P(x), \mu_A^P(y)\}$ $\forall (xy) \in \widetilde{V^2}$ and $\mu_B^N(x,y) \geqslant max\{\mu_A^N(x), \mu_A^N(y)\}, \forall (xy) \in \widetilde{V^2}$, and $\mu_B^P(x,y) = \mu_B^N(x,y) = 0, \forall (xy) \in \widetilde{V^2} - E$, then $G = (V, A, B)$ is called a bipolar fuzzy graph of the graph $G^* = (V, E)$.

Definition 2.5. [1, 3] Let $G = (V, A, B)$ be a bipolar fuzzy graph. If $\mu_B^P(x,y) = min\{\mu_A^P(x), \mu_A^P(y)\}$ and $\mu_B^N(x,y) = max\{\mu_A^N(x), \mu_A^N(y)\}, \forall x,y \in V$, then G is called a complete bipolar fuzzy graph and if $\mu_B^P(x,y) = min\{\mu_A^P(x), \mu_A^P(y)\}$ and $\mu_B^N(x,y) = max\{\mu_A^N(x), \mu_A^N(y)\}, \forall (xy) \in E$, then G is called strong bipolar fuzzy graph of the graph $G^* = (V, E)$.

Definition 2.6. [2,3] Let $G = (V, A, B)$ be a bipolar fuzzy graph and $x, y \in V$.

- A path $P : x = x_0, x_1, \ldots, x_{k-1}, x_k = y$ in G is a sequence of distinct vertices such that $(\mu_B^P(x_{i-1}, x_i) > 0, \mu_B^N(x_{i-1}, x_i) < 0), i = 1, 2, \ldots, k$ and the length of the path is k.
- If $P : x = x_0, x_1, \ldots, x_{k-1}, x_k = y$ be a path of length k between x and y, then $(\mu_B^P(x, y))^k$ and $(\mu_B^N(x, y))^k$ are defined as $(\mu_B^P(x, y))^k = \sup\{\mu_B^P(x, x_1) \bigwedge \mu_B^P(x_1, x_2) \bigwedge \ldots \bigwedge \mu_B^P(x_{k-1}, y)\}$ and $(\mu_B^N(x, y))^k = \inf\{\mu_B^N (x, x_1) \bigvee \mu_B^N(x_1, x_2) \bigvee \ldots \bigvee \mu_B^N(x_{k-1}, y)\}$. $(((\mu_B^P(x, y))^\infty, (\mu_B^N(x, y))^\infty)$ is said to be the strength of connectedness between two vertices x and y in G, where $(\mu_B^P(x, y))^\infty = \sup_{k\in\mathbb{N}}\{(\mu_B^P(x, y))^k\}$ and $(\mu_B^N(x, y))^\infty = \inf_{k\in\mathbb{N}}\{(\mu_B^N(x, y))^k\}$.
- If $\mu_B^P(x, y) \geq (\mu_B^P(x, y))^\infty$ and $\mu_B^N(x, y) \leq (\mu_B^N(x, y))^\infty$, then the arc xy in G is said to be a strong arc. A path $x - y$ is strong path if all arcs on the path are strong.
- If $((\mu_B^P(x, y))^\infty, (\mu_B^N(x, y))^\infty)$ lie on same arc of a path between x and y then this path is called the strongest path between x and y. If $\mu_B^P(x, y) > 0$ and $\mu_B^N(x, y) < 0, \forall(x, y)$ in G, then the bipolar fuzzy graph G is said to be a connected bipolar fuzzy graph.

Two nodes x and y are said to be a bipolar fuzzy neighbor's [34] if $\mu_B^P(x, y) > 0$ and $\mu_B^N(x, y) < 0$ and also x is said to be a bipolar fuzzy strong neighbors of y if (x, y) is a bipolar fuzzy strong arc. We denote the set of all bipolar fuzzy neighbors of x by $N_{B.F}(x)$ and the set of all bipolar fuzzy strong neighbors of x by $N_{B.F.S}(x)$.

Definition 2.7. [2] A connected bipolar fuzzy graph $G = (V, A, B)$ is said to be a bipolar fuzzy tree if G has a bipolar spanning subgraph $S = (A, E)$ which is also a bipolar fuzzy tree and $\forall$ arcs (a, b) not in S, $\mu_B^P(a, b) < (\mu_E^P)^\infty(a, b)$ and $\mu_B^N(a, b) > (\mu_E^N)^\infty(a, b)$. The bipolar spanning subgraph S of G is a maximum spanning subgraph of G if G has no bipolar spanning subgraph different from S contains S. Since G is a bipolar fuzzy tree, so G has a unique maximum spanning tree [2].

3. Bipolar fuzzy detour g-distance

Definition 3.1. The length of a $x - y$ strong path P between x and y in a connected bipolar fuzzy graph $G = (V, A, B)$ is said to be a bipolar fuzzy detour g-distance if there is no other strong path longer than P between x and y and we denote this by $B.F.D_g(x, y)$. Any $x-y$ strong path whose length is $B.F.D_g(x, y)$ is called a $x-y$ bipolar fuzzy g-detour.

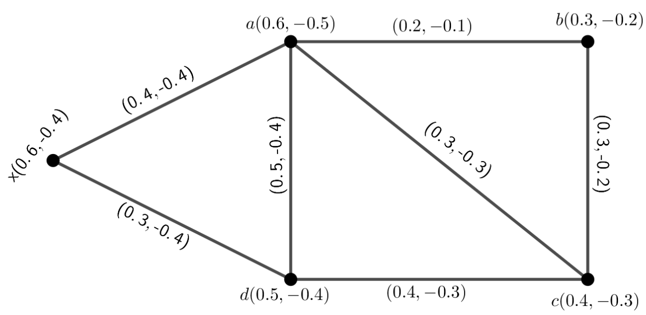

Figure 1. Connected bipolar fuzzy graph G

Example 3.2. Let $G = (V, A, B)$ be a connected bipolar fuzzy graph of the graph $G^* = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{(a, b), (b, c), (c, d), (d, a), (a, c), (a, x), (x, d)\}$ (see Figure 1). For the bipolar fuzzy graph of Figure 1, it is seen that all arcs except (a, b) and

(x,d) are strong arc and the bipolar fuzzy detour g-distance of two nodes are given below: $B.F.D_g(a,b)=3$, $B.F.D_g(a,c)=2$, $B.F.D_g(a,d)=2$, $B.F.D_g(b,c)=1$, $B.F.D_g(b,d)=3$, $B.F.D_g(c,d)=2$, $B.F.D_g(a,x)=1$, $B.F.D_g(x,d)=3$, $B.F.D_g(c,x)=3$, $B.F.D_g(b,x)=4$.

Definition 3.3. The length of any smallest strong path from x to y is called the bipolar fuzzy geodesic distance, denoted by $B.F.d_g(x,y)$.

The bipolar fuzzy detour g-eccentricity $e_{B.F.D_g}(a)$ for a node a is $\max(B.F.D_g(a,x)), \forall x \in G$. The set of all bipolar fuzzy detour g-eccentric nodes of a, denoted by $a^*_{B.F.D_g}$. The bipolar fuzzy detour g-radius of G, denoted by $rad_{B.F.D_g}(G)$, is defined as $\min e_{B.F.D_g}(x), \forall x \in G$. If $e_{B.F.D_g}(a) = rad_{B.F.D_g}(G)$, then the node a is the bipolar fuzzy detour g-central node of G. The bipolar fuzzy detour g-diameter of G, denoted by $diam_{B.F.D_g}(G)$, is defined as $\max e_{B.F.D_g}(x), \forall x \in G$. If $e_{B.F.D_g}(a) = diam_{B.F.D_g}(G)$, then the node a is called the bipolar fuzzy detour g-peripheral node of G.

Example 3.4. For the connected bipolar fuzzy graph G in the Figure 1, $e_{B.F.D_g}(a)=3$, $e_{B.F.D_g}(b)=4$, $e_{B.F.D_g}(c)=3$, $e_{B.F.D_g}(d)=3$, $e_{B.F.D_g}(x)=4$, and $rad_{B.F.D_g}(G)=3$, $diam_{B.F.D_g}(G)=4$.

Definition 3.5. A bipolar fuzzy graph $G=(V,A,B)$ is called a bipolar fuzzy g-detour graph if $B.F.D_g(x,y) = B.F.d_g(x,y), \forall (x,y) \in E$.

4. Bipolar fuzzy detour g-periphery $(Per_{B.F.D_g}(G))$ and bipolar fuzzy detour g-eccentric subgraph $(Ecc_{B.F.D_g}(G))$

Definition 4.1. The bipolar fuzzy subgraph of the bipolar fuzzy graph $G=(V,A,B)$, whose nodes are only the bipolar fuzzy detour g-peripheral nodes is called a bipolar fuzzy detour g-periphery of G and it is denoted by $Per_{B.F.D_g}(G)$.

Definition 4.2. If each node of a connected bipolar fuzzy graph $G=(V,A,B)$ is a bipolar fuzzy detour g-eccentric node, then G is said to be a bipolar fuzzy detour g-eccentric bipolar fuzzy graph. The bipolar fuzzy subgraph of G formed by the set of all bipolar fuzzy g-eccentric nodes of G is called a bipolar fuzzy detour g-eccentric bipolar fuzzy subgraph of G, it is denoted by $Ecc_{B.F.D_g}(G)$.

Example 4.3. For the bipolar fuzzy graph of Figure 2, nodes a,b,d are bipolar fuzzy detour g-periphery nodes since $e_{B.F.D_g}(a)=3$, $e_{B.F.D_g}(b)=3$, $e_{B.F.D_g}(c)=2$, $e_{B.F.D_g}(d)=3$, $diam_{B.F.D_g}(G)=3$. Here $Per_{B.F.D_g}(G)$ of connected bipolar fuzzy graph shown in Figure 2.

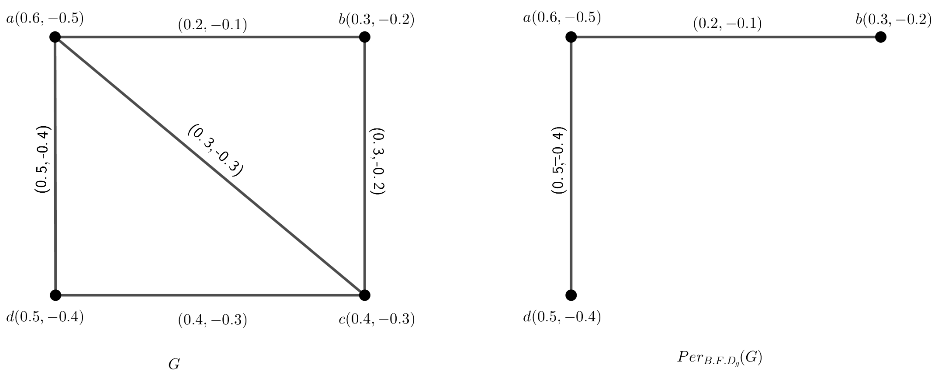

Figure 2. Connected bipolar fuzzy graph G and its $Per_{B.F.D_g}(G)$

Example 4.4. From Figure 2, we get $a^*_{B.F.D_g} = \{b\}$, $b^*_{B.F.D_g} = \{a, d\}$, $c^*_{B.F.D_g} = \{a, d\}$, $d^*_{B.F.D_g} = \{b\}$. Its $Ecc_{B.F.D_g}(G)$ is shown in Figure 2.

Theorem 4.5. *A bipolar fuzzy graph G is a bipolar fuzzy detour g-self centrad if and only if every node of G is a bipolar fuzzy detour g-eccentric.*

Proof. Suppose G is a bipolar fuzzy detour g-self centrad bipolar fuzzy graph and let b be a node in G. Let $a \in b^*_{B.F.D_g}$. So $e_{B.F.D_g}(b) = B.F.D_g(a, b)$. Since G is a bipolar fuzzy detour g-self centrad bipolar fuzzy graph, $e_{B.F.D_g}(a) = e_{B.F.D_g}(b) = B.F.D_g(a, b)$ and this implies that $b \in a^*_{B.F.D_g}$. Hence b is a bipolar fuzzy detour g-eccentric node of G.

Conversely, let every node of G is a bipolar fuzzy detour g-eccentric node. If possible, let G be not bipolar fuzzy detour g-self centrad bipolar fuzzy graph. Then $rad_{B.F.D_g}(G) \neq diam_{B.F.D_g}(G)$ and $\exists$ a node $r \in G$ such that $e_{B.F.D_g}(r) = diam_{B.F.D_g}(G)$. Also let $p \in r^*_{B.F.D_g}$. Let U be a $r - p$ bipolar fuzzy detour in G. So there must have a node q on U for which the node q is not a bipolar fuzzy detour g-eccentric node of U. Also q cannot be a bipolar fuzzy detour g-eccentric node of every other node. Again if q be a bipolar fuzzy detour g-eccentric node of a node a (say), means $q \in a^*_{B.F.D_g}$. Then $\exists$ an extension of $a - q$ bipolar fuzzy $g-$detour up to r or up to p. But this contradicts the facts that $q \in a^*_{B.F.D_g}$. Hence $rad_{B.F.D_g}(G) = diam_{B.F.D_g}(G)$ and G is a bipolar fuzzy detour g-self centrad bipolar fuzzy graph. □

Theorem 4.6. *If G is a bipolar fuzzy detour g-self centrad bipolar fuzzy graph, then $rad_{B.F.D_g}(G) = diam_{B.F.D_g}(G) = n - 1$, where n is the number of nodes of G.*

Proof. Let G be a bipolar fuzzy detour g-self centrad bipolar fuzzy graph. If possible, let $diam_{B.F.D_g}(G) = l < n - 1$.

Let U_1 and U_2 be two distinct bipolar fuzzy detour g-peripheral path. Let $p \in U_1, q \in U_2$. So $\exists$ a strong path between p and q, because of connectedness of G. Then $\exists$ nodes on U_1 and U_2, whose eccentricity $> l$, but this is impossible, because $diam_{B.F.D_g}(G) = l$. Hence U_1 and U_2 are not distinct. Since U_1 and U_2 are arbitrary, so $\exists$ node r in G such that r is a common in all bipolar fuzzy detour peripheral paths. So $e_{B.F.D_g}(r) < l$, which is impossible, because G is a bipolar fuzzy detour g-self centrad. Hence $diam_{B.F.D_g}(G) = n - 1 = rad_{B.F.D_g}(G)$. □

Corollary 4.7. *For a connected bipolar fuzzy graph G, $Per_{B.F.D_g}(G) = G$ if and only if the bipolar fuzzy detour g-eccentricity of each node of G is $n - 1$, $n =$ number of nodes in G.*

Proof. Let $Per_{B.F.D_g}(G) = G$. Then $e_{B.F.D_g}(p) = diam_{B.F.D_g}(G), \forall p \in G$. So every node of G is a bipolar fuzzy detour g-periphery node of G. Therefore G is a self centrad bipolar fuzzy graph and $rad_{B.F.D_g}(G) = diam_{B.F.D_g}(G) = n - 1$. So the bipolar fuzzy detour g-eccentricity of each node of G is $n - 1$.

Conversely, let the bipolar fuzzy detour g-eccentricity of each node of G is $n - 1$. So $rad_{B.F.D_g}(G) = diam_{B.F.D_g}(G) = n - 1$. All nodes of G are bipolar fuzzy detour g-peripheral nodes and hence $Per_{B.F.D_g}(G) = G$. □

Corollary 4.8. *For a connected bipolar fuzzy graph G, $Ecc_{B.F.D_g}(G) = G$ if and only if the bipolar fuzzy detour g-eccentricity of each node of G is $n - 1$, $n =$ number of nodes in G.*

Proof. Let $Ecc_{B.F.D_g}(G) = G$. So all nodes of G are bipolar fuzzy detour g-eccentric node. Therefore G is self centrad bipolar fuzzy graph and $diam_{B.F.D_g}(G) = n - 1$. Hence the bipolar fuzzy detour g-eccentricity of each node of G is $n - 1$.

Conversely, let the bipolar fuzzy detour g-eccentricity of each node of G is $n - 1$. So $rad_{B.F.D_g}(G) = diam_{B.F.D_g}(G) = n - 1$. So all nodes of G are bipolar fuzzy detour g-peripheral nodes as well as bipolar fuzzy detour g-eccentric node. Hence, $Ecc_{B.F.D_g}(G) = G$. □

Theorem 4.9. *In a connected bipolar fuzzy graph G, a node b is a bipolar fuzzy detour g-eccentric node if and only if b is a bipolar fuzzy detour g-peripheral node.*

Proof. Let b be a bipolar fuzzy detour g-eccentric node of G and let $b \in a^*_{B.F.D_g}$. Let x and y be two bipolar fuzzy detour g-peripheral nodes, then $B.F.D_g(x,y) = diam_{B.F.D_g}(G) = k(say)$. Let P_1 and P_2 be any $x-y$ and $a-b$ bipolar fuzzy g-detour in G respectively. There arise two cases.

Case 1: When b is not internal node in G i.e, there is only one node, say c which is adjacent to b. So $c \in P_2$. Since G is connected, c is connected to a node of P_1, say c'. So either $c' \in P_2$ or $c' \in (P_1 \cap P_2)$. Thus in any case the path from a to x or a to y through c and c' is longer than P_2. But it is impossible, since b is a bipolar fuzzy detour g-eccentric node of a. Hence $e_{B.F.D_g}(a) = diam_{B.F.D_g}(G)$ i.e, b is a bipolar fuzzy detour g-peripheral node of G.

Case 2: When b is internal node in G, then $\exists$ a connection between b to x and b to y, because of connectedness of G. Then $a-b$ bipolar fuzzy g-detour can be extend to x or y. This is impossible, because b is a bipolar fuzzy detour g-eccentric node of a. Hence $e_{B.F.D_g}(a) = diam_{B.F.D_g}(G)$ i.e, b is a bipolar fuzzy detour g-peripheral node of G.

Conversely, we assume that b be a bipolar fuzzy detour g-peripheral node G. So $\exists$ a bipolar fuzzy detour g-peripheral node, say a (distinct from b). Therefore b is a bipolar fuzzy detour g-eccentric node of a. □

5. Bipolar fuzzy detour g-boundary node of a bipolar fuzzy graph

Definition 5.1. In a connected bipolar fuzzy graph G, a node b is said to be a bipolar fuzzy detour g-boundary node of a node a if $B.F.D_g(a,b) \geq B.F.D_g(a,c)$ for each c in G, where c is a neighbor of b. The set of all bipolar fuzzy detour g-boundary nodes of a denoted by $a'_{B.F.D_g}$.

Example 5.2. For the connected bipolar fuzzy graph G in Figure 3,
$a'_{B.F.D_g} = \{c,x,z,t\}$, $b'_{B.F.D_g} = \{c,z,t\}$, $c'_{B.F.D_g} = \{z,t\}$, $x'_{B.F.D_g} = \{c,z,t\}$, $y'_{B.F.D_g} = \{c,x,z,t\}$, $z'_{B.F.D_g} = \{x,t\}$, $t'_{B.F.D_g} = \{c,x,z\}$. Here c,x,z,t are the bipolar fuzzy detour g-boundary nodes of G.

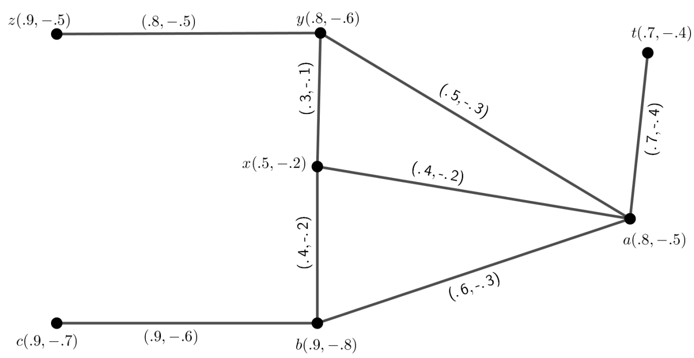

Figure 3. Connected bipolar fuzzy graph G

Definition 5.3. If the bipolar fuzzy subgraph formed by strong neighbor of a node b in a bipolar fuzzy graph G, form a complete bipolar fuzzy graph then the node b is said to be a complete node of G.

Theorem 5.4. *A node in a complete bipolar fuzzy graph is bipolar fuzzy detour g-boundary node of every other nodes if and only if the node is complete.*

Proof. Let a node b be a complete node in a connected bipolar fuzzy graph G. Let a be an another node of G. Each arc in G is strong, because of completeness of G [24]. So $B.F.D_g(a,b) = n-1 = B.F.D_g(a,c), \forall c \in N(b)$, where n = numbers of nodes in G. Therefore b is a bipolar fuzzy detour boundary node of a.

Conversely, let b be a bipolar fuzzy detour $g-$boundary node of every other node. Then each arc in G is strong, because of completeness of G [24]. Then $B.F.D_g(a,b) = n-1, \forall a \in G$. So all neighbor of b are strong neighbor. Hence by definition 5.3, the node b is complete. □

Theorem 5.5. *If a node in a connected bipolar fuzzy graph G is a complete node of G, then the node is a bipolar fuzzy detour g-boundary node of all other node.*

Proof. Let a node b be a complete node in a connected bipolar fuzzy graph G and let a be another node of G. Assume that $a = b_0, b_1, \ldots, b_{k-1}, b_k = b$ be a $a-b$ bipolar fuzzy g-detour and c be a strong neighbor of b. There arises two cases

Case 1: If $c = b_{k-1}$, then $B.F.D_g(a,c) \leq B.F.D_g(a,b)$. Hence b be a bipolar fuzzy detour g-boundary node of a.

Case 2: If $c \neq b_{k-1}$, since c is a strong neighbor of b, so the arc (c, b_{k-1}) is a strong arc and also $c \neq b_{k-1}$. So the length of the path $a = b_0, b_1, \ldots, b_{k-1}, c, b_k = b$ is greater than than the length of the path $a = b_0, b_1, \ldots b_{k-1}, b_k = b$. Hence $B.F.D_g(a,c) \leq B.F.D_g(a,b)$. Therefore b is a bipolar fuzzy detour g-boundary node of a. □

Remark 5.6. Converse of the above theorem may not be true. For example, consider the bipolar fuzzy graph of Figure 4. We see that

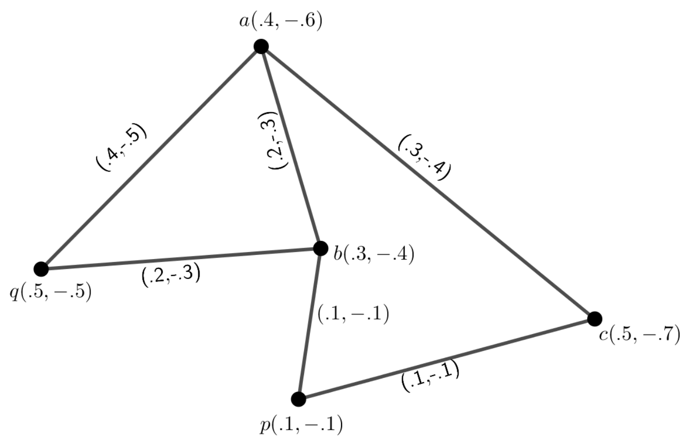

Figure 4. Connected bipolar fuzzy graph G

q is a bipolar fuzzy detour g-boundary node of every other nodes, but q is not a complete node.

Theorem 5.7. *A connected bipolar fuzzy graph G is a bipolar fuzzy tree if and only if G is bipolar fuzzy g-detour graph.*

Proof. Let G be a bipolar fuzzy tree. Then between any two nodes in G, there is exactly one bipolar fuzzy strong path. So $B.F.D_g(a,b) = B.F.d_g(a,b)$ for any two nodes a, b in G. Hence G is bipolar fuzzy $g-$detour graph.

Conversely, let G be a bipolar fuzzy g-detour graph, which has n nodes. Then $B.F.D_g(a,b) = B.F.d_g(a,b)$ for any two nodes a, b in G. If $n = 2$ then G is a bipolar fuzzy tree. □

Let $n \geq 3$. If possible, let G be not a bipolar fuzzy tree. So $\exists$ two nodes p, q in G for which there is at least two strong path between p and q. Let Q_1 and Q_2 be two $p-q$ bipolar fuzzy strong paths. So $Q_1 \cup Q_2$ has a cycle $C(say)$ in G. If node a and b are adjacent nodes in G, then we have $B.F.d_g(a,b) = 1$ and $B.F.D_g(a,b) > 1$. This contradicts the fact that $B.F.D_g(a,b) = B.F.d_g(a,b)$. Hence G is a bipolar fuzzy tree.

Theorem 5.8. *In a bipolar fuzzy tree G, a node b is a bipolar fuzzy detour g-boundary node of G if and only if b cannot be a bipolar fuzzy cut node of G.*

Proof. Let G be a bipolar fuzzy tree and a node b in G be a bipolar fuzzy detour $g-$boundary node of a node c in G. If possible, let b be a bipolar fuzzy cut node of G.

Let E be a bipolar fuzzy maximum spanning tree in G, which is unique in G. Since b is a bipolar fuzzy cut node, so b cannot be an internal node of E. Let $x \in N_{B.F.S}(b)$ such that x does not lie on the bipolar fuzzy detour in E. Therefore $B.F.D_g(p,q)$ is same when p, q be any two nodes of E and G both. But $B.F.D_g(c,x) = B.F.D_g(c,b) + B.F.D_g(b,x) > B.F.D_g(c,b)$. This contradicts the fact that b is a bipolar fuzzy detour $g-$boundary node of a node c in G. Therefore the node b cannot be a bipolar fuzzy cut node of G.

Conversely, let b be not a bipolar fuzzy cut node of the bipolar fuzzy graph G. So b is end node of maximum bipolar spanning tree, which is unique. Then b has a strong neighbor which is also unique [2]. So there does not exist any extension of any bipolar fuzzy g-detour for a node x to b. Hence b is a bipolar fuzzy detour g-boundary node of G. □

Definition 5.9. A node x in a bipolar fuzzy graph G is said to be a bipolar fuzzy end node of G if y is only strong neighbor of x, where $y \in G$.

Example 5.10. For the bipolar fuzzy graph G in Figure 3, the nodes c, z, t are bipolar fuzzy end node of G.

Theorem 5.11. *A node b in a bipolar fuzzy tree G is a bipolar fuzzy detour g-boundary node if and only if b is a bipolar fuzzy end node.*

Proof. Let a node b be a bipolar fuzzy detour g-boundary node for a node a in a bipolar fuzzy tree G. Let E be a maximum bipolar spanning tree in G, which is unique in G [2]. By Theorem 5.9, each node of G is a bipolar fuzzy cut node of G or a bipolar fuzzy end node of G [2]. So by Theorem 5.9, b must be a bipolar fuzzy end node of G.

Conversely, let b be a bipolar fuzzy end node of a bipolar fuzzy tree G. Let E be the maximum bipolar spanning tree of G. Then b is a bipolar fuzzy end node of E. Hence b is not a bipolar fuzzy cut node of G. Therefore by Theorem 5.9, b is a bipolar fuzzy detour g-boundary node of G. □

6. Bipolar fuzzy detour g-interior node of a bipolar fuzzy graph

In a connected bipolar fuzzy graph G, a node b lie between the nodes a and c in the sense of bipolar fuzzy detour g-distance if $B.F.D_g(a,c) = B.F.D_g(a,b) + B.F.D_g(b,c)$.

Definition 6.1. In a connected bipolar fuzzy graph G, a node b is said to be a bipolar fuzzy detour $g-$interior node if for each node a in G different from b, there is a node c in G for which $B.F.D_g(a,c) = B.F.D_g(a,b) + B.F.D_g(b,c)$.

Definition 6.2. The set of all bipolar fuzzy detour g-interior node of G, denoted by $Int_{B.F.D_g}(G)$, form a bipolar fuzzy subgraph of G.

Example 6.3. For the bipolar fuzzy graph in Figure 3, $Int_{B.F.D_g}(G) = \{a, b, y\}$.

Theorem 6.4. *A node in a connected bipolar fuzzy graph G is a bipolar fuzzy detour g-boundary node of G if and only if the node cannot be a bipolar fuzzy detour g-interior node of G.*

Proof. Let b be a bipolar fuzzy detour g-boundary node of a node a in a connected bipolar fuzzy graph G. If possible, let b be a bipolar fuzzy detour g-interior node of G. So there exist a node c different from a and b such that b lies between a and c. Let $U : a = b_1, b_2, \ldots, b = b_k, b_{k+1}, \ldots, b_l = c$ be a $a - c$ bipolar fuzzy $g-$detour and $1 < k < l$. Then $b_{k+1} \in N_{B.F.S}(b)$, and this implies $B.F.D_g(a, b_{k+1}) > B.F.D_g(a, b)$, this is a contradiction. Hence b cannot be a bipolar fuzzy detour g- interior node of G.

Conversely, let a node b in G, which is not a bipolar fuzzy detour g-interior node of G. Then there exist a node a in G for which any node c different from a and b, $B.F.D_g(a, c) \neq B.F.D_g(a, b) + B.F.D_g(b, c)$. Therefore $B.F.D_g(a, q) \leq B.F.D_g(a, b)$ where $q \in N_{B.F.S}(b)$. This implies that b is a bipolar fuzzy detour g-boundary node of a. □

Theorem 6.5. *A bipolar fuzzy end node of a connected bipolar fuzzy graph G cannot be a bipolar fuzzy detour g-interior node.*

Proof. Let q be a bipolar fuzzy end node of a bipolar fuzzy graph G. Then there is only one bipolar fuzzy strong neighbor of q. So there is no strong bipolar fuzzy g-detour for which b lies between a and c, where a and c be two node of G and also different from b. Hence b is not a bipolar fuzzy detour g-interior node of G. □

Example 6.6. For the bipolar fuzzy graph G in Figure 5, the node p is not a bipolar

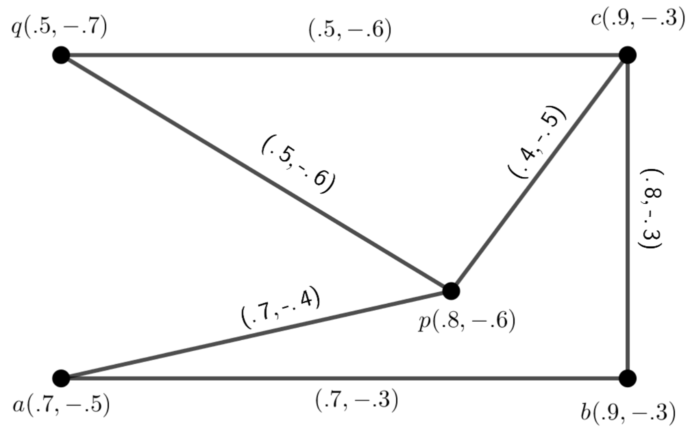

Figure 5. Connected bipolar fuzzy graph G

fuzzy detour g-interior node and also p is not a bipolar fuzzy end node of G.

So the converse of the Theorem 6.5 need not true.

7. Bipolar fuzzy detour g-interior node and bipolar fuzzy detour g-boundary node determined by bipolar fuzzy maximum spanning tree

Theorem 7.1. *In a connected bipolar fuzzy graph G, if a node b is a bipolar fuzzy detour g-interior node, then the node b is also an internal node for any maximum bipolar spanning tree of G.*

Proof. Let b be a bipolar fuzzy detour g-interior node in a connected bipolar fuzzy graph G. Then for each node a different from b, there exist a node c in G such that $B.F.D_g(a, c) = B.F.D_g(a, b) + B.F.D_g(b, c)$. So $a - b - c$ is a bipolar fuzzy g-detour and this path lie on each maximum bipolar spanning tree in G. □

Converse of this theorem need not true. For example, the bipolar fuzzy graph in Figure 5, the node p is a bipolar fuzzy internal node in each maximum bipolar fuzzy spanning tree in G, but p cannot be a bipolar fuzzy detour g-interior node in G.

Theorem 7.2. *In a bipolar fuzzy tree G, a node b is an bipolar fuzzy internal node for the unique maximum bipolar fuzzy spanning tree of G if and only if b is a bipolar fuzzy detour g-interior node of G.*

Proof. Let a node b be an internal node for the unique maximum bipolar fuzzy spanning tree of a bipolar fuzzy tree G. So b must be a bipolar fuzzy cut node of G [2]. Then b cannot be a bipolar fuzzy detour g-boundary node of G (by Theorem 5.9). Hence b is a bipolar fuzzy detour g-interior node of G (by Theorem 6.4).

The converse part also true, from the previous Theorem 7.1. □

Theorem 7.3. *In a connected bipolar fuzzy graph G, if a node x is an end node of a maximum bipolar spanning tree of G, then x is a bipolar fuzzy detour g-boundary node of G.*

Proof. Let x be an end node of a maximum bipolar spanning tree of a bipolar fuzzy graph G. Then for any maximum bipolar spanning tree of G, x is not an internal node. Hence x is not a bipolar fuzzy detour g-interior node of G, by Theorem 7.1. Therefore x is a bipolar fuzzy detour g-boundary node of G, by Theorem 6.4. □

8. Applications

8.1. Wi-Fi network connections in a town in terms of bipolar fuzzy graph

During the present time, the use of Wi-Fi network is very indispensable in such of many places like office, court, educational field, town, bus stand, railway station etc. The speed of Wi-Fi network become slow due to certain reasons like excessive users, natural disaster, mechanical disturbances. So it is uncertain. Here, we present a Wi-Fi network connection in Midnapore town as a bipolar fuzzy graph G which is shown in Figure 6. The nodes k, l, c, b, r, s of the bipolar fuzzy graph G in Figure 6 denotes the Wi-Fi network in places Keranitola, LIC more, Church school, Bus stand, Railway station, St. John's church in Midnapore town respectively. Each edge of G denote the roads between corresponding places.

The positive membership value of each node represents the power of a Wi-Fi networks in corresponding place of the town. Its value is 0 if the internet speed is $\leq 10\%$ and its value is 1 if the internet speed is $\geq 90\%$. So the positive membership value of each nodes lies in $(0, 1)$ if its internet speed lies between $> 10\%$ and $< 90\%$. The negative membership value of each node represents the possibility of slowing down internet speed by the reasons excessive users, natural disaster, mechanical disturbance etc. Its value is 0 if the possibility is $\leq 5\%$ and its value is -1 if the possibility is $\geq 80\%$. So the negative membership value of each nodes lies in $(-1, 0)$ if the possibility lies between $> 5\%$ and $< 80\%$. The positive membership value of each edge represents the average internet speed on corresponding road. Its value is 0 if the average internet speed is $\leq 10\%$ and its value is 1 if the average internet speed is $\geq 85\%$. So the positive membership of each edge lies in $(0, 1)$ if the average internet speed lies between $> 10\%$ and $< 85\%$. The negative membership value of each edge represents the average possibility of slowing down internet speed by the same reasons. Its value is 0 if the average possibility is $\leq 10\%$ and its value is -1 if the average possibility is $\geq 80\%$. So the negative membership value of each edge lies in $(-1, 0)$ if the average possibility lies between $> 10\%$ and $< 80\%$.

For the bipolar fuzzy graph G in Figure 6, the arcs $(k, l), (l, b), (b, c), (k, c), (c, s)$, (s, r) are strong arcs. The paths $r - s - c - b - l - k$ and $r - s - c - k$ are only two strong paths from r to k. So $B.F.D_g(r, k) = 5$ and $B.F.d_g(r, k) = 3$. So the path $r - s - c - k$ is the shortest strong path from Railway station to Keranitola. If a person wants to go from Railway station to Keranitola in shortest path with best average internet speed, then for him the path $r - s - c - k$ will be the best route to go.

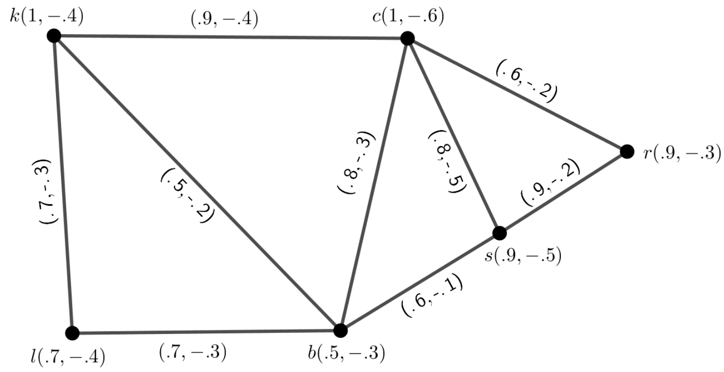

Figure 6. Bipolar fuzzy graph G corresponding to the Wi-Fi network in a town

8.2. Modeling of wireless sensor network in terms of bipolar fuzzy graph and determination of its boundary and interior stations

In a wireless sensor network, if the sensor failure or sensor give expansive errors or disconnection of network, then the capability of each station to capture the sense of occurrence and communication between them are uncertain. Here we present a bipolar fuzzy graph G (see Figure 7) which is applied on a wireless sensor network (W.S.N) to determine its boundary and interior station which is shown in Figure 7. The nodes u, v, w, x, y, z of G represents the stations and each edge represents the communication between corresponding stations.

The positive membership value of each nodes of G represents the capability of the station to capture the sense of occurrence. Its value is 0 if the capability is $\leq 5\%$ and its value is 1 if the capability is $\geq 80\%$. So the positive membership value of each node lies in $(0, 1)$ if the capability lies between $> 5\%$ and $< 80\%$. The negative membership value of each node of G represents the disability of the station to capture the sense of occurrence (disability means it gives expansive error, change in sensor position or disconnection of network). Its value is 0 if the disability is $\leq 10\%$ and its value is -1 if the disability is $\geq 75\%$. So the negative membership value of each node lies in $(-1, 0)$ if the disability lies between $> 10\%$ and $< 75\%$. The positive membership value of each edge represents the ability to communicate of two corresponding stations. Its value is 0 if the ability is $\leq 25\%$ and its value is 1 if the ability is $\geq 70\%$. So the positive membership value of each edge lies in $(0, 1)$ if the ability lies between $> 25\%$ and $< 70\%$. The negative membership value of each edge represents the disability to communicate of two corresponding stations. Its value is 0 if the disability is $\leq 30\%$ and its value is -1 if the disability is $\geq 80\%$. So the negative membership value of each edge lies in $(-1, 0)$ if the disability lies between $> 30\%$ and $< 80\%$.

In W.S.N connecting and covering the whole area are very essential. If the sensor failure or sensor give expansive errors or disconnection of network to coverage the whole area, then we have to find out the boundary stations and interior stations of the W.S.N, which is equivalent to determine the bipolar fuzzy detour g-boundary and g-interior nodes of G.

For the bipolar fuzzy graphs in Figure 7, $B.F.D_g(u, v) = 1$, $B.F.D_g(u, w) = 3$, $B.F.D_g(u, x) = 4$, $B.F.D_g(u, y) = 1$, $B.F.D_g(u, z) = 3$, $B.F.D_g(v, w) = 4$, $B.F.D_g(v, x) = 5$, $B.F.D_g(v, y) = 2$, $B.F.D_g(v, z) = 4$, $B.F.D_g(w, x) = 1$, $B.F.D_g(w, y) = 2$, $B.F.D_g(w, z) = 2$, $B.F.D_g(x, y) = 3$, $B.F.D_g(x, z) = 3$, $B.F.D_g(y, z) = 2$. So $u'_{B.F.D_g} = \{x\}$, $v'_{B.F.D_g} = \{x\}$, $w'_{B.F.D_g} = \{v\}$, $x'_{B.F.D_g} = \{v\}$, $y'_{B.F.D_g} = \{x\}$, $z'_{B.F.D_g} = \{v\}$.

Therefore v, x are bipolar fuzzy detour g-boundary nodes and u, w, x, y are bipolar fuzzy detour g-interior nodes of G. Hence the stations at v, x are boundary stations and the stations u, w, y, z are the interior stations of the W.S.N.

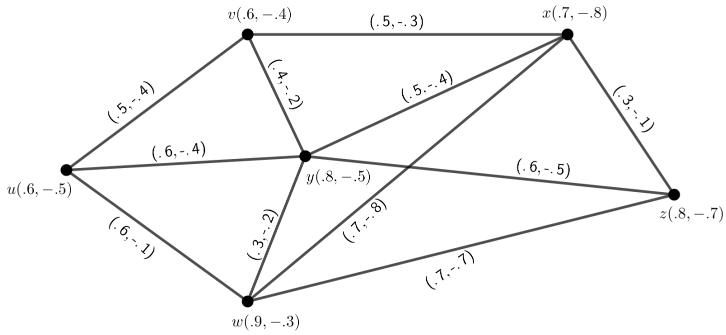

Figure 7. Bipolar fuzzy graph G corresponding to the W.S.N

9. Conclusion

An extensive number of application of fuzzy graph in manifold field of real life. The bipolar fuzzy graph give more exactness and malleable and it has application of the notion of the bipolar fuzzy graph in many cases. In geodesics using fuzzy graph and fuzzy detour graph we get more useful results in various field including electrical networks, computer science, engineering, operation research, optimization, wireless sensor network etc. In this article, we have introduced detour g-distance, detour g-boundary nodes, detour g-interior nodes in bipolar fuzzy graphs and properties of these. We initiated theorems on detour g-interior node, detour g-boundary node, cut node in bipolar fuzzy graph, using maximum bipolar fuzzy spanning tree. Applications of our research work are shown in Wi-Fi network in a town and in wireless sensor network.

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