Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 147 – 161
DOI : 10.15672/HJMS.2019.668

Research Article

# Characterization of automorphisms of Hom-biproducts

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## Abstract

We study certain subgroups of the full group of monoidal Hom-Hopf algebra automorphisms of a Hom-biproduct, which gives a Hom-version of Radford's results.

**Mathematics Subject Classification (2010).** 16W30

**Keywords.** monoidal Hom-Hopf algebra, Hom-biproduct, automorphism

## 1. Introduction

In the theory of the classical Hopf algebras, Radford's biproducts are very important Hopf algebras, which play a central role in the theory of classification of pointed Hopf algebra [1] and account for many examples of semisimple Hopf algebra. There has been many generalizations of Radford's biproducts such as [2] for quasi-Hopf algebra case, [9] for multiplier Hopf algebra case and [13] for monoidal Hom-Hopf algebra.

Let $B \times H$ be the Radford's biproduct, where $B$ is both a left $H$-module algebra and a left $H$-comodule coalgebra. Define $\pi : B \times H \to H$, $\pi(b \times h) = \varepsilon_B(b)h$ and $j(h) = 1_B \times h$, let $\mathrm{Aut}_{\mathrm{Hopf}}(B \times H, \pi)$ be the set of Hopf algebra automorphisms $F$ of $B \times H$ satisfying $\pi \circ F = \pi$. Radford [17] characterized the element of $\mathrm{Aut}_{\mathrm{Hopf}}(B \times H, \pi)$, and factorized $F \in \mathrm{Aut}_{\mathrm{Hopf}}(B \times H, \pi)$ into two suitable maps. Motivated by the idea in [17], the study of automorphisms of Radford's Hom-biproducts introduced in [13] is the focus of this paper.

This paper is organized as follows. In Section 2, we recall some definitions and basic results related to monoidal Hom-algebras, Hom-coalgebras, Hom-bialgebras (Hopf algebras), Hom-(co)module, Hom-module algebras, Hom-smash (co)products and Hom-biproducts.

In Section 3, we study the automorphisms of Radford's Hom-biproducts and show that the automorphism has a factorization closely related to the factors $B$ and $H$ of Radford's Hom-biproduct $B \times H$ in [13]. Finally, we characterize the automorphisms of a concrete example.

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Received: 15.11.2017; Accepted: 02.10.2018

## 2. Preliminaries

Throughout this paper, $k$ will be a field. More materials about monoidal Hom-(co)algebra, monoidal Hopf Hom-algebra, etc. can be found in ([3–8, 10–16, 18–20]). We denote $id_M$ for the identity map from $M$ to $M$.

Let $\mathcal{M} = (\mathcal{M}, \otimes, k, a, l, r)$ be the monoidal category of vector spaces over $k$. We can construct a new monoidal category $\mathcal{H}(\mathcal{M})$ whose objects are ordered pairs $(M, \mu)$ with $M \in \mathcal{M}$ and $\mu \in \mathrm{Aut}(M)$ and morphisms $f : (M, \mu) \to (N, \nu)$ are morphisms $f : M \to N$ in $\mathcal{M}$ satisfying $\nu \circ f = f \circ \mu$. The monoidal structure is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$ and $(k, id_k)$. All monoidal Hom-structures are objects in the tensor category $\widetilde{\mathcal{H}}(\mathcal{M}) = (\mathcal{H}(\mathcal{M}), \otimes, (k, id_k), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in [3] with the associativity and unit constraints given by

$$\widetilde{a}_{M,N,C}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \gamma^{-1}(c)),$$
$$\widetilde{l}(x \otimes m) = \widetilde{r}(m \otimes x) = x\mu(m),$$

for $(M, \mu), (N, \nu)$ and $(C, \gamma)$. The category $\widetilde{\mathcal{H}}(\mathcal{M})$ is termed Hom-category associated to $\mathcal{M}$. In the following, we recall some definitions about Hom-structures from [3] and [13].

### 2.1. Monoidal Hom-algebra

A monoidal Hom-algebra is an object $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with linear maps $m_A : A \otimes A \to A$, $m_A(a \otimes b) = ab$ and $\eta_A : k \to A$ such that

$$\alpha(ab) = \alpha(a)\alpha(b), \alpha(a)(bc) = (ab)\alpha(c), \tag{2.1}$$
$$\alpha(\eta(1)) = \eta(1),\ a\eta(1) = \alpha(a) = \eta(1)a, \tag{2.2}$$

for all $a, b, c \in A$. We shall write $\eta_A(1) = 1_A$.

A left $(A, \alpha)$-Hom-module consists of an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\psi : A \otimes M \to M, \psi(a \otimes m) = a \cdot m$ satisfying the following conditions:

$$(ab) \cdot \mu(m) = \alpha(a) \cdot (b \cdot m), 1_A \cdot m = \mu(m), \tag{2.3}$$

for all $m \in M$ and $a, b \in A$. For $\psi$ to be a morphism in $\widetilde{\mathcal{H}}(\mathcal{M})$, one needs

$$\mu(a \cdot m) = \alpha(a) \cdot \mu(m). \tag{2.4}$$

We call that $\psi$ is a left Hom-action of $(A, \alpha)$ on $(M, \mu)$.

Let $(M, \mu)$ and $(M', \mu')$ be two left $(A, \alpha)$-Hom-modules. We call a morphism $f : M \to M'$ right $(A, \alpha)$-linear, if $f \circ \mu = \mu' \circ f$ and $f(a \cdot m) = a \cdot f(m)$. $\widetilde{\mathcal{H}}({}_A\mathcal{M})$ denotes the category of all left $(A, \alpha)$-Hom-modules.

### 2.2. Monoidal Hom-coalgebras

A monoidal Hom-coalgebra is an object $(C, \gamma) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with two linear maps $\Delta_C : C \to C \otimes C$, $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\varepsilon_C : C \to k$ such that

$$\gamma^{-1}(c_{(1)}) \otimes \Delta_C(c_{(2)}) = c_{(1)(1)} \otimes (c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)})), \Delta_C(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}), \tag{2.5}$$
$$\varepsilon_C(\gamma(c)) = \varepsilon_C(c),\ c_{(1)}\varepsilon_C(c_{(2)}) = \gamma^{-1}(c) = \varepsilon_C(c_{(1)})c_{(2)}, \tag{2.6}$$

for all $c \in C$.

A left $(C, \gamma)$-Hom-comodule consists of an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\rho_M : M \to C \otimes M, \rho_M(m) = m_{[-1]} \otimes m_{[0]}$ (summation implicitly understood) satisfying the following conditions:

$$\Delta_C(m_{[-1]}) \otimes \mu^{-1}(m_{[0]}) = \gamma^{-1}(m_{[-1]}) \otimes (m_{[0][-1]} \otimes m_{[0][0]}), \tag{2.7}$$
$$\varepsilon_C(m_{[-1]})m_{[0]} = \mu^{-1}(m), \tag{2.8}$$
$$\rho_M(\mu(m)) = \gamma(m_{[-1]}) \otimes \mu(m_{[0]}), \tag{2.9}$$

for all $m \in M$. We call that $\rho_M$ is a left Hom-coaction of $(A, \alpha)$ on $(M, \mu)$.

Let $(M, \mu)$ and $(M', \mu')$ be two left $(C, \gamma)$-Hom-comodules. We call a morphism $f : M \to M'$ left $(C, \gamma)$-colinear, if $f \circ \mu = \mu' \circ f$ and $f(m)_{[0]} \otimes f(m)_{[-1]} = f(m_{[0]}) \otimes m_{[-1]}$. $\widetilde{\mathcal{H}}({}^C\mathcal{M})$ denotes the category of all left $(C, \gamma)$-Hom-comodules.

## 2.3. Monoidal Hom-Hopf algebra

A monoidal Hom-bialgebra $H = (H, \beta, m_H, 1_H, \Delta_H, \varepsilon_H)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M})$. This means that $(H, \beta, m_H, 1_H)$ is a monoidal Hom-algebra and $(H, \beta, \Delta_H, \varepsilon_H)$ is a monoidal Hom-coalgebra such that $\Delta_H$ and $\varepsilon_H$ are Hom-algebra maps, that is, for any $h, g \in H$,

$$\Delta_H(hg) = \Delta_H(h)\Delta_H(g), \Delta_H(1_H) = 1_H \otimes 1_H, \tag{2.10}$$

$$\varepsilon_H(hg) = \varepsilon_H(h)\varepsilon_H(g), \varepsilon_H(1_H) = 1. \tag{2.11}$$

A monoidal Hom-bialgebra $(H, \beta)$ is called a monoidal Hom-Hopf algebra, if there exists a morphism (called the Hom-antipode) $S_H : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ such that

$$S_H(h_{(1)})h_{(2)} = \varepsilon_H(h)1_A = h_{(1)}S(h_{(2)}), \tag{2.12}$$

for all $h \in H$.

## 2.4. Hom-module algebra

Let $(H, \beta)$ be a monoidal Hom-bialgebra. A monoidal Hom-algebra $(B, \alpha)$ is called a left $(H, \beta)$-Hom-module algebra, if $(B, \alpha)$ is a left $(H, \beta)$-Hom-module with the action $\cdot$ obeying the following axioms:

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b),\ h \cdot 1_B = \varepsilon_H(h)1_B, \tag{2.13}$$

for all $a, b \in A$ and $h \in H$.

Let $(B, \alpha)$ be a left $(H, \beta)$-Hom-module algebra. The Hom-smash product $(B\sharp H, \alpha\sharp\beta)$ of $(B, \alpha)$ and $(H, \beta)$ is defined as follows, for all $a, b \in B$, $h, g \in H$,

- as $k$-space, $B\sharp H = B \otimes H$,
- Hom-multiplication is given by

$$(a\sharp h)(b\sharp g) = a(h_{(1)} \cdot \alpha^{-1}(b))\sharp\beta(h_{(2)})g.$$

$(B\sharp H, 1_B\sharp 1_H, \alpha \otimes \beta)$ is a monoidal Hom-algebra.

## 2.5. Hom-comodule coalgebra

Let $(H, \beta)$ be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra $(B, \alpha)$ is called a left $(H, \beta)$-Hom-comodule coalgebra, if $(B, \alpha)$ is a left $(H, \beta)$-Hom-comodule with the coaction $\rho_B(b) = b_{[-1]} \otimes b_{[0]}$ obeying the following axioms:

$$b_{[-1]} \otimes \Delta_B(b_{[0]}) = b_{(1)[-1]}b_{(2)[-1]} \otimes b_{(1)[0]} \otimes b_{(2)[0]},\ b_{[-1]}\varepsilon_B(b_{[0]}) = \varepsilon_B(b)1_H, \tag{2.14}$$

for all $b \in B$.

Let $(B, \alpha)$ be a left $(H, \beta)$-Hom-comodule cocalgebra. The Hom-smash coproduct $(B\natural H, \alpha\natural\beta)$ of $(B, \alpha)$ and $(H, \beta)$ is defined as follows, for all $a, b \in B$, $h, g \in H$,

- as $k$-space, $B\natural H = B \otimes H$,
- Hom-comultiplication is given by

$$\Delta(b\natural h) = (b_{(1)}\natural b_{(2)[-1]}\beta^{-1}(h_{(1)})) \otimes (\alpha(b_{(2)[0]})\natural h_{(2)}).$$

$(B\natural H, \Delta, \varepsilon_B \otimes \varepsilon_H, \alpha \otimes \beta)$ is a monoidal Hom-coalgebra.

### 2.6. Hom-comodule algebra

Let $(H,\beta)$ be a monoidal Hom-bialgebra. A monoidal Hom-algebra $(B,\alpha)$ is called a left $(H,\beta)$-Hom-comodule algebra, if $(B,\alpha)$ is a left $(H,\beta)$-Hom-comodule with the coaction $\rho_B$ obeying the following axioms:

$$\rho_B(ab) = a_{[-1]}b_{[-1]} \otimes a_{[0]}b_{[0]},\ \rho_B(1_B) = 1_H \otimes 1_B, \tag{2.15}$$

for all $a,b \in B$.

### 2.7. Hom-module coalgebra

Let $(H,\beta)$ be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra $(B,\alpha)$ is called a left $(H,\beta)$-Hom-module coalgebra, if $(B,\alpha)$ is a left $(H,\beta)$-Hom-module with the action $\cdot$ obeying the following axioms:

$$\Delta_B(h\cdot b) = h_{(1)}\cdot b_{(1)} \otimes h_{(2)}\cdot b_{(2)},\ \varepsilon_B(h\cdot b) = \varepsilon_H(h)\varepsilon_B(b), \tag{2.16}$$

for all $b \in B$ and $h \in H$.

### 2.8. Radford's Hom-biproduct

Recall from [Theorem 3.5] that the vector space $B\otimes H$ with the Hom-smash product structure and the Hom-smash coproduct structure is a monoidal Hom-bialgebra if and only if the following conditions hold:

- $\varepsilon_B$ is an algebra map and $\Delta_B(1_B) = 1_B \otimes 1_B$,
- $(B,\alpha)$ is a left $(H,\beta)$-Hom-module coalgebra,
- $(B,\alpha)$ is a left $(H,\beta)$-Hom-comodule algebra,
- for $a,b \in B$,

$$\Delta_B(ab) = a_{(1)}(a_{(2)[-1]}\cdot \alpha^{-1}(b_{(1)})) \otimes \alpha(a_{(2)[0]})b_{(2)} \tag{2.17}$$

- for all $h \in H$ and $b \in B$,

$$(h_{(1)}\cdot \alpha^{-1}(b))_{[-1]}h_{(2)} \otimes \alpha((h_{(1)}\cdot\alpha^{-1}(b))_{[0]}) = h_{(1)}b_{[-1]} \otimes h_{(2)}\cdot b_{[0]}. \tag{2.18}$$

Under the assumption that $(H,S_H)$ is a monoidal Hom-Hopf algebra and $id_B$ has a convolution inverse in $\mathrm{End}(B)$, $B\otimes H$ is a monoidal Hom-Hopf algebra. The monoidal Hom-Hopf algebra$(B\otimes H,\alpha\otimes\beta)$ is called the Radford's Hom-biproduct and is denoted by $B\times H$.

## 3. Factorization of certain biproduct endomorphisms

Let $(B\times H,\alpha\otimes\beta)$ be the Radford's Hom-biproduct. We define $\pi : B\times H \to H$ by $\pi(b\times h) = \varepsilon_B(b)h$ for $b\in B$ and $h \in H$ and $j : H \to B\times H$ by $j(h) = 1_B \times h$ for $h\in H$ are monoidal Hom-Hopf algebra maps which satisfy $\pi\circ j = id_H$. Let $\mathrm{End}_{\text{Hom-Hopf}}(B\times H, H, \pi)$ be the set of all monoidal Hom-Hopf algebra endomorphisms $F$ of $A$ such that $\pi\circ F = \pi$ and let $\mathrm{Aut}_{\text{Hom-Hopf}}(B\times H,H,\pi)$ be its set of units. Thus $\mathrm{Aut}_{\text{Hom-Hopf}}(B\times H,H,\pi)$ is the group of monoidal Hom-Hopf algebra automorphisms $F$ of $B\times H$ such that $\pi\circ F = \pi$ under composition. We will write $\mathrm{End}_{\text{Hom-Hopf}}(B\times H,\pi)$ for $\mathrm{End}_{\text{Hom-Hopf}}(B\times H,H,\pi)$, and $\mathrm{Aut}_{\text{Hom-Hopf}}(B\times H,\pi)$ for $\mathrm{Aut}_{\text{Hom-Hopf}}(B\times H,H,\pi)$. The pupose of this section is to show that $F$ has a factorization closely related to the factors $B$ and $H$ of $B\times H$.

We define $\Pi : B\times H \to B$ and $J : B \to B\times H$ by $\Pi(b\times h) = b\varepsilon_H(h)$, for all $b\in B, h\in H$ and $J(b) = b\times 1_H$, for all $b\in B$. There is a fundamental relationship between these four maps given by:

$$J\circ\Pi = id_{B\times H} \star (j\circ S_H\circ\pi). \tag{3.1}$$

The factorization of $F$ is given in terms of $F_l : B \to B$ and $F_r : H \to B$ defined by:

$$F_l = \Pi\circ F\circ J \quad \text{and} \quad F_r = \Pi\circ F\circ j. \tag{3.2}$$

First, we shall reveal the relationships among $F$, $F_l$ and $F_r$ in the following lemma.

**Lemma 3.1.** *Let $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then:*

$$F_l(b) \times 1_H = F(b \times 1_H), \tag{3.3}$$

$$F_r(h) \times 1_H = F(1_B \times h_{(1)})(1_B \times S_H(h_{(2)})), \tag{3.4}$$

$$F(b \times h) = F_l(\alpha^{-1}(b))F_r(h_{(1)}) \times \beta(h_{(2)}), \tag{3.5}$$

*for all $b \in B$ and $h \in H$.*

***Proof.*** We need to calculate $J \circ \Pi \circ F$. For $b \in B$ and $h \in H$, we use (3.1) to compute

$$\begin{aligned}(J \circ \Pi)(F(b \times h)) &= F((b \times h)_{(1)})(j \circ S_H \circ \pi)(F((b \times h)_{(2)}))\\ &= F((b \times h)_{(1)})(j \circ S_H \circ \pi)((b \times h)_{(2)})\\ &= F(b_{(1)} \times b_{(2)[-1]}\beta^{-1}(h_{(1)}))(j \circ S_H \circ \pi)(\alpha(b_{(2)[0]}) \times h_{(2)})\\ &= F(\alpha^{-1}(b) \times h_{(1)})(1_A \times S_H(h_{(2)})).\end{aligned}$$

It follows that

$$(J \circ \Pi \circ F)(b \times h) = F(\alpha^{-1}(b) \times h_{(1)})(1_B \times S_H(h_{(2)})),$$

for all $b \in B$ and $h \in H$. Equations (3.3) and (3.4) follow from the above equation. As for (3.5), we calculate

$$\begin{aligned}&F(b \times h)\\ &= F(\alpha^{-1}(b) \times 1_H)F(1_B \times \beta^{-1}(h))\\ &= F(\alpha^{-1}(b) \times 1_H)[F(1_B \times \beta^{-1}(h_{(1)}))((1_B \times S_H(\beta^{-1}(h_{(2)(1)})))(1_B \times \beta^{-1}(h_{(2)(2)})))]\\ &= F(\alpha^{-1}(b) \times 1_H)[(F(1_B \times \beta^{-2}(h_{(1)}))(1_B \times S_H(\beta^{-1}(h_{(2)(1)}))))(1_B \times h_{(2)(2)})]\\ &= F(\alpha^{-1}(b) \times 1_H)[(F(1_B \times \beta^{-1}(h_{(1)(1)}))(1_B \times S_H(\beta^{-1}(h_{(1)(2)}))))(1_B \times \beta^{-1}(h_{(2)}))]\\ &= [F(\alpha^{-2}(b) \times 1_H)(F(1_B \times \beta^{-1}(h_{(1)(1)}))(1_B \times S_H(\beta^{-1}(h_{(1)(2)}))))](1_B \times h_{(2)})\\ &= [F_l(\alpha^{-2}(b) \times 1_H)(F_r(\beta^{-1}(h_{(1)})) \times 1_H)](1_B \times h_{(2)})\\ &= F_l(\alpha^{-1}(b))F_r(h_{(1)}) \times \beta(h_{(2)}),\end{aligned}$$

as desired. □

By (3.3) and (3.4) of Lemma 3.1:

$$(id_{B\times H})_l = id_B \quad \text{and} \quad (id_{B\times H})_r = \eta_B \circ \varepsilon_H. \tag{3.6}$$

Since $F_l(1_B) = 1_B$ by (3.3) of Lemma 3.1. By (3.5) of Lemma 3.1:

$$F(1_B \times h) = F_r(\beta(h_{(1)})) \times \beta(h_{(2)}), \tag{3.7}$$

for all $h \in H$. We are now able to compute the factors of a composite.

**Corollary 3.2.** *Let $F, G \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then*

(1) $(F \circ G)_l = F_l \circ G_l$,
(2) $(F \circ G)_r = (F_l \circ G_r) \star F_r$

***Proof.*** For $b \in B$, by (3.3) of Lemma 3.1, we have

$$(F \circ G)_l(b) \times 1_H = (F \circ G)(b \times 1) = F(G_l(b) \times 1_H) = (F_l \circ G_l)(b) \times 1_H.$$

Thus, it follows that part (1) holds. Let $h \in H$. Using (3.7), the fact that $F$ is multiplicative, and part (1) of (3.3), we obtain that:

$$\begin{aligned}&(F\circ G)_r(\beta(h_{(1)})) \times \beta(h_{(2)})\\ (3.7)\ &=(F \circ G)(1_B \times h)\\ (3.7)\ &=F(G_r(\beta(h_{(1)})) \times \beta(h_{(2)}))\\ &=F(G_r(h_{(1)}) \times 1_H)F(1_B \times h_{(2)})\end{aligned}$$

$$\begin{aligned}(3.3) =&(F_lG_r(h_{(1)}) \times 1_H)(F_r(\beta(h_{(2)(1)})) \times \beta(h_{(2)(2)}))\\ =&F_lG_r(h_{(1)})F_r(\beta(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})\\ =&F_lG_r(\beta(h_{(1)(1)}))F_r(\beta(h_{(1)(2)})) \times \beta(h_{(2)}),\end{aligned}$$

i.e.,

$$(F \circ G)_r(\beta(h_{(1)})) \times \beta(h_{(2)}) = F_lG_r(\beta(h_{(1)(1)}))F_r(\beta(h_{(1)(2)})) \times \beta(h_{(2)}).$$

Applying $id_B \otimes \varepsilon_H$ to both sides of the above equation, we can get part (2). □

By virtue of Lemma 3.1, to characterize $F$ is a matter of characterizing $F_l$ and $F_R$. Note in particular part (5) of the following describes a commutation relation between $F_l$ and $F_R$. First, we shall characterize $F_l$ in the following lemma.

**Lemma 3.3.** *Let* $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$*. Then:*

(1) $F_l : B \to B$ *is a monoidal Hom-algebra endomorphism.*

(2) $\varepsilon_B \circ F_l = \varepsilon_B$.

(3) *for all* $b \in B$,

$$\Delta(F_l(b)) = F_l(\alpha^{-1}(b_{(1)}))F_r(b_{(2)[-1]}) \otimes F_l(\alpha(b_{(2)[0]})), \tag{3.8}$$

(4) *for all* $b \in B$,

$$\rho(F_l(b)) = b_{[-1]} \otimes F_l(b_{[0]}), \tag{3.9}$$

(5) *for all* $b \in B$ *and* $h \in H$,

$$F_l(h_{(1)} \cdot b)F_r(\beta(h_{(2)})) = F_r(\beta(h_{(1)}))(h_{(2)} \cdot F_l(b)). \tag{3.10}$$

***Proof.*** In order to prove (1), we need to check three aspects. From the above discussion, we have known that $F_l(1_B) = 1_B$. It is easy to check that $F_l \circ \alpha = \alpha \circ F_l$. Finally, we shall check that $F_l$ preserves the multiplication. In fact, for $a, b \in B$, we have

$$\begin{aligned}F_l(ab) =&(\Pi \circ F \circ J)(ab)\\ =&\Pi(F(ab \times 1_H)) = \Pi(F(a \times 1_H)F(b \times 1_H))\\ (3.3) =&\Pi((F_l(a) \times 1_H)(F_l(b) \times 1_H))\\ =&F_l(a)F_l(b).\end{aligned}$$

It is easy to check part (2). Next, we shall check that parts (3) and (4) hold. As a matter of fact, we compute the coproduct of $F_l(b) \times 1_H = F(b \times 1_H)$ in two ways. First of all,

$$\Delta(F_l(b) \times 1_H) = (F_l(b)_{(1)} \times \beta(F_l(b)_{(2)[-1]})) \otimes (\alpha(F_l(b)_{(2)[0]}) \times 1_H)$$

and secondly, since $F$ is a coalgebra map, we have

$$\begin{aligned}&\Delta(F(b \times 1_H))\\ &= F((b \times 1_H)_{(1)}) \otimes F((b \times 1_H)_{(2)})\\ &= F(b_{(1)} \times \beta(b_{(2)[1]})) \otimes F(\alpha(b_{(2)[0]}) \times 1_H)\\ &= [F_l(\alpha^{-1}(b_{(1)}))F_r(\beta(b_{(2)[-1](1)})) \times \beta^2(b_{(2)[-1](2)})] \otimes (F_l(\alpha(b_{(2)[0]})) \times 1_H).\end{aligned}$$

It follows that

$$\begin{aligned}&(F_l(b)_{(1)} \times \beta(F_l(b)_{(2)[-1]})) \otimes (\alpha(F_l(b)_{(2)[0]}) \times 1_H)\\ &=[F_l(\alpha^{-1}(b_{(1)}))F_r(\beta(b_{(2)[-1](1)})) \times \beta^2(b_{(2)[-1](2)})]. \otimes (F_l(\alpha(b_{(2)[0]})) \times 1_H)\end{aligned} \tag{3.11}$$

Applying $id_B \otimes \varepsilon_H \otimes id_B \otimes \varepsilon_H$ to both sides of (3.11) yields (3.8). It follows easily that $\varepsilon_B \circ F_r = \varepsilon_B$ from (3.7). Applying $\varepsilon_B \otimes id_H \otimes id_B \otimes \varepsilon_H$ to both sides of (3.11) again, we can gain (3.9).

Finally, it is left to us to check part (5). Indeed, for $b \in B$ and $h \in H$, we have

$$F((1_B \times h)(b \times 1_H)) = F((\beta(h_{(1)}) \cdot b) \times \beta^2(h_{(2)})) = F_l(h_{(1)} \cdot \alpha^{-1}(b))F_r(\beta^2(h_{(2)(1)})) \times \beta^3(h_{(2)(2)}).$$

On the other hand, Since $F$ preserves the multiplication, we compute:

$$\begin{aligned}F((1_B\times h)(b\times 1_H)) &= F(1_B\times h)F(b\times 1_H)\\ &= (F_r(\beta(h_{(1)}))\times\beta(h_{(2)}))(F_l(b)\times 1_H)\\ &= F_r(\beta(h_{(1)}))(\beta(h_{(2)(1)})\cdot F_l(\alpha^{-1}(b)))\times\beta^3(h_{(2)(2)}).\end{aligned}$$

Applying $id_B\otimes\varepsilon_H$ to both expressions for $F((1_B\times h)(b\times 1_H))$, we obtain (3.10). □

As the reader might suspect, whether or not $F_l$ is a coalgebra map is explained in terms of $F_R$.

**Corollary 3.4.** *Let $F\in\mathrm{End}_{\text{Hom-Hopf}}(B\times H,\pi)$. Then, $F_l$ is a monoidal Hom-coalgebra if and only if $F_r(c_{[-1]})\otimes c_{[0]}=1_B\otimes\alpha^{-1}(c)$, for all $c\in\mathrm{Im}(F_l)$.*

***Proof.*** Suppose $F_r(c_{[-1]})\otimes c_{[0]}=1_B\otimes\alpha^{-1}(c)$, for all $c\in Im(F_l)$. Then we have

$$\begin{aligned}\Delta(F_l(b)) =&F_l(\alpha^{-1}(b_{(1)}))F_r(b_{(2)[-1]})\otimes F_l(\alpha(b_{(2)[0]}))\\ (3.9)\ =&F_l(\alpha^{-1}(b_{(1)}))F_r(F_l(b_{(2)})_{[-1]})\otimes\alpha(F_l(b_{(2)})_{[0]})\\ =&F_l(b_{(1)})\otimes F_l(b_{(2)}).\end{aligned}$$

Conversely, suppose that $F_l$ is a monoidal Hom-coalgebra map. For all $b\in B$, we compute

$$\begin{aligned}&F_r(F_l(b)_{[-1]})\otimes F_l(b)_{[0]}\\ (3.9)\ =&F_r(b_{[-1]})\otimes F_l(b_{[0]})\\ =&F_l(\varepsilon_B(\alpha^{-1}(b_{(1)}))1_B)F_r(b_{(2)[-1]})\otimes F_l(\alpha(b_{(2)[0]}))\\ =&F_l(S_B(\alpha^{-1}(b_{(1)(1)}))\alpha^{-1}(b_{(1)(2)}))F_r((b_{(2)[-1]})\otimes F_l(\alpha(b_{(2)[0]}))\\ =&[F_l(S_B(\alpha^{-1}(b_{(1)(1)})))F_l(\alpha^{-1}(b_{(1)(2)}))]F_r((b_{(2)[-1]})\otimes F_l(\alpha(b_{(2)[0]}))\\ (2.5)\ =&[F_l(S_B(\alpha^{-2}(b_{(1)})))F_l(\alpha^{-1}(b_{(2)(1)}))]F_r(\beta(b_{(2)(2)[-1]}))\otimes F_l(\alpha^2(b_{(2)(2)[0]}))\\ (2.1)\ =&F_l(S_B(\alpha^{-1}(b_{(1)})))[F_l(\alpha^{-1}(b_{(2)(1)}))F_r(b_{(2)(2)[-1]})]\otimes\alpha(F_l(\alpha(b_{(2)(2)[0]})))\\ (3.8)\ =&F_l(S_B(\alpha^{-1}(b_{(1)})))F_l(b_{(2)})_{(1)}\otimes\alpha(F_l(b_{(2)})_{(2)})\\ =&F_l(S_B(\alpha^{-1}(b_{(1)})))F_l(b_{(2)(1)})\otimes F_l(\alpha(b_{(2)(2)}))\\ =&F_l(S_B(b_{(1)(1)}))F_l(b_{(1)(2)})\otimes F_l(b_{(2)})\\ =&1_B\otimes\alpha^{-1}(F_l(b)),\end{aligned}$$

as desired. □

From Lemma 3.3, we have characterize the conditions that $F_l$ satisfies. It is left to us to characterize $F_r$ as follows.

**Lemma 3.5.** *Let $F\in\mathrm{End}_{\text{Hom-Hopf}}(B\times H,\pi)$. Then,*

(1) *$F_r(1_H)=1_B$.*
(2) *for all $h,g\in H$,*

$$F_r(hg)=F_r(\beta(h_{(1)}))(h_{(2)}\cdot F_r(\beta^{-1}(g))), \tag{3.12}$$

(3) *$F_r: H\to B$ is a monoidal Hom-coalgebra map,*
(4) *for all $h\in H$,*

$$\rho(F_r(h))=h_{(1)(1)}S(\beta^{-1}(h_{(2)}))\otimes F_r(\beta(h_{(1)(2)})). \tag{3.13}$$

***Proof.*** It is easy to check part (1). We shall check that part (2) holds, for all $h,g\in H$, we calculate on one hand,

$$F(1_B\times hg)=F_r(\beta(h_{(1)})\beta(g_{(1)}))\times\beta(h_{(2)})\beta(g_{(2)}),$$

and on the other hand,

$$\begin{aligned}F(1_B\times hg)&=F(1_B\times h)F(1_B\times g)\\&=(F_r(\beta(h_{(1)}))\times\beta(h_{(2)}))(F_r(\beta(g_{(1)}))\times\beta(g_{(2)}))\\&=F_r(\beta(h_{(1)}))(\beta(h_{(2)(1)})\cdot F_r(g_{(1)}))\times\beta^2(h_{(2)(2)})\beta(g_{(2)}).\end{aligned}$$

Applying $id_B\otimes\varepsilon_H$ to both expressions for $F(1_B\times hg)$, it follows that (3.12) holds.

Let $h\in H$. To show parts (3) and (4), we compute $\Delta(F(1_B\times h))$ in two ways as follows.

$$\begin{aligned}\Delta(F(1_B\times h))&=F(1_B\times h_{(1)})\otimes F(1_B\times h_{(2)})\\&=(F_r(\beta(h_{(1)(1)}))\times\beta(h_{(1)(2)}))\otimes(F_r(\beta(h_{(2)(1)}))\times\beta(h_{(2)(2)})).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta(F(1_B\times h))=&\Delta(F_r(\beta(h_{(1)}))\times\beta(h_{(2)}))\\=&(F_r(\beta(h_{(1)}))_{(1)}\times F_r(\beta(h_{(1)}))_{(2)[-1]}h_{(2)(1)})\\&\otimes(\alpha(F_r(\beta(h_{(1)}))_{(2)[0]})\times\beta(h_{(2)(2)})).\end{aligned}$$

Applying $id_B\otimes\varepsilon_H\otimes id_B\otimes\varepsilon_H$ to the expressions for $\Delta(F(1_B\times h))$ gives part (2). Applying $\varepsilon_B\otimes id_H\otimes id_B\otimes\varepsilon_H$ to the expressions for $\Delta(F(1_B\times h))$ again yields

$$\beta(h_{(1)})\otimes F_r(h_{(2)})=F_r(\beta(h_{(1)}))_{[-1]}h_{(2)}\otimes F_r(\beta(h_{(1)}))_{[0]}. \tag{3.14}$$

Therefore,

$$\begin{aligned}\rho(F_r(h))=&\beta^{-1}(F_r(\beta(h_{(1)}))_{[-1]})\varepsilon_H(\beta^{-1}(h_{(2)}))1_H\otimes F_r(\beta(h_{(1)}))_{[0]}\\=&\beta^{-1}(F_r(\beta(h_{(1)}))_{[-1]})[\beta^{-1}(h_{(2)(1)})S(\beta^{-1}(h_{(2)(2)}))]\otimes F_r(\beta(h_{(1)}))_{[0]}\\(2.1)=&[\beta^{-2}(F_r(\beta(h_{(1)}))_{[-1]})\beta^{-1}(h_{(2)(1)})]S(h_{(2)(2)})\otimes F_r(\beta(h_{(1)}))_{[0]}\\(2.5)=&[\beta^{-2}(F_r(\beta^2(h_{(1)(1)}))_{[-1]})\beta^{-1}(h_{(1)(2)})]S(\beta^{-1}(h_{(2)}))\otimes F_r(\beta^2(h_{(1)(1)}))_{[0]}\\=&\beta^{-2}(F_r(\beta^2(h_{(1)(1)}))_{[-1]}\beta(h_{(1)(2)}))S(\beta^{-1}(h_{(2)}))\otimes F_r(\beta^2(h_{(1)(1)}))_{[0]}\\(3.14)=&h_{(1)(1)}S(\beta^{-1}(h_{(2)}))\otimes F_r(\beta(h_{(1)(2)})),\end{aligned}$$

which shows that (3.13) holds. □

**Corollary 3.6.** *Let $F\in\mathrm{End}_{\text{Hom-Hopf}}(B\times H,\pi)$. Then $F_l$ is a left $(H,\beta)$-Hom-module map if and only if the condition $F_l(h_{(1)}\cdot b)F_r(\beta(h_{(2)}))=F_r(\beta(h_{(1)}))F_l(h_{(2)}\cdot b)$. hlods.*

***Proof.*** The necessary condition can be followed easily from (3.10) of Lemma 3.3. Now, we will prove the sufficient part. Suppose that the condition holds. Note that $F_r$ is a Hom-coalgebra map by (3) of Lemma 3.5. Using this fact and (5) of Lemma 3.3, for all $h\in H$ and $b\in B$, we have

$$\begin{aligned}h\cdot F_l(b)=&\varepsilon_B(F_r(\beta(h_{(1)})))1_B(h_{(2)}\cdot F_l(\alpha^{-1}(b)))\\=&(S_B(F_r(\beta(h_{(1)(1)})))F_r(\beta(h_{(1)(2)})))(h_{(2)}\cdot F_l(\alpha^{-1}(b)))\\=&S_B(F_r(\beta^2(h_{(1)(1)})))[F_r(\beta(h_{(1)(2)}))(\beta^{-1}(h_{(2)})\cdot F_l(\alpha^{-2}(b)))]\\(2.5)=&S_B(F_r(\beta(h_{(1)})))[F_r(\beta(h_{(2)(1)}))(h_{(2)(2)}\cdot F_l(\alpha^{-2}(b)))]\\(3.10)=&S_B(F_r(\beta(h_{(1)})))[F_l(h_{(2)(1)}\cdot\alpha^{-2}(b))F_r(\beta(h_{(2)(2)}))]\\=&S_B(F_r(\beta(h_{(1)})))[F_r(\beta(h_{(2)(1)}))F_l(h_{(2)(2)}\cdot\alpha^{-2}(b))]\\=&[S_B(F_r(h_{(1)}))F_r(\beta(h_{(2)(1)}))]F_l(\beta(h_{(2)(2)})\cdot\alpha^{-1}(b))\\(2.5)=&[S_B(F_r(\beta(h_{(1)(1)})))F_r(\beta(h_{(1)(2)}))]F_l(h_{(2)}\cdot\alpha^{-1}(b))\\=&\varepsilon_B(F_r(h_{(1)}))F_l(\beta(h_{(2)})\cdot b)\\=&F_l(h\cdot b),\end{aligned}$$

which shows that $F_l$ is a left $(H,\beta)$-Hom-module map. □

**Lemma 3.7.** *Let $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then, $F_r$ is a monoidal Hom-algebra map if and only if $h \cdot F_r(g) = \varepsilon_H(h)F_r(\beta(g))$, for all $h, g \in H$.*

***Proof.*** Suppose that $F_r$ is a monoidal Hom-algebra map. Using (2) and (3) of Lemma 3.5, for $h, g \in H$, we have:

$$\begin{aligned} h \cdot F_r(g) =& [S(F_r(\beta(h_{(1)(1)})))F_r(\beta(h_{(1)(2)}))](h_{(2)} \cdot F_r(\beta^{-1}(g))) \\ =& S(F_r(\beta^2(h_{(1)(1)})))[F_r(\beta(h_{(1)(2)}))(\beta^{-1}(h_{(2)}) \cdot F_r(\beta^{-2}(g)))] \\ (2.5) =& S(F_r(\beta(h_{(1)})))[F_r(\beta(h_{(2)(1)}))(h_{(2)(2)} \cdot F_r(\beta^{-2}(g)))] \\ (3.12) =& S(F_r(\beta(h_{(1)})))F_r(h_{(2)}\beta^{-1}(g)) \\ =& S(F_r(\beta(h_{(1)})))(F_r(h_{(2)})F_r(\beta^{-1}(g))) \\ =& [S(F_r(h_{(1)}))F_r(h_{(2)})]F_r(g) \\ =& \varepsilon_H(h)F_r(\beta(g)). \end{aligned}$$

If $h \cdot F_r(g) = \varepsilon_H(h)F_r(\beta(g))$ holds, by using (2) of Lemma 3.5, we have

$$\begin{aligned} F_r(hg) &= F_r(\beta(h_{(1)}))(h_{(2)} \cdot F_r(\beta^{-1}(g))) \\ &= F_r(\beta(h_{(1)}))\varepsilon_H(h_{(2)})F_r(g)) \\ &= F_r(h)F_r(g). \end{aligned}$$

□

**Corollary 3.8.** *Let $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then $F_r$ has a convolution inverse $J_r$ defined by $J_r(h) = h_{(1)} \cdot F_r(S_H(h_{(2)}))$*

***Proof.*** Let $h \in H$. Then by parts (1) and (2) of Lemma 3.5, we have

$$\begin{aligned} F_r \star J_r(h) =& F_r(h_{(1)})J_r(h_{(2)}) \\ =& F_r(h_{(1)})(h_{(2)(1)} \cdot F_r(S_H(h_{(2)(2)}))) \\ (2.5) =& F_r(\beta(h_{(1)(1)}))(h_{(1)(2)} \cdot F_r(\beta^{-1}(S_H(h_{(2)})))) \\ (3.12) =& F_r(h_{(1)}S_H(h_{(2)})) \\ =& \varepsilon_H(h)1_B, \end{aligned}$$

and using the fact that $(B,\alpha)$ is a left $(H,\beta)$-Hom-module algebra, we have

$$\begin{aligned} J_r \star F_r(h) =& J_r(h_{(1)})F_r(h_{(2)}) \\ =& (h_{(1)(1)} \cdot F_r(S_H(h_{(1)(2)})))F_r(h_{(2)}) \\ (2.5) =& (\beta(h_{(1)(1)(1)}) \cdot F_r(S_H(h_{(1)(2)}))) \\ & \times ([\beta(h_{(1)(1)(2)(1)})S_H(\beta(h_{(1)(1)(2)(2)}))] \cdot F_r(\beta^{-1}(h_{(2)}))) \\ =& (\beta(h_{(1)(1)(1)}) \cdot F_r(S_H(h_{(1)(2)}))) \\ & \times (\beta^2(h_{(1)(1)(2)(1)}) \cdot (S_H(\beta(h_{(1)(1)(2)(2)})) \cdot F_r(\beta^{-2}(h_{(2)})))) \\ =& (\beta^2(h_{(1)(1)(1)(1)}) \cdot F_r(S_H(h_{(1)(2)}))) \\ & \times (\beta^2(h_{(1)(1)(1)(2)}) \cdot (S_H(h_{(1)(1)(2)}) \cdot F_r(\beta^{-2}(h_{(2)})))) \\ =& \beta^2(h_{(1)(1)(1)}) \cdot [F_r(S_H(h_{(1)(2)}))(S_H(h_{(1)(1)(2)}) \cdot F_r(\beta^{-2}(h_{(2)})))] \\ (2.5) =& \beta(h_{(1)(1)}) \cdot [F_r(S_H(\beta(h_{(1)(2)(2)})))(S_H(h_{(1)(2)(1)}) \cdot F_r(\beta^{-2}(h_{(2)})))] \\ =& \beta(h_{(1)(1)}) \cdot [F_r(\beta(S_H(h_{(1)(2)})_{(1)}))(S_H(h_{(1)(2)})_{(2)} \cdot F_r(\beta^{-2}(h_{(2)})))] \\ (3.12) =& \beta(h_{(1)(1)}) \cdot F_r(S_H(h_{(1)(2)})\beta^{-1}(h_{(2)})) \\ =& \varepsilon_H(h)1_B. \end{aligned}$$

The proof is completed. □

Using the above lemmas and corollaries what we have got, we can gain the main result.

**Theorem 3.9.** *Let $B \times H$ be a Hom-biproduct, let $\pi : B \times H \to H$ be the projection from $B \times H$ onto $H$, and let $\mathcal{F}_{B,H}$ be the set of pairs $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L} : B \to B$, $\mathcal{R} : H \to B \in \widetilde{\mathcal{H}}(\mathcal{M})$ are maps which satisfy the conclusions of Lemma 3.3 and Lemma 3.5 for $F_l$ and $F_r$, respectively. Then*

(1) *The function $\Phi : \mathcal{F}_{B,H} \to \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$, described by $(\mathcal{L}, \mathcal{R}) \mapsto F$, where $F(b \times h) = \mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)}) \times \beta(h_{(2)})$, for all $b \in B$ and $h \in H$, is a bijection. Furthermore, $F_l = \mathcal{L}$ and $F_r = \mathcal{R}$.*

(2) *Suppose $(\mathcal{L}, \mathcal{R}) \in \mathcal{F}_{B,H}$, then, $F \in \mathrm{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)$ if and only if $\mathcal{L}$ is a bijection.*

***Proof.*** In order to prove (1), we define $\Psi : \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi) \to \mathcal{F}_{B,H}$ by $\Psi(F) = (\Pi \circ F \circ J, \Pi \circ F \circ j)$. It is easily proved that $\Phi$ and $\Psi$ are mutually inverse.

According the definition of $F$, we shall check that $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. It is easy to see that $\pi \circ F = \pi$. Note that $F(1_B \times 1_H) = 1_B \times 1_H$ and

$$\begin{aligned}
\varepsilon(F(b \times h)) &= \varepsilon(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)}) \times \beta(h_{(2)})) \\
&= \varepsilon_B(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)}))\varepsilon_H(\beta(h_{(2)})) \\
&= \varepsilon_B(\mathcal{L}(\alpha^{-1}(b)))\varepsilon_B(\mathcal{R}(h_{(1)}))\varepsilon_H(\beta(h_{(2)}) \\
&= \varepsilon_B(b)\varepsilon_H(h),
\end{aligned}$$

for $b \in B$ and $h \in H$ which means $\varepsilon \circ F = \varepsilon$.

Let $b, b' \in B$ and $h, h' \in H$. Then,

$$\begin{aligned}
&F((b \times h)(b' \times h')) \\
&= F(b(h_{(1)} \cdot \alpha^{-1}(b')) \times \beta(h_{(2)})h') \\
&= \mathcal{L}(\alpha^{-1}(b)(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b')))\mathcal{R}(\beta(h_{(2)(1)})h'_{(1)}) \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&= [\mathcal{L}(\alpha^{-1}(b))\mathcal{L}(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b'))]\underline{\mathcal{R}(\beta(h_{(2)(1)})h'_{(1)})} \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&\overset{(3.12)}{=} [\mathcal{L}(\alpha^{-1}(b))\mathcal{L}(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b'))] \\
&\quad [\mathcal{R}(\beta^2(h_{(2)(1)(1)}))(\beta(h_{(2)(1)(2)}) \cdot \mathcal{R}(\beta^{-1}(h'_{(1)})))] \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&\overset{(2.1)}{=} \{\mathcal{L}(\alpha^{-1}(b))[\mathcal{L}(\beta^{-2}(h_{(1)}) \cdot \alpha^{-3}(b'))\mathcal{R}(\beta(h_{(2)(1)(1)}))]\}(\beta^2(h_{(2)(1)(2)}) \cdot \mathcal{R}(h'_{(1)})) \\
&\quad \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&\overset{(2.5)}{=} \{\mathcal{L}(\alpha^{-1}(b))\underline{[\mathcal{L}(\beta^{-1}(h_{(1)(1)}) \cdot \alpha^{-3}(b'))\mathcal{R}(h_{(1)(2)})]}\}(\beta(h_{(2)(1)}) \cdot \mathcal{R}(h'_{(1)})) \\
&\quad \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&\overset{(3.10)}{=} \{\mathcal{L}(\alpha^{-1}(b))[\mathcal{R}(h_{(1)(1)})(\beta^{-1}(h_{(1)(2)}) \cdot \mathcal{L}(\alpha^{-3}(b')))]\}(\beta(h_{(2)(1)}) \cdot \mathcal{R}(h'_{(1)})) \\
&\quad \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&\overset{(2.1)}{=} [\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(\beta(h_{(1)(1)}))][(h_{(1)(2)} \cdot \mathcal{L}(\alpha^{-2}(b')))(h_{(2)(1)} \cdot \mathcal{R}(\beta^{-1}(h'_{(1)})))] \\
&\quad \times \beta^2(h_{(2)(2)})\beta(h'_{(2)}) \\
&= [\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)})][(\beta(h_{(2)(1)(1)}) \cdot \mathcal{L}(\alpha^{-2}(b')))(\beta(h_{(2)(1)(2)}) \cdot \mathcal{R}(\beta^{-1}(h'_{(1)})))] \\
&\quad \times \beta^2(h_{(2)(2)})\beta(h'_{(2)})
\end{aligned}$$

$$\overset{(2.1)}{=} [\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)})][\beta(h_{(2)(1)})\cdot(\mathcal{L}(\alpha^{-2}(b'))\mathcal{R}(\beta^{-1}(h'_{(1)})))]\times\beta^2(h_{(2)(2)})\beta(h'_{(2)})$$
$$= F(b\times h)F(b'\times h').$$

Therefore, $F$ is a monoidal Hom-algebra morphism. Next, we shall check that $\Delta\circ F = (F\otimes F)\circ\Delta$ holds. Indeed, for all $b\in B, h\in H$,

$$\begin{aligned}
&\Delta(F(b\times h))\\
&= \Delta(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)})\times\beta(h_{(2)}))\\
&= ((\underline{\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)}))_{(1)}}\times(\underline{\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)}))_{(2)[-1]}}h_{(2)(1)})\\
&\quad\otimes((\alpha(\underline{(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)}))_{(2)[0]}}))\times\beta(h_{(2)(2)}))\\
&\overset{(2.17)}{=} [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\beta^{-1}(\mathcal{R}(h_{(1)})_{(1)}))\\
&\quad\times(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]})\mathcal{R}(h_{(1)})_{(2)})_{[-1]}h_{(2)(1)}]\\
&\quad\otimes[\alpha((\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]})\mathcal{R}(h_{(1)})_{(2)})_{[0]})\times\beta(h_{(2)(2)})]\\
&= [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-1}(h_{(1)(1)})))\\
&\quad\times(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]})\mathcal{R}(h_{(1)(2)}))_{[-1]}h_{(2)(1)}]\\
&\quad\otimes[\alpha((\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]})\mathcal{R}(h_{(1)(2)}))_{[0]})\times\beta(h_{(2)(2)})] \qquad \text{(By (3) of Lemma 3.5)}\\
&= [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-1}(h_{(1)(1)})))\\
&\quad\times(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]})_{[-1]}\mathcal{R}(h_{(1)(2)})_{[-1]})h_{(2)(1)}]\\
&\quad\otimes[\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]})_{[0]}\mathcal{R}(h_{(1)(2)})_{[0]})\times\beta(h_{(2)(2)})]\\
&= [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-1}(h_{(1)(1)})))\\
&\quad\times(\beta(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][-1]})\underline{\mathcal{R}(h_{(1)(2)})_{[-1]}})h_{(2)(1)}]\\
&\quad\otimes[\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][0]})\underline{\mathcal{R}(h_{(1)(2)})_{[0]}})\times\beta(h_{(2)(2)})]\\
&\overset{(3.13)}{=} [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-1}(h_{(1)(1)})))\\
&\quad\times(\beta(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][-1]})\underline{(h_{(1)(2)(1)(1)}S_H(\beta^{-1}(h_{(1)(2)(2)}))))h_{(2)(1)}}]\\
&\quad\otimes[\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][0]})\mathcal{R}(\beta(h_{(1)(2)(1)(2)})))\times\beta(h_{(2)(2)})]\\
&\overset{(2.1)}{=} [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-1}(h_{(1)(1)})))\\
&\quad\times(\beta(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][-1]})\beta(h_{(1)(2)(1)(1)}))(S_H(h_{(1)(2)(2)})\beta^{-1}(h_{(2)(1)}))]\\
&\quad\otimes[\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][0]})\mathcal{R}(\beta(h_{(1)(2)(1)(2)})))\times\beta(h_{(2)(2)})]\\
&\overset{(2.5)}{=} [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-2}(h_{(1)})))\\
&\quad\times(\beta(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][-1]})h_{(2)(1)(1)})(\underline{S_H(\beta(h_{(2)(2)(1)(1)}))\beta(h_{(2)(2)(1)(2)})})]\\
&\quad\otimes[\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][0]})\mathcal{R}(h_{(2)(1)(2)}))\times\beta^2(h_{(2)(2)(2)})]\\
&\overset{(2.12)}{=} [\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[-1]}\cdot\mathcal{R}(\beta^{-2}(h_{(1)})))\times(\beta^2(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][-1]})\beta(h_{(2)(1)(1)}))]\\
&\quad\otimes[\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0][0]})\mathcal{R}(h_{(2)(1)(2)}))\times\beta(h_{(2)(2)})]\\
&\overset{(3.8)}{=} [(\mathcal{L}(\alpha^{-2}(b_{(1)}))\mathcal{R}(\beta^{-1}(b_{(2)[-1]})))(\underline{\mathcal{L}(b_{(2)[0]})_{[-1]}}\cdot\mathcal{R}(\beta^{-2}(h_{(1)})))\\
&\quad\times(\beta^2(\underline{\mathcal{L}(b_{(2)[0]})_{[0][-1]}})\beta(h_{(2)(1)(1)}))]\\
&\quad\otimes[\alpha(\alpha(\underline{\mathcal{L}(b_{(2)[0]})_{[0][0]}})\mathcal{R}(h_{(2)(1)(2)}))\times\beta(h_{(2)(2)})]
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.9)}{=} [(\mathcal{L}(\alpha^{-2}(b_{(1)}))\mathcal{R}(\beta^{-1}(b_{(2)[-1]})))(b_{(2)[0][-1]} \cdot \mathcal{R}(\beta^{-2}(h_{(1)})))\\
&\quad \times (\beta^2(\mathcal{L}(b_{(2)[0][0]})_{[-1]})\beta(h_{(2)(1)(1)}))]\\
&\quad \otimes [\alpha(\alpha(\mathcal{L}(b_{(2)[0][0]})_{[0]})\mathcal{R}(h_{(2)(1)(2)})) \times \beta(h_{(2)(2)})]\\
&\overset{(2.7)}{=} \underline{[(\mathcal{L}(\alpha^{-2}(b_{(1)}))\mathcal{R}(b_{(2)[-1](1)}))(b_{(2)[-1](2)} \cdot \mathcal{R}(\beta^{-2}(h_{(1)})))}\\
&\quad \times (\beta^2(\mathcal{L}(\alpha^{-1}(b_{(2)[0]}))_{[-1]})\beta(h_{(2)(1)(1)}))]\\
&\quad \otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b_{(2)[0]}))_{[0]})\mathcal{R}(h_{(2)(1)(2)})) \times \beta(h_{(2)(2)})]\\
&\overset{(2.1)}{=} [\mathcal{L}(\alpha^{-1}(b_{(1)}))(\underline{\mathcal{R}(b_{(2)[-1](1)})(\beta^{-1}(b_{(2)[-1](2)}) \cdot \mathcal{R}(\beta^{-3}(h_{(1)})))})\\
&\quad \times (\beta^2(\mathcal{L}(\alpha^{-1}(b_{(2)[0]}))_{[-1]})\beta(h_{(2)(1)(1)}))]\\
&\quad \otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b_{(2)[0]}))_{[0]})\mathcal{R}(h_{(2)(1)(2)})) \times \beta(h_{(2)(2)})]\\
&\overset{(3.12)}{=} [\mathcal{L}(\alpha^{-1}(b_{(1)}))\mathcal{R}(\beta^{-1}(b_{(2)[-1]})\beta^{-2}(h_{(1)})) \times (\beta^2(\underline{\mathcal{L}(\alpha^{-1}(b_{(2)[0]}))_{[-1]}})\beta(h_{(2)(1)(1)}))]\\
&\quad \otimes [\alpha(\alpha(\underline{\mathcal{L}(\alpha^{-1}(b_{(2)[0]}))_{[0]}})\mathcal{R}(h_{(2)(1)(2)})) \times \beta(h_{(2)(2)})]\\
&\overset{(3.9)}{=} [\mathcal{L}(\alpha^{-1}(b_{(1)}))\mathcal{R}(\beta^{-1}(b_{(2)[-1]})\beta^{-2}(h_{(1)})) \times (\beta(b_{(2)[0][-1]})\beta(h_{(2)(1)(1)}))]\\
&\quad \otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b_{(2)[0][0]})))\mathcal{R}(h_{(2)(1)(2)})) \times \beta(h_{(2)(2)})]\\
&\overset{(2.7)}{=} [\mathcal{L}(\alpha^{-1}(b_{(1)}))\mathcal{R}(b_{(2)[-1](1)}\beta^{-2}(h_{(1)})) \times (\beta(b_{(2)[-1](2)})\beta(h_{(2)(1)(1)}))]\\
&\quad \otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-2}(b_{(2)[0]})))\mathcal{R}(h_{(2)(1)(2)})) \times \beta(h_{(2)(2)})]\\
&\overset{(2.5)}{=} [\mathcal{L}(\alpha^{-1}(b_{(1)}))\mathcal{R}(b_{(2)[-1](1)}\beta^{-1}(h_{(1)(1)})) \times (\beta(b_{(2)[-1](2)})h_{(1)(2)})]\\
&\quad \otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-2}(b_{(2)[0]})))\mathcal{R}(\beta^{-1}(h_{(2)(1)}))) \times \beta(h_{(2)(2)})]\\
&= [\mathcal{L}(\alpha^{-1}(b_{(1)}))\mathcal{R}(b_{(2)[-1](1)}\beta^{-1}(h_{(1)(1)})) \times (\beta(b_{(2)[-1](2)})h_{(1)(2)})]\\
&\quad \otimes [\mathcal{L}(b_{(2)[0]})\mathcal{R}(h_{(2)(1)}) \times \beta(h_{(2)(2)})]\\
&= F((b \times h)_{(1)}) \otimes F((b \times h)_{(2)}).
\end{aligned}
$$

The other conditions which make $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$ can be checked easily. Thus the proof of part (1) is completed.

As for (2), suppose $F \in \mathrm{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)$. it is easily showed that $F_l$ and $(F^{-1})_l$ are inverses. Thus $F_l$ is bijective and $(F_l)^{-1} = (F^{-1})_l$.

Conversely, suppose that $F_l$ is bijective. Set $G_l = (F_l)^{-1}$. From Corollary 3.8, $F_r$ has a convolution inverse $J_r$. $G_r = G_l \circ J_r = (F_l)^{-1} \circ J_r$. Define $G(b \times h) = G_l(\alpha^{-1}(b))G_r(h_{(1)}) \times \beta(h_{(2)})$. For all $b \in B$ and $h \in H$, we compute

$$
\begin{aligned}
G(F(b \times h)) &= G(F_l(\alpha^{-1}(b))F_r(h_{(1)}) \times \beta(h_{(2)}))\\
(3.5) &= G_l(F_l(\alpha^{-2}(b))F_r(\beta^{-1}(h_{(1)})))(G_l \circ J_r)(\beta(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})\\
&= \alpha^{-1}(b)G_l(F_r(\beta^{-1}(h_{(1)}))J_r(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})\\
(2.5) &= \alpha^{-1}(b)G_l(\underline{F_r(h_{(1)(1)})J_r(h_{(1)(2)})}) \times \beta(h_{(2)})\\
&= \alpha^{-1}(b)1_B\varepsilon_H(h_{(1)}) \times \beta(h_{(2)}) \quad \text{(By Corollary 3.8)}\\
&= b \times h.
\end{aligned}
$$

$$
\begin{aligned}
F(G(b \times h)) &= F(G_l(\alpha^{-1}(b))G_r(h_{(1)}) \times \beta(h_{(2)}))\\
(3.5) &= F_l(\alpha^{-1}(G_l(\alpha^{-1}(b))G_r(h_{(1)})))F_r(\beta(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})\\
&= (\alpha^{-2}(b)(F_l \circ G_r)(\beta^{-1}(h_{(1)})))F_r(\beta(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})
\end{aligned}
$$

$$
\begin{aligned}
&=(\alpha^{-2}(b)J_r(\beta^{-1}(h_{(1)})))F_r(\beta(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})\\
&=\alpha^{-1}(b)(J_r(\beta^{-1}(h_{(1)}))F_r(h_{(2)(1)})) \times \beta^2(h_{(2)(2)})\\
(2.5)\ &=\alpha^{-1}(b)(\underline{J_r(h_{(1)(1)})F_r(h_{(1)(2)})}) \times \beta(h_{(2)})\\
&=b \times h. \quad \text{(By Corollary 3.8)}
\end{aligned}
$$

Thus we have shown that $G \circ F = id_{B\times H} = F \circ G$. □

Let $\mathcal{F}^{\bullet}_{B,H}$ denote the set of $(\mathcal{L}, \mathcal{R}) \in \mathcal{F}_{B,H}$ such that $\mathcal{L}$ is bijective. Then, the corresponding of part induces a bijection $\mathcal{F}^{\bullet}_{B,H} \to \mathrm{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)$.

Let $(B, \alpha)$ be a monoidal Hom-algebra and $(C, \beta)$ be a monoidal Hom-coalgebra over $k$. Let $\mathrm{Hom}(C, B)$ be the set of linear maps $f : C \to B$ satisfying $\alpha \circ f = f \circ \beta$. Then $\mathrm{Hom}(C, B)$ is an ordinary associative algebra with the unit $\eta_B \circ \varepsilon_C$ under the convolution $\star$. Indeed, for $f, \phi, \varphi \in \mathrm{Hom}(C, B)$ and $c \in C$,

$$
\begin{aligned}
((f \star \phi) \star \varphi)(c) &=(f(c_{(1)(1)})\phi(c_{(1)(2)}))\varphi(c_{(2)})\\
&=(f(\beta^{-1}(c_{(1)}))\phi(c_{(2)(1)}))\varphi(\beta(c_{(2)(2)}))\\
&=\alpha(f(\beta^{-1}(c_{(1)})))(\phi(c_{(2)(1)})\alpha^{-1}(\varphi(\beta(c_{(2)(2)}))))\\
&=f(c_{(1)})(\phi(c_{(2)(1)})\varphi(c_{(2)(2)})\\
&=(f \star (\phi \star \varphi))(c).
\end{aligned}
$$

and

$$(f \star (\eta_B \circ \varepsilon_C))(c) = f(c_{(1)})\varepsilon_C(c_{(2)})1_B = \alpha(f(\beta^{-1}(c))) = f(c).$$

Thus it follows that $f \star (\eta_B \circ \varepsilon_C) = f$. That $(\eta_B \circ \varepsilon_C) \star f = f$ can be checked similarly.

The group $\mathcal{G}(B) = \mathrm{Aut}_{\text{Hom-algebra}}(B)$ acts on the convolution algebra $\mathrm{Hom}(C, B)$ by $f \blacktriangleright g = f \circ g$ for all $f \in \mathcal{G}(B)$ and $g \in \mathrm{Hom}(C, B)$. This action satisfies:

$$f \blacktriangleright (\eta \circ \varepsilon_C) = \eta \circ \varepsilon_C \quad \text{and} \quad f \blacktriangleright (\phi \star \varphi) = (f \blacktriangleright \phi) \star (f \blacktriangleright \varphi),$$

for all $f \in \mathcal{G}(B)$ and $\phi, \varphi \in \mathrm{Hom}(C, B)$. Let $\mathcal{U}(C, B)$ be the group of units of the algebra $\mathrm{Hom}(C, B)$. Then, $\mathcal{G}(B) \blacktriangleright \mathcal{U}(C, B) \subseteq \mathcal{U}(C, B)$; thus there is a group homomorphism,

$$\Gamma : \mathcal{G}(B) \to \mathrm{Aut}_{\text{Group}}(\mathcal{U}(C, B)) \tag{3.15}$$

given by $\Gamma(f)(\phi) = f \blacktriangleright \phi$, for all $f \in \mathcal{G}(B)$ and $\phi \in \mathrm{Hom}(C, B)$. The resulting group $\mathcal{U}(C, B) \rtimes_\Gamma \mathcal{G}(B)$ has product given by

$$(\phi, f)(\varphi, f') = (\phi \star (f \circ \varphi), f \circ f').$$

**Theorem 3.10.** *Suppose that $B \times H$ is a Hom-biproduct. Then, there is a one-to-one group homomorphism $\mathrm{Aut}_{\text{Hom-Hopf}}(B \times H, \pi) \to \mathcal{U}(C, B)^{op} \rtimes \mathcal{G}(B)$, wich is given by $F \mapsto (F_r, F_l)$, for all $F \in \mathrm{End}_{\text{Hom-Hopf}}(B \times H, \pi)$.*

As the end of this paper, we consider an example in [13]. Let $B = k < 1_B, x >$ and the automorphism $\alpha : B \to B$, $1_B \mapsto 1_B$ and $x \mapsto -x$. $(B, \alpha)$ is both a monoidal Hom-algebra and a monoidal Hom-coalgebra with multiplication, comultiplication and counit given by

$$
\begin{gathered}
1_B 1_B = 1_B, 1_B x = x 1_B = -x, x^2 = 0,\\
\Delta_B(1_B) = 1_B \otimes 1_B,\ \Delta_B(x) = (-x) \otimes 1_B + 1_B \otimes (-x),\\
\varepsilon_B(1_B) = 1,\ \varepsilon_B(x) = 0.
\end{gathered}
$$

We define $S_B : B \to B$, $S_B(1_B) = 1_B$, $S_B(x) = -x$, which is the convolution inverse of $\mathrm{id}_B$.

Let $H = k < 1_H, g >$ be the group Hopf algebra with $g^2 = 1_H$ and $\Delta_H(g) = g \otimes g$, $S_H(g) = g = g^{-1}$. Then $(H, id_H)$ is a monoidal Hom-Hopf algebra. $(B, \alpha)$ is $(H, id_H)$-module algebra and module coalgebra with the action $\cdot : H \otimes B \to B$ given by

$$1_H \cdot 1_B = 1_B,\ 1_H \cdot x = -x,\ g \cdot 1_B = 1_B \text{ and } g \cdot x = x.$$

Also, $(B,\alpha)$ is a left $(H, id_H)$-comodule algebra and comodule coalgebra with the coaction $\rho_B : B \to H \otimes B$ given by

$$\rho_B(1_B) = 1_H \otimes 1_B, \quad \rho_B(x) = g \otimes (-x).$$

Then $(B\times H = \{1_B\otimes 1_H,\ 1_B\otimes g,\ x\otimes 1_H,\ x\otimes g,\ \alpha\otimes id_H\})$ is a Radford's Hom-biproduct with multiplication, comultiplication, counit and antipode defined as follows:

- Multiplication

| $m$ | $1_B\times 1_H$ | $1_B\times g$ | $x\times 1_H$ | $x\times g$ |
|---|---|---|---|---|
| $1_B\times 1_H$ | $1_B\times 1_H$ | $1_A\times g$ | $-x\times 1_H$ | $-x\times g$ |
| $1_A\times g$ | $1_B\times g$ | $1_B\times 1_H$ | $x\times g$ | $x\times 1_H$ |
| $x\times 1_H$ | $-x\times 1_H$ | $-x\times g$ | 0 | 0 |
| $x\times g$ | $-x\times g$ | $-x\times 1_H$ | 0 | 0 |

- Comultiplication

$$\Delta(1_B\times 1_H) = (1_B\times 1_H)\otimes(1_B\times 1_H),\ \Delta(1_B\times g) = (1_B\times g)\otimes(1_B\times g),$$
$$\Delta(x\times 1_H) = (-x\times 1_H)\otimes(1_B\times 1_H) + (1_B\times g)\otimes(-x\times 1_H),$$
$$\Delta(x\times g) = (-x\times g)\otimes(1_B\times g) + (1_B\times 1_H)\otimes(-x\times g)$$

- Counit

$$\varepsilon(1_B\times 1_H) = 1 = \varepsilon(1_B\times g),\ \varepsilon(x\times 1_H) = 0 = \varepsilon(x\times g).$$

- Hom-antipode

$$S(1_B\times 1_H) = 1_B\times 1_H,\ S(1_B\times g) = 1_B\times g,\ S(x\times 1_H) = x\times g,\ S(x\times g) = -x\times 1_H.$$

Now, we compute the morphisms $\mathcal{L}\in \mathrm{End}(B)$ satisfying the conclusions of lemma 3.1. Taking a base of $\mathrm{End}(B)$ $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3$ and $\mathcal{L}_4$ given respectively by

$$\mathcal{L}_1 : 1_B \mapsto 1_B,\ x\mapsto 0,$$
$$\mathcal{L}_2 : 1_B \mapsto 0,\ x\mapsto 1_B,$$
$$\mathcal{L}_3 : 1_B \mapsto x,\ x\mapsto 0,$$
$$\mathcal{L}_4 : 1_B \mapsto 0,\ x\mapsto x.$$

Let $\mathcal{L} = t_1\mathcal{L}_1 + t_2\mathcal{L}_2 + t_3\mathcal{L}_3 + t_4\mathcal{L}_4$. If $\mathcal{L}$ satisfies part (2) of, we can get $t_1 = 1$ and $t_2 = 0$. Thus $\mathcal{L} = \mathcal{L}_1 + t_3\mathcal{L}_3 + t_4\mathcal{L}_4$. By part (4) of Lemma 3.1, it follows that $t_3 = 0$. So $\mathcal{L} = \mathcal{L}_1 + t_4\mathcal{L}_4$. So there is a bijection between the set of the morphisms $\mathcal{L}\in\mathrm{End}(B)$ satisfying the conclusions of lemma 2.1 and the set $\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} | t\in k\}$.

Now, We will discuss the morphisms of $\mathrm{Hom}(H,B)$ which satisfy Lemma 3.3 in similar way as above. Taking a base of $\mathrm{Hom}(H,B)$ $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ and $\mathcal{R}_4$ given respectively by

$$\mathcal{R}_1 : 1_H \mapsto 1_B,\ g\mapsto 0,$$
$$\mathcal{R}_2 : 1_H \mapsto 0,\ g\mapsto 1_B,$$
$$\mathcal{R}_3 : 1_H \mapsto x,\ g\mapsto 0,$$
$$\mathcal{R}_4 : 1_B \mapsto 0,\ g\mapsto x.$$

Let $\mathcal{R} = k_1\mathcal{R}_1 + k_2\mathcal{R}_2 + k_3\mathcal{R}_3 + k_4\mathcal{R}_4$. If $\mathcal{R}$ satisfies part (1) of Lemma 3.3, it follows that $k_1 = 1$ and $k_3 = 0$. Thus $\mathcal{R} = \mathcal{R}_1 + k_2\mathcal{R}_2 + k_4\mathcal{R}_4$. By (3.13) of Lemma 3.3, we can get $k_4 = 0$ and furthermore $\mathcal{R} = \mathcal{R}_1 + k_2\mathcal{R}_2$. Using part (4), we can obtain $k_2 = 1$. Thus $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, i.e., $\mathcal{R}(1_H) = 1_B$ and $\mathcal{R}(g) = 1_B$. Hence $\mathcal{F}^{\bullet}_{B,H} \cong \{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} |0\neq t\in k\} \cong k^*$. We can give the concrete characterization of $\mathrm{Aut}_{\text{Hom-Hopf}}(B\times H,\pi)$. Let $\mathcal{F}\in\mathrm{Aut}_{\text{Hom-Hopf}}(B\times H,\pi)$. By Theorem 3.9, we have

$$\mathcal{F}(1_B\times 1_H) = 1_B\times 1_H, \quad \mathcal{F}(1_B\times g) = 1_B\otimes g,$$

$$\mathcal{F}(x \times 1_H) = tx \times 1_H, \quad \mathcal{F}(x \times g) = tx \otimes g.$$

where $t \in k^*$.

**Acknowledgment.** The authors sincerely thank the referee for his/her numerous very valuable comments and suggestions on this article. This work was supported by the National Natural Science Foundation of China (Nos. 11871301, 11471186 , 11501317 and11801304 ) and the Natural Foundation of Shandong Province (No. ZR2018MA012).

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