ON THE PERIODICITY OF THE SOLUTION OF A RATIONAL DIFFERENCE EQUATION

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ABSTRACT. In this paper, some cases on the periodicity of the rational difference equation

$$S_{n+1} = S_{n-p} \left(\frac{aS_{n-q} + bS_{n-r} + cS_{n-s}}{dS_{n-q} + eS_{n-r} + fS_{n-s}} \right),$$

are investigated, where a, b, c, d, e, $f \in (0, \infty)$. The initial conditions S_{-p}, $S_{-p+1},...,S_{-q}$, $S_{-q+1},...,S_{-r}$, $S_{-r+1},...,S_{-s},...,S_{-s+1},...,S_{-1}$ and S_0 are arbitrary positive real numbers such that $p > q > r > s \geq 0$. Some numerical examples are provided to illustrate the theoretical discussion.

1. INTRODUCTION

Difference equations appear naturally as discrete analogues, numerical solutions of differential and delay differential equations as they are applied in biology, ecology, economy, physics, and so on. Although difference equations appear simpler in its form, the behaviors of their solutions are hard to be comprehended thoroughly. Recently, immense effort has been enacted in studying the qualitative analysis of rational difference equations. To put as an example, refer to [1-6]. Many researchers have investigated periodic solutions of difference equations, and they have proposed various methods for the existence and qualitative properties of the solutions [7-10].

The main aim of this study is to exhibit some cases on the periodic character of the positive solutions of the rational difference equation

$$S_{n+1} = S_{n-p} \left(\frac{aS_{n-q} + bS_{n-r} + cS_{n-s}}{dS_{n-q} + eS_{n-r} + fS_{n-s}} \right), \tag{1}$$

where $a, b, c, d, e, f \in (0, \infty)$. The initial conditions $S_{-p}, S_{-p+1},...,S_{-q}, S_{-q+1},...,S_{-r}, S_{-r+1},...,S_{-s},...,S_{-s+1},...,S_{-1}$ and S_0 are arbitrary positive real numbers such that $p > q > r > s \geq 0$.

Definition 1. *(Periodicity). A sequence $\{y_n\}_{n=-s}^{\infty}$ is said to be periodic with period p if*

$$y_{n+p} = y_n \quad for \quad all \quad n \geq -s.$$

Moreover, it is said to have prime period p if it is periodic with period p and p is the smallest positive integer satisfying this condition.

2. Main Result

In this section, we are going to be presented with the study of some cases on the periodic character of the positive solutions of Equation (1) for the first time.

Theorem 1. *Equation (1) has no positive solutions of prime period two $\forall\ a, b, c, d, e, f \in (0, \infty)$ in all the following cases:*
1) The positive integers p,r,s are even and the positive integer q is odd.
2) The positive integers p,q,r and s are even.
3) The positive integer p,q,r are even and the positive integer s is odd.
4) The positive integers p,q are even and the positive integers r,s are odd.
5) The positive integers p,s are even and the positive integers q,r are odd.
6) The positive integers p is even and the positive integers q,r,s are odd.
7) The positive integer r,s are even and the positive integers p,q are odd with $d \neq a$.
8) The positive integers q is even and the positive integers p,r,s are odd with $(e + f) \neq (b + c)$.
9) The positive integers s is even and the positive integers p,q,r are odd with $(d + e) \neq (a + b)$.
10) The positive integers p,q,r and s are odd with $(d + e + f) \neq (a + b + c)$.

Proof. Suppose that there is an existence of two positive distinctive solutions of prime period which two having such $...,\phi, \psi, \phi, \psi,...$ of Equation (1). Then the following cases are discussed;
Case 1. Suppose the positive integers p,r,s are even and the positive integer q is odd. In this case

$$S_n = S_{n-p} = S_{n-r} = S_{n-s} \ and \ S_{n+1} = S_{n-q}.$$

From Equation (1), we have

$$\phi = \psi \left(\frac{a\phi + (b+c)\psi}{c\phi + (d+f)\psi} \right),$$

$$\psi = \phi \left(\frac{a\psi + (b+c)\phi}{c\psi + (d+f)\phi} \right).$$

Thus

$$c\phi^2 + (d+f)\phi\psi = a\phi\psi + (b+c)\psi^2, \tag{2}$$

and

$$c\psi^2 + (d+f)\phi\psi = a\phi\psi + (b+c)\phi^2. \tag{3}$$

By subtracting (2) from (3), we deduce that

$$c(\phi^2 - \psi^2) + (b + c)(\phi^2 - \psi^2) = 0.$$

Hence, we have

$$(b + 2c)(\phi^2 - \psi^2) = 0.$$

If $\phi \neq \psi$, then

$$(b + 2c) = 0.$$

Since b and c are nonzero positive real numbers, thus $(b + 2c) \neq 0$. This implies $\phi = \psi$. This contradicts the hypothesis $\phi \neq \psi$.

Case 2. Suppose the positive integers p,q,r and s are even. In this case

$$S_n = S_{n-p} = S_{n-q} = S_{n-r} = S_{n-s}.$$

From Equation (1), we have

$$\phi = \psi \left(\frac{(a + b + c)\psi}{(d + e + f)\psi} \right),$$

$$\psi = \phi \left(\frac{(a + b + c)\phi}{(d + e + f)\phi} \right).$$

Thus,

$$(d + e + f)\phi\psi = (a + b + c)\psi^2, \qquad (4)$$

and

$$(d + e + f)\phi\psi = (a + b + c)\phi^2. \qquad (5)$$

By subtracting (4) from (5), we deduce that

$$(a + b + c)(\phi^2 - \psi^2) = 0.$$

Hence, we have

$$(a + b + c)(\phi^2 - \psi^2) = 0.$$

If $\phi \neq \psi$, then

$$(a + b + c) = 0.$$

Since a, b and c are nonzero positive real numbers, thus $(a+b+c) \neq 0$. This implies $\phi = \psi$. This contradicts the hypothesis $\phi \neq \psi$.

Case 3. Suppose the positive integer p,q,r are even and the positive integer s is odd. In this case

$$S_n = S_{n-p} = S_{n-q} = S_{n-r} \text{ and } S_{n+1} = S_{n-s}.$$

From Equation (1), we have

$$\phi = \psi \left(\frac{(a + b)\psi + c\phi}{(d + e)\psi + f\phi} \right),$$

$$\psi = \phi \left(\frac{(a + b)\phi + c\psi}{(d + e)\phi + f\psi} \right).$$

Thus,

$$(d + e)\phi\psi + f\phi^2 = (a + b)\psi^2 + c\phi\psi, \qquad (6)$$

and

$$(d + e)\phi\psi + f\psi^2 = (a + b)\phi^2 + c\phi\psi. \tag{7}$$

By subtracting (6) from (7), we deduce that

$$f(\phi^2 - \psi^2) + (a + b)(\phi^2 - \psi^2) = 0.$$

Hence, we have

$$(a + b + f)(\phi^2 - \psi^2) = 0.$$

If $\phi \neq \psi$, then

$$(a + b + f) = 0.$$

Since $a, b,$ and f are nonzero positive real numbers, thus $(a + b + f) \neq 0$. This implies $\phi = \psi$. This contradicts the hypothesis $\phi \neq \psi$.

Equivalently, we ratiocinate that there is no positive solution of prime period two derived from cases 4, 5 and 6.

Case 7. Suppose the positive integers r, s are even and the positive integers p, q are odd. In this case

$$S_n = S_{n-r} = S_{n-s} \text{ and } S_{n+1} = S_{n-p} = S_{n-q}.$$

From Equation (1), we have

$$\phi = \phi\left(\frac{a\phi + (b + c)\psi}{d\phi + (e + f)\psi}\right),$$

$$\psi = \psi\left(\frac{a\psi + (b + c)\phi}{d\psi + (e + f)\phi}\right).$$

Thus,

$$d\phi^2 + (e + f)\phi\psi = a\phi^2 + (b + c)\phi\psi, \tag{8}$$

and

$$d\psi^2 + (e + f)\phi\psi = a\psi^2 + (b + c)\phi\psi. \tag{9}$$

By subtracting (8) from (9), we deduce that

$$d(\phi^2 - \psi^2) - a(\phi^2 - \psi^2) = 0.$$

Hence, we have

$$[d - a](\phi^2 - \psi^2) = 0.$$

Since a and d are nonzero positive real numbers, and $\phi \neq \psi$.
This implies $[d = a]$, This contradicts the condition $[d \neq a]$.

Case 8. Suppose the positive integers q is even and the positive integers p, r, s are odd. In this case

$$S_n = S_{n-q} \text{ and } S_{n+1} = S_{n-p} = S_{n-r} = S_{n-s}.$$

From Equation (1), we have

$$\phi = \phi\left(\frac{a\psi + (b + c)\phi}{d\psi + (e + f)\phi}\right),$$

$$\psi = \psi \left(\frac{a\phi + (b+c)\psi}{d\phi + (e+f)\psi} \right).$$

Thus,

$$d\phi\psi + (e+f)\phi^2 = a\phi\psi + (b+c)\phi^2, \tag{10}$$

and

$$d\phi\psi + (e+f)\psi^2 = a\phi\psi + (b+c)\psi^2. \tag{11}$$

By subtracting (10) from (11), we deduce that

$$(e+f)(\phi^2 - \psi^2) - (b+c)(\phi^2 - \psi^2) = 0.$$

Hence, we have

$$[(e+f) - (b+c)](\phi^2 - \psi^2) = 0.$$

Since b, c, e and f are nonzero positive real numbers, and $\phi \neq \psi$. This implies $[(e+f) = (b+c)]$, This contradicts the condition $[(e+f) \neq (b+c)]$.
Equivalently, we ratiocinate that there is no positive solution of prime period two derived from cases 9 and 10. Therefore, the proof of the theorem is now complete.

$$\square$$

3. Numerical Examples

In this section, the previous results are illustrated through some numerical examples.

Example 1. *Figure 1, points out that the solution of Equation (1) holds no positive prime period two solutions on the assumption that the positive integers p,r and s are even and the positive integer q is odd. Select*
$p = 6, q = 5, r = 4, s = 2,\ S_{-6} = 1,\ S_{-5} = 2,\ S_{-4} = 3,\ S_{-3} = 4,\ S_{-2} = 5,\ S_{-1} = 6,\ S_0 = 7,\ a = 10, b = 20, c = 15, d = 25, e = 30, f = 35.$
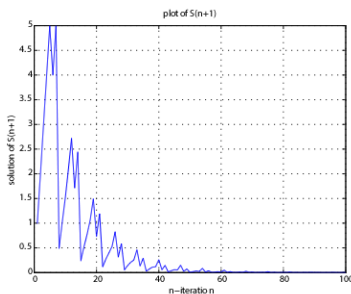

FIGURE 1. $\left(S_{n+1} = S_{n-6} \left(\frac{10 S_{n-5} + 20 S_{n-4} + 15 S_{n-2}}{25 S_{n-5} + 30 S_{n-4} + 35 S_{n-2}} \right) \right)$

Example 2. *Figure 2, points out that the solution of Equation (1) holds no positive prime period two solutions on the assumption that the positive integers p,q,r and s are even. Select*
$p = 8, q = 6, r = 4, s = 2, \ S_{-8} = 1, S_{-7} = 2, S_{-6} = 3, \ S_{-5} = 4, \ S_{-4} = 5, \ S_{-3} = 6, \ S_{-2} = 7, \ S_{-1} = 8, \ S_0 = 9 \ \ a = 10, \ b = 20, \ c = 15, \ d = 25, \ e = 30, \ f = 35.$
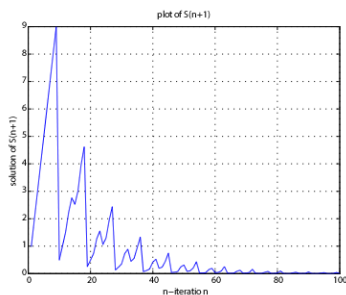

FIGURE 2. $\left(S_{n+1} = S_{n-8} \left(\frac{10S_{n-6}+20S_{n-4}+15S_{n-2}}{25S_{n-6}+30S_{n-4}+35S_{n-2}} \right) \right)$

Example 3. *Figure 3, points out that the solution of Equation (1) holds no positive prime period two solutions on the assumption that the positive integers p,q and r are even and the positive integer s is odd. Select*
$p = 6, q = 4, r = 2, s = 1, \ S_{-6} = 1, \ S_{-5} = 2, \ S_{-4} = 3, \ S_{-3} = 4, \ S_{-2} = 5, \ S_{-1} = 6, \ S_0 = 7, \ a = 10, \ b = 20, \ c = 15, \ d = 25, \ e = 30, \ f = 35.$
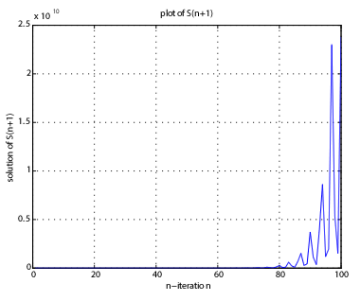

FIGURE 3. $\left(S_{n+1} = S_{n-6} \left(\frac{10S_{n-4}+20S_{n-2}+15S_{n-1}}{25S_{n-4}+30S_{n-2}+35S_{n-1}} \right) \right)$

Example 4. *Figure 4, points out that the solution of Equation (1) holds no positive prime period two solutions on the assumption that the positive integers r,s are even and the positive integer p,q are odd. Select*

$p = 7, q = 5, r = 4, s = 2,\ S_{-7} = 1,\ S_{-6} = 2,\ S_{-5} = 3,\ S_{-4} = 4,\ S_{-3} = 5,\ S_{-2} = 6,\ S_{-1} = 7,\ S_0 = 8,\ a = 3,\ b = 4,\ c = 15,\ d = 2,\ e = 3,\ f = 5.$
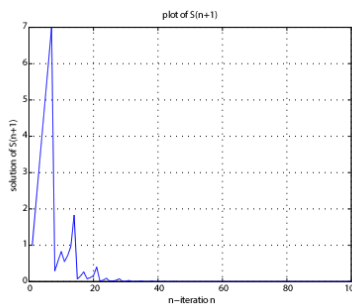

FIGURE 4. $\left(S_{n+1} = S_{n-7}\left(\frac{3S_{n-5}+4S_{n-4}+15S_{n-2}}{2S_{n-5}+3S_{n-4}+5S_{n-2}}\right)\right)$

Example 5. *Figure 5, points out that the solution of Equation (1) holds no positive prime period two solutions on the assumption that the positive integers q is even and the positive integer p,r,s are odd. Select*
$p = 5, q = 4, r = 3, s = 1,\ S_{-5} = 1,\ S_{-4} = 2,\ S_{-3} = 3,\ S_{-2} = 4, S_{-1} = 5,\ S_0 = 6,$
$a = 3,\ b = 4,\ c = 5,\ d = 25,\ e = 30,\ f = 35.$

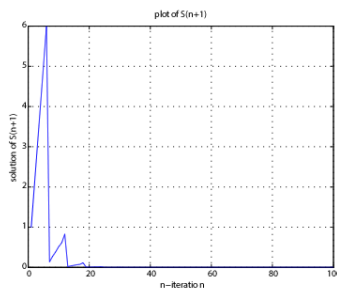FIGURE 5. $\left(S_{n+1} = S_{n-5}\left(\frac{3S_{n-4}+4S_{n-3}+5S_{n-1}}{25S_{n-4}+30S_{n-3}+35S_{n-1}}\right)\right)$

4. CONCLUSION

In this paper, the periodicity of some cases of the positive solutions of the rational difference Equation (1) is introduced and proved. It was evidently apparent that Equation (1) has no positive solutions of prime period two in the discussed cases. Therefore, in order to illustrate the results, some numerical examples were given.

References

[1] Abdul, Khaliq, Faris, Alzahrani and Elsayed, E. M., Global attractivity of a rational difference equation of order ten, *Journal of Nonlinear Sciences and Applications*, 9 (2016), 4465–4477.

[2] Alotaibi, A. M., El-Moneam, M. A. and Noorani, M. S. M., On the rational difference equation $y_{n+1} = \frac{\alpha_0 y_n + \alpha_1 y_{n-p} + \alpha_2 y_{n-q} + \alpha_3 y_{n-r} + \alpha_4 y_{n-s}}{\beta_0 y_n + \beta_1 y_{n-p} + \beta_2 y_{n-q} + \beta_3 y_{n-r} + \beta_4 y_{n-s}}$, *Journal of Nonlinear Sciences and Applications*, 11 (2018), 80–97.

[3] Alotaibi, A. M., Noorani M. S. M. and El-Moneam, M. A., On the Asymptotic Behavior Of Some Nonlinear Difference Equations, *Journal of Computational Analysis and Applications*, 26 (2019), 604–627.

[4] El-Moneam, M. A. and Alamoudy, S. O., On study of the asymptotic behavior of some rational difference equations, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysi*, 21 (2014), 89–109.

[5] El-Moneam, M. A., On the dynamics of the higher order nonlinear rational difference equation, *Mathematical Sciences Letter*, 3 (2014) 121–129.

[6] Elsayed, E. M., On the global attractivity and periodic character of a recursive sequence, *Opuscula Mathematica*, 30 (2010), 431–446.

[7] Grove, E. A. and Ladas, G., Periodicities in nonlinear difference equations, Vol.4, Chapman & Hall / CRC, 2005.

[8] Ibrahim, T. F., Periodicity and global attractivity of difference equation of higher order. *Journal of Computational Analysis and Applications*, 16 (2014).

[9] Ocalan, O., Global dynamics of a non-autonomous rational difference equation. *Journal of Applied Mathematics & Informatics*, 32, (2014) 843–848.

[10] Touafek, N. and Elsayed, E. M., On the periodicity of some systems of nonlinear difference equations, *Bulletin mathematique de la Sociat des Sciences Mathematiques de Roumanie*, 55, (2012), 217–224.

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