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Authors: Metin Yaman

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Long-time Behaviour of Solutions to Inverse Problem for Higher-order Parabolic Equation

Metin Yaman*

ABSTRACT

We consider an inverse problem for the fourth-order parabolic equation. Long-time behavior of the solution for the higher-order nonlinear inverse problem is established. Additional condition is given in the form of integral overdetermination.

Keywords: long-time behavior, integral overdetermination, higher-order parabolic equation, inverse problem

1. INTRODUCTION

In this study it is considered the following problem for the fourth-order parabolic equation:

$$u_t + \Delta^2 u - \sum_{i=1}^{n} b_i u_{x_i x_i} + |u|^p u = f(t)w(x), x \in \Omega, t > 0 \quad (1)$$

$$u = \Delta u = 0, x \in \partial\Omega, \mathrm{t} \geq 0 \quad (2)$$

$$u(x,0) = u_0(x), x \in \Omega \quad (3)$$

$$\int_\Omega u(x,t)w(x)dx = \varphi(t) \quad (4)$$

where $\Omega \subset \mathbb{R}^n$ is the bounded region which has a smooth boundary $\partial\Omega$ and $p > 0$. u_0, w, φ are given functions and they satisfy the following conditions

$$w \in H_0^2(\Omega) \cap L^{\mathrm{p}+2}, \int_\Omega w^2 dx = 1 \quad (A1)$$

$$B_0 = \max_{x \in \Omega} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}, x \in \Omega, b_i \in C(\overline{\Omega}). \quad (A2)$$

The inverse problem consists of finding a pair of functions $\{u(x,t), f(t)\}$ satisfying (1)-(4) when

$$u_0 \in H_0^1(\Omega) \cap L^{\mathrm{p}+2}(\Omega), \int_\Omega u_0 w dx = \varphi(0). \quad (A3)$$

Condition (4) is the *overdetermination condition* for the inverse problem and is given sometimes point-wise or integral form. Here is the integral form. Physically it means measurements of the temperature $u(x,t)$ by a device averaging over the domain Ω [4].

Inverse problems are known as ill-posed problems in Hadamard's sense and have many application areas in physics and engineering. For example inverse scattering problems in quantum physics, inverse problems in geophysics[2].

There are some papers devoted to the study of existence and uniqueness of solutions of inverse problems for various parabolic type equations with unknown source functions [1,3,5]. In [6,7,8,9,10]

* myaman@sakarya.edu.tr Sakarya University Faculty of Arts and Sciences Department of Mathematics, Orcid: 0000-0003-2208-5730

authors studied global nonexistence and blow-up solution to fourth-order equations.

Here, we used the following notations:

$(u,v)=\int_\Omega uvdx$, $\|u(t)\|=\|u(t)\|_{L^2(\Omega)}$,

$\|u(t)\|_p=\|u(t)\|_{L^p(\Omega)}$.

Cauchy inequality with epsilon and Young inequality for $a,b>0$ are as follows;

$$ab\le\frac{\epsilon}{2}a^2+\frac{1}{2\epsilon}b^2\ ,\ ab\le\beta a^p+C(p,\beta)b^q \qquad (5)$$

with $1/p+1/q=1$, $C(p,\beta)=1/q(\beta p)^{q/p}$.

$$\lambda_1\|u(t)\|^2\le\|\Delta u(t)\|^2\ ,u\in H_0^2(\Omega) \qquad (6)$$

where λ_1 is the least eigenvalue of the following eigenvalue problem

$$\lambda u=\Delta^2u, x\in\Omega,$$
$$u=\frac{\partial u}{\partial n}=0, x\in\partial\Omega\ \ [11].$$

Definition 1 *The pair of functions* $\{u(x,t),f(t)\}$ *is called the weak solution of the inverse problem* (1)-(4) *if*

$u\in L^2(0,T;H^4(\Omega)\cap H_0^2(\Omega))\cap L^{p+2}(\Omega)$,
$f\in L^2(0,T)$

satisfying the identiy

$$(u_t,\psi)+(\Delta\psi,\Delta u)-(\psi,\textstyle\sum_{i=1}^n b_iu_{x_ix_i})$$
$$+(\psi,|u|^pu)=f(t)(\psi,w) \qquad (7)$$

where $\psi\in C_0^\infty(\Omega)$.

Multiplying both sides of (1) by w and integrating the resulting equation over Ω leads to the following relation with the conditions (3), (4) and (A1)

$$f(t)=\varphi'(t)+(\Delta w,\Delta u)-(w,\textstyle\sum_{i=1}^n b_iu_{x_ix_i})$$
$$+(w,|u|^pu)\ . \qquad (8)$$

Substituting (8) into the equation (1), the problem (1)-(3) yields a direct problem given by [1].

2. A PRIORI ESTIMATES

Theorem 1. *Suppose that the conditions* (A2) *and* (A3) *are satisfied and assume that* φ *and* φ' *are continuous functions defined on* $[0,\infty)$ *which tends to zero as* $t\to\infty$ *. Then*

$$\lim_{t\to\infty}\left(\|\Delta u\|^2+\frac{1}{p+2}\|u\|_{p+2}^{p+2}\right)=0 \qquad (9)$$

with a constant $1>\frac{(4+\lambda_1)B_0{}^2}{4\lambda_1}$, *where* λ_1 is the constant in (6).

Proof. Let us multiply the equation (1) by $u+u_t$ and integrate over Ω then we get the relation

$$\frac{d}{dt}\eta(t)+\|\Delta u\|^2+\|u\|_{p+2}^{p+2}+\|u_t\|^2+\|\Delta u_t\|^2$$
$$=(\varphi+\varphi')f(t)+(u+u_t,\textstyle\sum_{i=1}^n b_iu_{x_ix_i}\ ) \quad (10)$$

where

$$\eta(t)=\frac{1}{2}\|u(t)\|^2+\|\Delta u(t)\|^2+\frac{1}{p+2}\|u(t)\|_{p+2}^{p+2}\ .$$

Replacing (8) into (10), we obtain

$$\frac{d}{dt}\eta(t)+\|\Delta u\|^2+\|u\|_{p+2}^{p+2}+\|u_t\|^2+\|\Delta u_t\|^2$$
$$=(\varphi+\varphi')[\varphi'(t)+(\Delta w,\Delta u)+(w,|u|^pu)$$
$$-(w,\textstyle\sum_{i=1}^n b_iu_{x_ix_i})]+(u+u_t,\sum_{i=1}^n b_iu_{x_ix_i}).\quad (11)$$

Using the inequalities (5) and (6) on the right-hand side of (11), we have the following estimates

$$\left|(u,\textstyle\sum_{i=1}^n b_iu_{x_ix_i})\right|\le\frac{B_0{}^2}{\lambda_1}\|\Delta u\|^2 \qquad (12)$$

$$\left|(u_t,\textstyle\sum_{i=1}^n b_iu_{x_ix_i})\right|\le\|u_t\|^2+\frac{B_0{}^2}{4}\|\Delta u\|^2 \qquad (13)$$

$$\left|(\Delta w,\Delta u)(\varphi+\varphi')\right|\le\frac{\xi}{2}\|\Delta u\|^2$$
$$+\frac{1}{\xi}\|\Delta w\|^2(|\varphi|^2+|\varphi'|^2) \qquad (14)$$

$$\left|(w,\textstyle\sum_{i=1}^n b_iu_{x_ix_i})(\varphi+\varphi')\right|\le\frac{\xi}{2}\|\Delta u\|^2+$$

$$+\frac{B_0{}^2}{\xi}\|\Delta w\|^2(|\varphi|^2+|\varphi'|^2) \quad (15)$$

$$\left|(w,|u|^p u)(\varphi+\varphi')\right| \le \xi\|u(t)\|_{p+2}^{p+2}$$

$$+C(\xi,p)\|w\|_{p+2}^{p+2}(|\varphi|^2+|\varphi'|^2). \quad (16)$$

Rewriting (11) with estimates (12)-(16), we get the following differential inequality

$$\frac{d}{dt}\eta(t)+\left(1-\xi-\frac{(4+\lambda_1)B_0{}^2}{4\lambda_1}\right)\|\Delta u\|^2$$

$$+(1-\xi)\|u\|_{p+2}^{p+2} \le D(t) \quad (17)$$

where

$$D(t)=(|\varphi|^2+|\varphi'|^2)\left(\frac{\xi}{2}\|\Delta w\|^2+\frac{B_0{}^2}{\xi}\|\Delta w\|^2+(\xi,p)\|w\|_{p+2}^{p+2}\right).$$

We choose $\xi_0>0$ such that

$$\xi_0 \le \xi \le 1-\frac{(4+\lambda_1)B_0{}^2}{4\lambda_1}$$

and take

$$K_1=min\left\{1-\xi_0,\frac{2}{3}\left(1-\xi_0-\frac{(4+\lambda_1)B_0{}^2}{4\lambda_1}\right)\right\}.$$

So, (17) follows

$$\frac{d}{dt}\eta(t)+K_1\left(\frac{3}{2}\|\Delta u\|^2+\|u\|_{p+2}^{p+2}\right) \le D(t) \quad . \quad (18)$$

The last term on the left-hand side of (18) can be written

$$\frac{3}{2}\|\Delta u\|^2+\|u\|_{p+2}^{p+2} \ge \frac{\lambda_1}{2}\|u\|^2+\|\Delta u\|^2+\|u\|_{p+2}^{p+2}$$

$$\ge K_2\left(\frac{1}{2}\|u\|^2+\|\Delta u\|^2+\frac{1}{p+2}\|u\|_{p+2}^{p+2}\right) \quad (19)$$

where $K_2=min\left\{\lambda_1,\frac{1}{p+2}\right\}$. It follows from (18) and (19)

$$\frac{d}{dt}\eta(t)+K_3\eta(t) \le D(t). \quad (20)$$

where $K_3=K_1K_2$. After solving first-order differential inequality (20), it concludes that

$$\lim_{t\to\infty}\left(\|\Delta u\|^2+\frac{1}{p+2}\|u\|_{p+2}^{p+2}\right)=0 \ .$$

The proof is completed.

3. CONCLUSION

Long-time behaviour of the solutions to the inverse problem (1)-(4) is asymptotically stable. It means that $\|\Delta u\|$ and $\|u\|_{p+2}$ goes to zero $as\ t\to\infty$.

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