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Gaussian (s,t)-Pell and Pell-Lucas Sequences and Their Matrix Representations

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Abstract
In this study, the Gaussian (s,t)-Pell and Pell-Lucas sequences are defined. Moreover, by using these sequences, the Gaussian (s,t)-Pell and Pell-Lucas matrix sequences are defined. Furthermore, generating functions, Binet's formulas and some summation formulas of these sequences are given. Finally, some relationships between Gaussian (s,t)-Pell and Pell-Lucas matrix sequences are obtained.

Keywords: (s,t)-Pell sequence, Gaussian Pell sequence, Gaussian (s,t)-Pell sequence, Gaussian (s,t)-Pell matrix sequence.

Gauss (s,t)-Pell ve Pell-Lucas Dizileri ve Matris Gösterimleri

Öz
Bu çalışmada, Gauss (s,t)-Pell ve Pell-Lucas dizileri tanımlanmıştır. Dahası, bu dizileri kullanarak Gauss (s,t)-Pell ve Pell-Lucas matris dizileri tanımlanmıştır. Ayrıca, bu dizilerin üreteç fonksiyonları, Binet formülleri ve bazı toplam formülleri verilmiştir. Son olarak, Gauss (s,t)-Pell ve Pell-Lucas matris dizileri arasında bazı ilişkiler elde edilmiştir.

Anahtar Kelimeler: (s,t)-Pell dizisi, Gauss Pell dizisi, Gauss (s,t)-Pell dizisi, Gauss (s,t)-Pell matris dizisi.

1. Introduction

In recent years, we have seen so many studies on the different number sequences such as Fibonacci, Lucas, Pell, Pell-Lucas, modified Pell sequences. We refer the reader to [1,6,9,14,16,19,20]. For $n \geq 2$, the well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$, Pell $\{P_n\}$, Pell-Lucas $\{Q_n\}$ and modified Pell $\{q_n\}$ sequences are defined as $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$, $P_n = 2P_{n-1} + P_{n-2}$, $Q_n = 2Q_{n-1} + Q_{n-2}$ and $q_n = 2q_{n-1} + q_{n-2}$ where $F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1, P_0 = 0, P_1 = 1, Q_0 = 2, Q_1 = 2$ and $q_0 = 1, q_1 = 1$, respectively.

In [4,5,8], two-parameters generalizations of the Fibonacci and Pell sequences are given. In [4,5], Civciv and Türkmen introduced (s,t)-Fibonacci $\{F_n(s,t)\}_{n=0}^{\infty}$ and (s,t)-Lucas $\{L_n(s,t)\}_{n=0}^{\infty}$ sequences by using Fibonacci and Lucas sequences. On the other hand, the matrix sequences that concern this special number sequences have taken so much interest [4,5,8]. In [8], Güleç and Taşkara introduced (s,t)-Pell $\{MP_n(s,t)\}_{n\in N}$ and (s,t)-Pell-Lucas $\{MQ_n(s,t)\}_{n\in N}$ matrix sequences by using (s,t)-Pell and (s,t)-Pell-Lucas sequences. They showed some properties of these matrix sequences using essentialy a matrix approach in [4,5,8].

Moreover, many authors studied on Gaussian Fibonacci, Lucas, Pell, Pell-Lucas and modified Pell sequences. We refer the reader to [2,3,7,10,12,13,15,18,21]. Halıcı and Öz [10] introduced the Gaussian Pell and Gaussian Pell-Lucas numbers respectively by

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$$GP_0 = i, \;\; GP_1 = 1; \;\; GP_n = 2GP_{n-1} + GP_{n-2},$$

$$GQ_0 = 2 - 2i, \;\; GQ_1 = 2 + 2i; \;\; GQ_n = 2GQ_{n-1} + GQ_{n-2}.$$

The authors also studied Gaussian Pell polynomials and their some properties in [11]. In addition, Yağmur and Karaaslan [21] defined the Gaussian modified Pell numbers by

$$Gq_0 = 1 - i, \;\; Gq_1 = 1 + i; \;\; Gq_n = 2Gq_{n-1} + Gq_{n-2}.$$

Also, they studied their properties in the same paper. In addition, Catarino and Campos [3] studied the Gaussian modified Pell numbers. Moreover, in [17], Pektaş gave the definition of (s,t)-Gaussian Fibonacci and (s,t)-Gaussian-Lucas numbers and then presented their matrix representatios.

In this study, we firstly define Gaussian (s,t)-Pell and Pell-Lucas sequences. Then, by using these sequences, we also define Gaussian (s,t)-Pell and Pell-Lucas matrix sequences. In the last of the study, we investigate the relationships between Gaussian (s,t)-Pell and Pell-Lucas matrix sequences.

2. The Gaussian (s,t)-Pell and Pell-Lucas Sequences

In this section, we first give the definitions of the Gaussian (s,t)-Pell and Pell-Lucas sequences, and then we obtain Binet's formulas, generating functions and sum formulas for these sequences.

Firstly, we give the fundamental definitions and properties for these sequences.

Definition 2.1. Let s and t be any real numbers satisfying that $s > 0$, $t \neq 0$ and $s^2 + t > 0$. By the aid of the reference [10], let us define the Gaussian (s,t)-Pell sequence $\{GP_n(s,t)\}_{n \epsilon N}$ is defined recursively by

$$GP_n(s,t) = 2sGP_{n-1}(s,t) + tGP_{n-2}(s,t), \qquad n \geq 2,$$

with initial values $GP_0(s,t) = i$ and $GP_1(s,t) = 1$.

One can see that

$$GP_n(s,t) = P_n(s,t) + itP_{n-1}(s,t)$$

where $P_n(s,t)$ is the nth (s,t)-Pell number.

Particular case of Gaussian (s,t)-Pell sequence is:

- If $s = \frac{1}{2}$ and $t = 1$, the classical Gaussian Fibonacci sequence is obtained in [2, 15].
- If $s = 1$ and $t = 1$, the classical Gaussian Pell sequence is obtained in [10].

Definition 2.2. Let s and t be any real numbers satisfying that $s > 0$, $t \neq 0$ and $s^2 + t > 0$. By the aid of the reference [10], let us define the Gaussian (s,t)-Pell-Lucas sequence $\{GQ_n(s,t)\}_{n \epsilon N}$ is defined recursively by

$$GQ_n(s,t) = 2sGQ_{n-1}(s,t) + tGQ_{n-2}(s,t), \qquad n \geq 2,$$

with initial values $GQ_0(s,t) = 2 - 2is$ and $GQ_1(s,t) = 2s + 2it$.

Also, it can be seen that

$$GQ_n(s,t) = Q_n(s,t) + itQ_{n-1}(s,t)$$

where $Q_n(s,t)$ is the nth (s,t)-Pell-Lucas number.

Particular case of Gaussian (s,t)-Pell-Lucas sequence is:

- If $s = \frac{1}{2}$ and $t = 1$, the classical Gaussian Lucas sequence is obtained in [15].
- If $s = 1$ and $t = 1$, the classical Gaussian Pell-Lucas sequence is obtained in [10].

In this paper, we only present the proofs of the results given for the Gaussian (s, t)-Pell sequence, because those for the Gaussian (s, t)-Pell-Lucas sequence are similar.

Now, we give the generating functions for these sequences by the following theorem.

Theorem 2.1. *The generating functions for the Gaussian (s, t)-Pell and Pell-Lucas sequences are*

$$f(x) = \frac{x + i(1 - 2sx)}{1 - 2sx - tx^2},$$

$$h(x) = \frac{(2 - 2sx) + i(4s^2x + 2tx - 2s)}{1 - 2sx - tx^2}$$

respectively.

Proof. Let $f(x)$ be the generating function of the Gaussian (s, t)-Pell sequence $\{GP_n(s, t)\}$. Then, we can write

$$f(x) = \sum_{i=0}^{\infty} GP_i(s, t)x^i = GP_0(s, t) + GP_1(s, t)x + GP_2(s, t)x^2 + \cdots + GP_n(s, t)x^n + \cdots.$$

Also, we can write by the recursive relations

$$f(x)(1 - 2sx - tx^2) = GP_0(s, t) + [GP_1(s, t) - 2sGP_0(s, t)]x.$$

Thus, we obtain

$$f(x) = \frac{x + i(1 - 2sx)}{1 - 2sx - tx^2}$$

which is desired. ■

It must be noted that for $s = t = 1$, these functions generalise the formulas in the reference [10]. That is

$$f(x) = \frac{x + i(1 - 2x)}{1 - 2x - x^2},$$

$$g(x) = \frac{(2 - 2x) + i(6x - 2)}{1 - 2x - x^2}$$

respectively.

Theorem 2.2. *Let $\alpha = s + \sqrt{s^2 + t}$ and $\beta = s - \sqrt{s^2 + t}$ be the roots of the equation $x^2 - 2sx - t = 0$. The Binet's formulas for nth Gaussian (s, t)-Pell and Pell-Lucas number are*

$$GP_n(s, t) = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta} \quad n \geq 0$$

and

$$GQ_n(s, t) = (\alpha^n + \beta^n) - i(\alpha\beta^n + \beta\alpha^n) \quad n \geq 0,$$

respectively.

Proof. We know that the general solution for the recurrence relation is given by

$$GP_n(s,t) = c\alpha^n + d\beta^n$$

for some coefficients c and d.
Using the initial values $GP_0(s,t) = i$ and $GP_1(s,t) = 1$,

$$c = \frac{1-i\left(s-\sqrt{s^2+t}\right)}{2\sqrt{s^2+t}}\,,\ d = \frac{-1+i\left(s+\sqrt{s^2+t}\right)}{2\sqrt{s^2+t}}$$

can be written.
Hence, the Binet's formula for $GP_n(s,t)$ is obtained as

$$GP_n(s,t) = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\,\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}.$$

So, the proof is completed. ■

Theorem 2.3. *For $2s + t \neq 1$, the sums of the Gaussian (s,t)-Pell and -Pell-Lucas sequences are given as*

$$(\boldsymbol{i})\ \sum_{i=1}^{n} GP_i(s,t) = \frac{1}{2s+t-1}\left[GP_{n+1}(s,t) + tGP_n(s,t) - 1 - it\right],$$

$$(\boldsymbol{ii})\ \sum_{i=1}^{n} GQ_i(s,t) = \frac{1}{2s+t-1}\left[GQ_{n+1}(s,t) + tGQ_n(s,t) - 2(s+t) + 2it(s-1)\right].$$

Proof. By the definition of Gaussian (s,t)-Pell sequence recurrence relation, we have

$$GP_{i-1}(s,t) = \frac{1}{2s}GP_i(s,t) - \frac{t}{2s}GP_{i-2}(s,t).$$

From this equation

$$GP_1(s,t) = \frac{1}{2s}GP_2(s,t) - \frac{t}{2s}GP_0(s,t),$$

$$GP_2(s,t) = \frac{1}{2s}GP_3(s,t) - \frac{t}{2s}GP_1(s,t),$$

$$GP_3(s,t) = \frac{1}{2s}GP_4(s,t) - \frac{t}{2s}GP_2(s,t),$$

$$\vdots$$

$$GP_n(s,t) = \frac{1}{2s}GP_{n+1}(s,t) - \frac{t}{2s}GP_{n-1}(s,t)$$

can be written.
Then, we have

$$\sum_{i=1}^{n} GP_i(s,t) = \frac{1}{2s}\sum_{i=2}^{n+1} GP_i(s,t) - \frac{t}{2s}\sum_{i=0}^{n-1} GP_i(s,t).$$

After necessary calculations we get

$$\sum_{i=1}^{n} GP_i(s,t) = \frac{1}{2s+t-1}\left[GP_{n+1}(s,t) + tGP_n(s,t) - 1 - it\right].$$

So, the proof is completed. ■

Theorem 2.4. *Let X be odd indexed Gaussian (s,t)-Pell numbers and Y be even indexed Gaussian (s,t)-Pell numbers. Then the following equalities hold:*

$$X = \sum_{i=1}^{n} GP_{2i-1}(s,t) = \frac{(1-t)[GP_{2n+1}(s,t) - 1] + 2st[GP_{2n}(s,t) - i]}{4s^2 - (1-t)^2},$$

$$Y = \sum_{i=1}^{n} GP_{2i}(s,t) = \frac{GP_{2n+2}(s,t) - t^2 GP_{2n}(s,t) - 2s + it(t-1)}{4s^2 - (1-t)^2}.$$

Proof. Terms of odd index of $GP_n(s,t)$ are

$$GP_1(s,t) = 2sGP_0(s,t) - tGP_{-1}(s,t),$$

$$GP_3(s,t) = 2sGP_2(s,t) - tGP_1(s,t),$$

$$GP_5(s,t) = 2sGP_4(s,t) - tGP_3(s,t),$$

$$\vdots$$

$$GP_{2n-1}(s,t) = 2sGP_{2n-2}(s,t) - tGP_{2n-3}(s,t).$$

Then, we obtain

$$X = \frac{2sY + 1 - GP_{2n+1}(s,t)}{1-t}. \qquad (1)$$

Similarly, terms of even index of $GP_n(s,t)$ are

$$GP_2(s,t) = 2sGP_1(s,t) - tGP_0(s,t),$$

$$GP_4(s,t) = 2sGP_3(s,t) - tGP_2(s,t),$$

$$GP_6(s,t) = 2sGP_5(s,t) - tGP_4(s,t),$$

$$\vdots$$

$$GP_{2n}(s,t) = 2sGP_{2n-1}(s,t) - tGP_{2n-2}(s,t).$$

Then, we get

$$Y = \frac{2sX - tGP_{2n}(s,t) + it}{1-t}. \qquad (2)$$

By considering Eq. (1) and (2), we obtain

$$X = \sum_{i=1}^{n} GP_{2i-1}(s,t) = \frac{(1-t)[GP_{2n+1}(s,t)-1] + 2st[GP_{2n}(s,t)-i]}{4s^2-(1-t)^2},$$

$$Y = \sum_{i=1}^{n} GP_{2i}(s,t) = \frac{GP_{2n+2}(s,t) - t^2 GP_{2n}(s,t) - 2s + it(t-1)}{4s^2-(1-t)^2}.$$

■

Theorem 2.5. *Let X be odd indexed Gaussian* (s,t)*-Pell-Lucas numbers and Y be even indexed Gaussian* (s,t)*-Pell-Lucas numbers. Then the following equalities hold:*

$$X = \sum_{i=1}^{n} GQ_{2i-1}(s,t) = \frac{(1-t)[GQ_{2n+1}(s,t)-2s-2it] + 2st[GQ_{2n}(s,t)-2+2is]}{4s^2-(1-t)^2},$$

$$Y = \sum_{i=1}^{n} GQ_{2i}(s,t) = \frac{GQ_{2n+2}(s,t) - t^2 GQ_{2n}(s,t) - 2(2s^2+t+ist) + t^2(2-2is)}{4s^2-(1-t)^2}.$$

Proof. This theorem is easily obtained by proceeding as in the proof of Theorem 2.4. ■

We now investigate some identities of the Gaussian (s,t)-Pell and Pell-Lucas sequences.

Theorem 2.6. *Let* $n \geq 0$ *and* $n \geq r$*. Then Catalan's identity for the Gaussian* (s,t)*-Pell and Pell-Lucas is*

$$(\boldsymbol{i})\ GP_{n-r}(s,t)GP_{n+r}(s,t) - GP_n{}^2(s,t) = \frac{(t+1-2is)(-t)^{n-r}[4(-t)^r-(\alpha^r+\beta^r)^2]}{4(s^2+t)},$$

$$(\boldsymbol{ii})\ GQ_{n-r}(s,t)GQ_{n+r}(s,t) - GQ_n{}^2(s,t) = (t+1-2is)(-t)^{n-r}[(\alpha^r+\beta^r)^2-4(-t)^r].$$

Proof. Let $A = 1-\beta i$ and $B = 1-\alpha i$. Then, using Theorem 2.2, we can write

$$GP_{n-r}(s,t)GP_{n+r}(s,t) - GP_n{}^2(s,t) = \left(\frac{A\alpha^{n-r}-B\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{A\alpha^{n+r}-B\beta^{n+r}}{\alpha-\beta}\right) - \left(\frac{A\alpha^{n}-B\beta^{n}}{\alpha-\beta}\right)^2.$$

After necessary calculations, we get

$$GP_{n-r}(s,t)GP_{n+r}(s,t) - GP_n{}^2(s,t) = \frac{AB\alpha^{n-r}\beta^{n-r}(2\alpha^r\beta^r-\alpha^{2r}-\beta^{2r})}{(\alpha-\beta)^2}.$$

Hence, from $AB = t+1-2is$, we have

$$GP_{n-r}(s,t)GP_{n+r}(s,t) - GP_n{}^2(s,t) = \frac{(t+1-2is)(-t)^{n-r}[4(-t)^r-(\alpha^r+\beta^r)^2]}{4(s^2+t)},$$

as required. ■

By setting $r = 1$ in Theorem 2.6, we obtain the following corollary which is Cassini's identity of the Gaussian (s,t)-Pell and Pell-Lucas sequences.

Corollary 2.1. *For positive integer* n*, we have*

$$(\boldsymbol{i})\ GP_{n-1}(s,t)GP_{n+1}(s,t) - GP_n{}^2(s,t) = -(t+1-2is)(-t)^{n-1},$$

$$(\boldsymbol{ii})\ GQ_{n-1}(s,t)GQ_{n+1}(s,t) - {GQ_n}^2(s,t) = 4(s^2+t)(t+1-2is)(-t)^{n-1}.$$

In the rest of paper, the Gaussian (s,t)-Pell and Pell-Lucas sequences will be denoted by GP_n and GQ_n instead of $GP_n(s,t)$ and $GQ_n(s,t)$, respectively.

3. The Matrix Representations For The Gaussian (s,t)-Pell and Pell-Lucas Sequences

In this section, we give the definitions of the Gaussian (s, t)-Pell and Pell-Lucas matrix sequences. Then, we obtain Binet's formulas, generating functions and sum formulas for these matrix sequences. We also investigate their properties.

Definition 3.1. Let s and t be any real numbers satisfying that $s>0$, $t\neq 0$ and $s^2+t>0$. The Gaussian (s,t)-Pell matrix sequence $\{MGP_n(s,t)\}_{n\in N}$ is defined recursively by

$$MGP_n(s,t) = 2sMGP_{n-1}(s,t) + tMGP_{n-2}(s,t) \quad n\geq 2$$

with initial values $MGP_0(s,t) = \begin{pmatrix} 1 & i \\ it & 1-2is \end{pmatrix}$ and $MGP_1(s,t) = \begin{pmatrix} 2s+it & 1 \\ t & it \end{pmatrix}$.

Also, it is easily seen that

$$MGP_n(s,t) = MP_n(s,t) + itMP_{n-1}(s,t)$$

where $MP_n(s,t)$ is the nth (s,t)-Pell matrix sequence.

Definition 3.2. Let s and t be any real numbers satisfying that $s>0$, $t\neq 0$ and $s^2+t>0$. The Gaussian (s,t)-Pell-Lucas matrix sequence $\{MGQ_n(s,t)\}_{n\in N}$ is defined recursively by

$$MGQ_n(s,t) = 2sMGQ_{n-1}(s,t) + tMGQ_{n-2}(s,t) \quad n\geq 2$$

with initial values $MGQ_0(s,t) = \begin{pmatrix} 2s+2it & 2-2is \\ 2t-2ist & -2s+4is^2+2it \end{pmatrix}$ and

$$MGQ_1(s,t) = \begin{pmatrix} 4s^2+2t+2ist & 2s+2it \\ 2st+2it^2 & 2t-2ist \end{pmatrix}.$$

Also, it is easily seen that

$$MGQ_n(s,t) = MQ_n(s,t) + itMQ_{n-1}(s,t)$$

where $MQ_n(s,t)$ is the nth (s,t)-Pell-Lucas matrix sequence.

In the rest of paper, the Gaussian (s,t)-Pell and Pell-Lucas matrix sequences will be denoted by MGP_n and MGQ_n instead of $MGP_n(s,t)$ and $MGQ_n(s,t)$, respectively.

Theorem 3.1. *Let* $n\geq 0$. *We have*

$(\boldsymbol{i})$ $MGP_n = \begin{pmatrix} GP_{n+1} & GP_n \\ tGP_n & tGP_{n-1} \end{pmatrix}$,

$(\boldsymbol{ii})$ $MGQ_n = \begin{pmatrix} GQ_{n+1} & GQ_n \\ tGQ_n & tGQ_{n-1} \end{pmatrix}$.

Proof. By induction on n we can prove the theorem. For $n=0$, we get

$$MGP_0 = \begin{pmatrix} GP_1 & GP_0 \\ tGP_0 & tGP_{-1} \end{pmatrix} = \begin{pmatrix} 1 & i \\ it & 1-2is \end{pmatrix}.$$

Now, assume that the theorem holds for $n = k$, that is

$$MGP_k = \begin{pmatrix} GP_{k+1} & GP_k \\ tGP_k & tGP_{k-1} \end{pmatrix}.$$

Then, for $n = k+1$, we obtain

$$\begin{aligned} MGP_{k+1} &= 2sMGP_k + tMGP_{k-1} \\ &= 2s\begin{pmatrix} GP_{k+1} & GP_k \\ tGP_k & tGP_{k-1} \end{pmatrix} + t\begin{pmatrix} GP_k & GP_{k-1} \\ tGP_{k-1} & tGP_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} 2sGP_{k+1} + tGP_k & 2sGP_k + tGP_{k-1} \\ 2stGP_k + t^2GP_{k-1} & 2stGP_{k-1} + t^2GP_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} GP_{k+2} & GP_{k+1} \\ tGP_{k+1} & tGP_k \end{pmatrix}. \end{aligned}$$

So, we obtain the desired result. ■

Theorem 3.2. *For* $n \geq 0$, *we have*

(***i***) $MGP_n = \left(\frac{MGP_1 - \beta MGP_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}\right)\beta^n$,

(***ii***) $MGQ_n = \left(\frac{MGQ_1 - \beta MGQ_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{MGQ_1 - \alpha MGQ_0}{\alpha - \beta}\right)\beta^n$.

Proof. We know that the general solution for the recurrence relation

$$MGP_n = c\alpha^n + d\beta^n$$

for some coefficients c and d.

Then, using the initial conditions imply that $MGP_0 = c + d$ and $MGP_1 = c\alpha + d\beta$.
Solving the system, we get

$$c = \frac{MGP_1 - \beta MGP_0}{\alpha - \beta} \text{ and } d = -\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}.$$

Thus, the Binet's formula for MGP_n is obtained as

$$MGP_n = \left(\frac{MGP_1 - \beta MGP_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}\right)\beta^n.$$

So, the proof is completed. ■

Theorem 3.3. *The generating functions for the Gaussian* (s,t)*-Pell and Pell-Lucas matrix sequences are*

$$g(x) = \frac{1}{1 - 2sx - tx^2}\left[\begin{pmatrix} 1 & i \\ it & 1-2is \end{pmatrix} + x\begin{pmatrix} it & 1-2is \\ t - 2ist & -2s + i(4s^2 + t) \end{pmatrix}\right],$$

$$n(x)=\frac{1}{1-2sx-tx^2}\left[\begin{pmatrix}2s+2it & 2-2is\\ 2t-2ist & -2s+2it+4is^2\end{pmatrix}+x\begin{pmatrix}2t-2sti & -2s+2it+4is^2\\ -2st+2it^2+4is^2t & 4s^2+2t-6ist-8is^3\end{pmatrix}\right]$$

respectively.

Proof. Let $g(x)$ be the generating function of the Gaussian (s,t)-Pell sequence $\{MGP_n\}$. Then we can write

$$g(x)=\sum_{i=0}^{\infty}MGP_i x^i=MGP_0+MGP_1x+MGP_2x^2+\cdots+MGP_nx^n+\cdots.$$

Also, we can write by the recursive relations

$$g(x)(1-2sx-tx^2)=MGP_0+[MGP_1-2sMGP_0]x.$$

Thus, we obtain

$$g(x)=\frac{1}{1-2sx-tx^2}\left[\begin{pmatrix}1 & i\\ it & 1-2is\end{pmatrix}+x\begin{pmatrix}it & 1-2is\\ t-2ist & -2s+i(4s^2+t)\end{pmatrix}\right].$$

The proof is completed. ■

Theorem 3.4. *For $2s+t\neq 1$, the sums of the Gaussian (s,t)-Pell and Pell-Lucas matrix sequences are given as*

$$(\boldsymbol{i})\ \sum_{i=1}^{n}MGP_i=\frac{1}{2s+t-1}[MGP_{n+1}+tMGP_n-MGP_2+(2s-1)MGP_1],$$

$$(\boldsymbol{ii})\ \sum_{i=1}^{n}MGQ_i=\frac{1}{2s+t-1}[MGQ_{n+1}+tMGQ_n-MGQ_2+(2s-1)MGQ_1].$$

Proof. By the definition of Gaussian (s,t)-Pell matrix sequence recurrence relation, we have

$$-tMGP_{i-2}=-MGP_i+2sMGP_{i-1}.$$

From this equation

$$-tMGP_1=-MGP_3+2sMGP_2,$$

$$-tMGP_2=-MGP_4+2sMGP_3,$$

$$-tMGP_3=-MGP_5+2sMGP_4,$$

$$\vdots$$

$$-tMGP_n=-MGP_{n+2}+2sMGP_{n+1}$$

can be written.
Then, we have

$$-t\sum_{i=1}^{n}MGP_i=(2s-1)(MGP_3+MGP_4+\cdots+MGP_{n+1})-MGP_{n+2}+2sMGP_2.$$

After necessary calculations we get

$$\sum_{i=1}^{n} MGP_i = \frac{1}{2s+t-1}[MGP_{n+1} + tMGP_n - MGP_2 + (2s-1)MGP_1].$$

So, the proof is completed. ■

Theorem 3.5. *For* $2s + t \neq 1$, *we have*

$$(\boldsymbol{i})\ \sum_{k=1}^{n} MGP_k = \frac{1}{2s+t-1}\begin{pmatrix} GP_{n+2} + tGP_{n+1} - 2s - t - it & GP_{n+1} + tGP_n - 1 - it \\ tGP_{n+1} + t^2GP_n - t - it^2 & tGP_n + t^2GP_{n-1} - t + it(2s-1) \end{pmatrix},$$

$$(\boldsymbol{ii})\ \sum_{k=1}^{n} MGQ_k = \frac{1}{2s+t-1}\begin{pmatrix} GQ_{n+2} + tGQ_{n+1} - 2(2s^2 + st + t) - i(t^2 + 2st) & GQ_{n+1} + tGQ_n - 2(s+t) + 2it(s-1) \\ tGQ_{n+1} + t^2GQ_n - 2t(s+t) + 2it^2(s-1) & tGQ_n + t^2GQ_{n-1} + 2t(s-1) - 2it(2s^2 - s + t \end{pmatrix}.$$

Proof. From the Theorem 3.1, we have

$$\sum_{k=1}^{n} MGP_k = \sum_{k=1}^{n} \begin{pmatrix} GP_{k+1} & GP_k \\ tGP_k & tGP_{k-1} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} GP_{k+1} & \sum_{k=1}^{n} GP_k \\ \sum_{k=1}^{n} tGP_k & \sum_{k=1}^{n} tGP_{k-1} \end{pmatrix}.$$

Then, by using the Theorem 2.3 we obtain

$$\sum_{k=1}^{n} MGP_k = \frac{1}{2s+t-1}\begin{pmatrix} GP_{n+2} + tGP_{n+1} - 2s - t - it & GP_{n+1} + tGP_n - 1 - it \\ tGP_{n+1} + t^2GP_n - t - it^2 & tGP_n + t^2GP_{n-1} - t + it(2s-1) \end{pmatrix}.$$

This completes the proof. ■

Theorem 3.6. *Let* X, Y *be odd indexed Gaussian* (s,t)*-Pell and Pell-Lucas numbers and* Z, T *be even indexed Gaussian* (s,t)*-Pell and Pell-Lucas numbers. Then the following equalities hold:*

$$X = \sum_{i=1}^{n} MGP_{2i-1} = \frac{(1-t)[MGP_{2n+1} - MGP_1] + 2st[MGP_{2n} - MGP_0]}{4s^2 - (1-t)^2},$$

$$Y = \sum_{i=1}^{n} MGQ_{2i-1} = \frac{(1-t)[MGQ_{2n+1} - MGQ_1] + 2st[MGQ_{2n} - MGQ_0]}{4s^2 - (1-t)^2},$$

$$Z = \sum_{i=1}^{n} MGP_{2i} = \frac{MGP_{2n+2} - t^2MGP_{2n} - MGP_2 + t^2MGP_0}{4s^2 - (1-t)^2},$$

$$T = \sum_{i=1}^{n} MGQ_{2i} = \frac{MGQ_{2n+2} - t^2MGQ_{2n} - MGQ_2 + t^2MGQ_0}{4s^2 - (1-t)^2}.$$

Proof. This theorem is easily obtained by proceeding as in the proof of Theorem 2.4. ■

Theorem 3.7. *Let us consider,* $s^2 + t > 0$, $s > 0$ *and* $t \neq 0$. *We get*

$(\boldsymbol{i})$ $\sum_{k=0}^{n} \frac{MGP_k}{x^k} = \frac{1}{x^2-2sx-t}(xMGP_1 + (x^2 - 2sx)MGP_0) - \frac{1}{x^n(x^2-2sx-t)}(xMGP_{n+1} + tMGP_n)$,

$(\boldsymbol{ii})$ $\sum_{k=0}^{n} \frac{MGQ_k}{x^k} = \frac{1}{x^2-2sx-t}(xMGQ_1 + (x^2 - 2sx)MGQ_0) - \frac{1}{x^n(x^2-2sx-t)}(xMGQ_{n+1} + tMGQ_n)$.

Proof. From Theorem 3.2, we get

$$\sum_{k=0}^{n} \frac{MGP_k}{x^k} = \left(\frac{MGP_1 - \beta MGP_0}{\alpha - \beta}\right)\sum_{k=0}^{n}\left(\frac{\alpha}{x}\right)^k - \left(\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}\right)\sum_{k=0}^{n}\left(\frac{\beta}{x}\right)^k.$$

By considering the definition of a geometric sequence, we have

$$\sum_{k=0}^{n} \frac{MGP_k}{x^k} = \left(\frac{MGP_1 - \beta MGP_0}{\alpha - \beta}\right)\left(\frac{x^{n+1} - \alpha^{n+1}}{x^{n+1}\left(\frac{x-\alpha}{x}\right)}\right) - \left(\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}\right)\left(\frac{x^{n+1} - \beta^{n+1}}{x^{n+1}\left(\frac{x-\beta}{x}\right)}\right)$$

$$= \frac{1}{x^n(x^2 - 2sx - t)}\left[\left(\frac{MGP_1 - \beta MGP_0}{\alpha - \beta}\right)(x^{n+1} - \alpha^{n+1})(x - \beta) - \left(\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}\right)(x^{n+1} - \beta^{n+1})(x - \alpha)\right]$$

$$= \frac{1}{x^n(x^2-2sx-t)}\left[\left(\frac{MGP_1 - \beta MGP_0}{\alpha - \beta}\right)(x^{n+2} - x^{n+1}\beta - x\alpha^{n+1} + \alpha^{n+1}\beta)\right.$$

$$\left. - \left(\frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}\right)(x^{n+2} - x^{n+1}\alpha - x\beta^{n+1} + \beta^{n+1}\alpha)\right].$$

Since $\alpha + \beta = 2s, \alpha\beta = -t$ and also by using the Binet's formula of Gaussian (s,t)-Pell matrix sequence, we get

$$\sum_{k=0}^{n} \frac{MGP_k}{x^k} = \frac{1}{x^n(x^2 - 2sx - t)}[x^{n+2}MGP_0 - x^{n+1}(2sMGP_0 - MGP_1) - xMGP_{n+1} - tMGP_n].$$

After necessary calculations we obtain

$$\sum_{k=0}^{n} \frac{MGP_k}{x^k} = \frac{1}{x^2 - 2sx - t}(xMGP_1 + (x^2 - 2sx)MGP_0) - \frac{1}{x^n(x^2 - 2sx - t)}(xMGP_{n+1} + tMGP_n).$$

■

Theorem 3.8. *For* $j > m$, *we get*

$(\boldsymbol{i})$ $\sum_{i=0}^{n} MGP_{mi+j} = \frac{MGP_{mn+m+j} + (-t)^m MGP_{j-m} - (-t)^m MGP_{mn+j} - MGP_j}{\alpha^m + \beta^m - (-t)^m - 1}$,

$(\boldsymbol{ii})$ $\sum_{i=0}^{n} MGQ_{mi+j} = \frac{MGQ_{mn+m+j} + (-t)^m MGQ_{j-m} - (-t)^m MGQ_{mn+j} - MGQ_j}{\alpha^m + \beta^m - (-t)^m - 1}$.

Proof. Let us consider $A = \frac{MGP_1 - \beta MGP_0}{\alpha - \beta}$ and $B = \frac{MGP_1 - \alpha MGP_0}{\alpha - \beta}$. After, we write

$\sum_{i=0}^{n} MGP_{mi+j} = \sum_{i=0}^{n} \frac{A\alpha^{mi+j} - B\beta^{mi+j}}{\alpha - \beta}$

$$= \frac{1}{\alpha-\beta}\left(A\alpha^j \sum_{i=0}^{n} \alpha^{mi} - B\beta^j \sum_{i=0}^{n} \beta^{mi}\right)$$

$$= \frac{1}{\alpha-\beta}\left[A\alpha^j\left(\frac{1-\alpha^{mn+m}}{1-\alpha^m}\right) - B\beta^j\left(\frac{1-\beta^{mn+m}}{1-\beta^m}\right)\right].$$

After necessary calculations we obtain

$$\sum_{i=0}^{n} MGP_{mi+j} = \frac{MGP_{mn+m+j} + (-t)^m MGP_{j-m} - (-t)^m MGP_{mn+j} - MGP_j}{\alpha^m + \beta^m - (-t)^m - 1}.$$

So, the proof is completed. ■

By using the matrix representation in the following theorem, we have given some equations for these newly defined numbers.

Theorem 3.9. *For* $m, n \geq 0$, *we get*

$(\boldsymbol{i})$ $MGP_m MGP_n = MGP_n MGP_m$,

$(\boldsymbol{ii})$ $MGQ_m MGQ_n = MGQ_n MGQ_m$,

$(\boldsymbol{iii})$ $MGP_m MGP_{n+1} = MGP_{m+1} MGP_n$,

$(\boldsymbol{iv})$ $MGQ_m MGQ_{n+1} = MGQ_{m+1} MGQ_n$.

Proof. $MGQ_m MGQ_n = (MQ_m + itMQ_{m-1})(MQ_n + itMQ_{n-1})$

$$= MQ_m MQ_n + itMQ_m MQ_{n-1} + itMQ_{m-1} MQ_n - t^2 MQ_{m-1} MQ_{n-1}$$

Since $MQ_m MQ_{n+1} = MQ_{m+1} MQ_n$ (see [8, Theorem 13]) and $MQ_m MQ_n = MQ_n MQ_m$ (see [8, Proposition 9]) where MQ_n is the *n*th Pell-Lucas matrix, we have

$$MGQ_m MGQ_n = MQ_n MQ_m + itMQ_n MQ_{m-1} + itMQ_{n-1} MQ_m - t^2 MQ_{n-1} MQ_{m-1}$$

$$= (MQ_n + itMQ_{n-1})(MQ_m + itMQ_{m-1})$$

$$= MGQ_n MGQ_m.$$

The proof is completed. ■

Theorem 3.10. (Catalan's Identity) *For* $n \geq 0$ *and* $n \geq r$, *the following results hold.*

$(\boldsymbol{i})$ $MGP_{n-r} MGP_{n+r} = {MGP_n}^2$,

$(\boldsymbol{ii})$ $MGQ_{n-r} MGQ_{n+r} = {MGQ_n}^2$.

Proof. Let $A = MGP_1 - \beta MGP_0$ and $B = MGP_1 - \alpha MGP_0$. Then, using Theorem 3.2, we can write

$$MGP_{n-r} MGP_{n+r} - {MGP_n}^2 = \left(\frac{A\alpha^{n-r} - B\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{A\alpha^{n+r} - B\beta^{n+r}}{\alpha-\beta}\right) - \left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right)^2.$$

After necessary calculations, we get

$$MGP_{n-r}MGP_{n+r} - MGP_n{}^2 = \frac{AB\alpha^{n-r}\beta^{n-r}(2\alpha^r\beta^r - \alpha^{2r} - \beta^{2r})}{(\alpha-\beta)^2}.$$

Hence, from $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have $MGP_{n-r}MGP_{n+r} = MGP_n{}^2$, as required.

This completes the proof. ■

Theorem 3.11. *For* $n \geq 0$, *following equalities are valid:*

$(\boldsymbol{i})$ $MGQ_n = 2sMGP_n + 2tMGP_{n-1}$,

$(\boldsymbol{ii})$ $MGQ_n = MGP_{n+1} + tMGP_{n-1}$.

Proof.

$(\boldsymbol{i})$ $2sMGP_n + 2tMGP_{n-1} = 2s(MP_n + itMP_{n-1}) + 2t(MP_{n-1} + itMP_{n-2})$

$$= 2sMP_n + 2tMP_{n-1} + it(2sMP_{n-1} + 2tMP_{n-2}).$$

Since $MQ_n = 2sMP_n + 2tMP_{n-1}$ (see [8, Theorem 10]), we have

$$2sMGP_n + 2tMGP_{n-1} = MQ_n + itMQ_{n-1}$$

$$= MGQ_n.$$

$(\boldsymbol{ii})$ $MGP_{n+1} + tMGP_{n-1} = (MP_n + itMP_{n-1}) + t(MP_{n-1} + itMP_{n-2})$

$$= (MP_{n+1} + tMP_{n-1}) + it(MP_n + tMP_{n-1}).$$

Since $MQ_n = MP_{n+1} + tMP_{n-1}$ (see [8, Theorem 10]), we get

$$MGP_{n+1} + tMGP_{n-1} = MQ_n + itMQ_{n-1}$$

$$= MGQ_n.$$

■

4. Conclusion

We firstly introduce the Gaussian (s,t)-Pell and Pell-Lucas sequences. By using these sequences, we define Gaussian (s,t)-Pell and Pell-Lucas matrix sequences. We also give some results, such as Binet's formulas, generating functions and summation formulas for these sequences. Moreover, we obtain some relationships between these matrix sequences.

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