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On superfluous subgroups and fully invariant subgroups

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Abstract

This paper is mainly concerned with Abelian groups having the lifting property with respect to fully invariant and projection invariant subgroups.

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Introduction

The notion of lifting is by now well-established object of module theory over associative unital rings and is useful in some rings (see [7],[8] and [10]). In the present paper, we try to obtain some possible results on superfluous subgroups, fully invariant subgroups, projection invariant subgroups and lifting groups.

Let us recall some standard terminology and elementary properties, which will be necessary in the sequel. They concern projective groups, small subgroups, lifting groups, fully invariant subgroups and projection invariant subgroups. Throughout this paper G denotes an abelian group and *group* means *abelian group*. Then, "$\leq$" will denote a subgroup, "$\leq_d$" a group direct summand and "$S(G)$" the set of subgroups of G. We will refer to [4] and [3] for all undefined notions used in the text.

A subgroup A of a group G is called *essential* in G, denoted by $A \leq_e G$, if for every $B \in S(G)$, $A \cap B = 0$ implies $B = 0$. The intersection of all essential subgroups of G is said to be the *socle* of G, or equivalently, the sum of all simple subgroups of G, denoted by $soc(G)$. It is easy to see that if A is an essential subgroup of G then $soc(G) \leq A$, and $soc(G) = 0$ if G has no simple subgroups.

Dually, a subgroup A of a group G is called *superfluous* (or *small*; the notation will be $A \ll G$), if $G \neq A + B$ for every proper subgroup B of G. Notice that the zero subgroup is small subgroup of any group G, also when $G = 0$. The intersection of all

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maximal subgroups of G is said to be the *radical* of G, or equivalently, the sum of small subgroups of G, denoted by $rad(G)$. It is easy to see that if A is a small subgroup of G then $A \leq rad(G)$, and $\mathrm{rad}(G) = G$ if G has no maximal subgroups. Some properties of small subgroups and radical are given in [2]. In Section 1, we will obtain new results for small subgroups.

Let G be a group. According to [2], G is called *lifting* , if for every $A \in S(G)$, there exists a direct summand B of G such that $B \leq A$ and A/B is small in G/B, equivalently G has a decomposition $G = G_1 \oplus G_2$ with $G_1 \leq A$ and $G_2 \cap A$ small in G_2, where $G_1, G_2 \in S(G)$. In module theory, it is well known that, the class of lifting modules is not closed under taking submodules, factor modules and sums. By this vein, in [9], the author proved that;

• Assume that M is a lifting module and X a submodule of M. If for every direct summand K of M, $(X+K)/X$ is a direct summand of M/X then M/X is lifting (see [9, Theorem 2.1]).

In Section 2, we investigate conditions which ensure that a quotient group of a lifting group will be a lifting group.

Recall that a subgroup A of G that is carried into itself by every endomorphism of G is said to be a *fully invariant* subgroup of G, i.e. if $f(A)$ is contained in A for every $f \in End(G)$. Clearly, 0 and G are fully invariant subgroups of G. We call a finite group G *FI-lifting* if for every fully invariant subgroup A of G, there exists a direct summand B of G such that $B \subseteq A$ and A/B small in G/B. The last part of Section 2, we will obtain some properties of the finite FI-lifting group and proved that

• Let G be a group and A be a fully invariant subgroup of G. If G is FI-lifting then G/A is FI-lifting.

Let G be a group. A subgroup $A \in S(G)$ is called a *projection invariant subgroup* if every projection π of G onto a direct summand maps A into itself, i.e. A is invariant under any projection of G (see [3]). In section 3, we will characterize these subgroups, and we will define projection invariant lifting groups, similarly to FI-lifting.

1. Properties of small subgroups

Recall that, a subgroup A of a group G is called *superfluous* (or *small*; the notation will be $A \ll G$), if $G \neq A + B$ for every proper subgroup B of G.

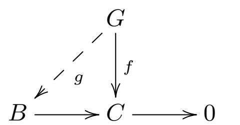
(1) We consider the group $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, i.e. the group $G =< a, b, c : 4a = 0, 2b = 0, 2c = 0 >$. The subgroups $H_1 = 0$ and $H_2 =< 2a >$ are small subgroups of G.
(2) Let $\mathbb{Z}$ denote the set of all integers. Then $\mathbb{Z}^{(\mathbb{N})} = \oplus_{n\geq 1}\mathbb{Z}x_n$ is a free group of countable rank and does not have small subgroups. But, in the Prüfer group $\mathbb{Z}(p^{\infty})$, all proper subgroups are small.

Lemma 1.1([1, Result 2.2]) *If $f : G \to H$ is a homomorphism of groups and $A \ll G$, then $f(A) \ll H$.*

It is easily seen that:

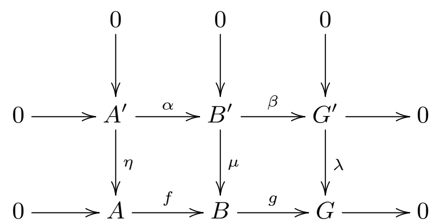
Lemma 1.2 *Let $A; B \in S(G)$ with $A \subseteq B$. If A be a small subgroup of G, then the small subgroups of G/A are the groups of the form B/A with $B \ll G$.*

Let us briefly recall the notion of projective group. A group G is called *projective* if every diagram



with exact row can be completed by a homomorphism $h : G \to B$ ([3, Page 74]).

Proposition 1.3 *We consider the following commutative diagram;*



where $\eta(A') \ll A$ *and* $\lambda(G') \ll G$. *If* G *is a projective group then* $\mu(B') \ll B$.

Proof. Clear. □

Let $\mathcal{A}$ be an arbitrary group category and $\mathcal{A}'$ be a group category with, for $G, H \in \mathcal{A}$, morphisms $\mathrm{Hom}(G, H)$ in $\mathcal{A}'$:

$$\mathrm{Hom}(G, H) = \varinjlim \ \mathrm{Hom}(G, H/B).$$

where $\mathrm{Hom}(G, H/B)$ is in $\mathcal{A}$ and the direct limit is taken over the upwards direct family of small subobjects B of H. Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}'$ be the canonical functor which is identity on objects. We shall denote the image $\mathcal{F}(f)$ of a morphism f in $\mathcal{A}$ by $\overline{f}$, i.e. if $\overline{f} : \overline{G} \to \overline{H}$ and $\overline{g} : \overline{H} \to \overline{U}$ be two morphisms in $\mathcal{A}'$, then we have $f : G \to H/H'$ and $G : H \to U/U'$ for suitable $H' \ll H$ and $U' \ll U$.

Lemma 1.4 *Let* G, H *be two groups and* $f : G \to H/H'$ *be a morphism with* $H' \ll H$. *Then the following statements are equivalent.*

(1) $\overline{f} : \overline{G} \to \overline{H}$ *is the zero morphism.*
(2) $f(G) \ll H/H'$.

Proof. (1) ⇒ (2) Since $\overline{f} : \overline{G} \to \overline{H}$ is the zero morphism, there is a small subgroup H'' of H with $H' \subseteq H''$ and the canonical projection $\pi : H/H' \to H/H''$ such that $\pi \circ f$ is the zero morphism. This implies that $f(G) \subseteq H''/H'$, so that $H''/H' \ll H/H'$ by Lemma 1.2.
(2) ⇒ (1) Assume $f(G) \ll H/H'$. By Lemma 1.2, $f(G) = H''/H'$ with $H'' \ll H$. Since $\overline{f} = Im(\pi \circ f)$, then $\overline{f} = 0$. □

Theorem 1.5 *Let* G, H *be two groups and* $f : G \to H/H'$ *be a morphism with* $H' \ll H$. *Then:*

(1) $\overline{f} : \overline{G} \to \overline{H}$ *is a monomorphism if and only if the inverse image of* f *of all small subgroups of* H/H' *are small subgroup of* G.
(2) $\overline{f} : \overline{G} \to \overline{H}$ *is an epimorphism if and only if* f *is an epimorphism.*

(3) $\overline{f}:\overline{G}\to\overline{H}$ *is an epimorphism if and only if* f *is an epimorphism with* $Ker(f)\ll G$.

Proof. (1) ($:\Rightarrow$) Let $A/H'\ll H/H'$, $D=f^{-1}(A/H')$, and consider the embedding $\varepsilon:D\to A$. Then $Im(f\varepsilon)\subseteq A/H'$, and so $Im(f\varepsilon)\ll H/H'$. By Lemma 1.4, $\overline{f}\circ\overline{\varepsilon}=\overline{f\circ\varepsilon}=\overline{0}$. It follows that $\overline{\varepsilon}=\overline{0}$. That is $D=f^{-1}(A/H')$ is small in G.
($\Leftarrow$:) Let $G'\ll G$, we consider the homomorphisms $\alpha:A\to G/G'$ and $\beta:G/G'\to(H/H')/f(A)$. Assume that $\overline{f}\overline{\alpha}=\overline{0}$. Now $f'\circ\alpha(A)\ll(H/H')/f(G')$ by Lemma 1.4. Let $\alpha(A)=C/G'$ for some $C\in S(G)$ with $G'\subseteq C$. Then $f'\circ\alpha(A)=f(C)/f(G')$. By Lemma 1.2, we can see that $f(C)\ll H/H'$. By hypothesis, $f^{-1}(f(C))\ll G$. Since $C\subseteq f^{-1}(f(C))$, then $C\ll G$. Hence $\alpha(A)=C/G'\ll G/G'$ and so $\overline{\alpha}=\overline{0}$ by Lemma 1.4.
(2) ($:\Rightarrow$) We consider the canonical projection $\pi:H/H'\to(H/H')/f(G')$. Then $\overline{\pi}\circ\overline{f}=\overline{0}$. By hypothesis, $\overline{\pi}=\overline{0}$. Since $Im(\pi)$ is small and π is onto, we have $(H/H')/f(G')=0$.
($\Leftarrow$:) Let $f:G\to H/H'$ be an epimorphism with $H'\ll H$, $\alpha:H\to S/S'$ be a homomorphism with $S'\ll S$ and $\overline{\alpha}\circ\overline{f}=\overline{0}$. Consider $\beta:H/H'\to(S/S')/\alpha(H')$ be an induced homomorphism by α. Then $\beta\circ f:G\to(S/S')/\alpha(H')$ is an homomorphism, so that $Im(\beta\circ f)\ll(S/S')/\alpha(H')$. Hence $Im(\beta\circ f)=\beta(H)/\beta(H')$, and so $\beta(H)\ll S/S'$. This implies that $\overline{\beta}=\overline{0}$.
(3) Follows from (1) and (2). □

We know that $\mathbb{Z}^{(\mathbb{N})}=\oplus_{n\geq1}\mathbb{Z}x_n$ does not have small subgroups. But, in the Prüfer group $\mathbb{Z}(p^\infty)$, all proper subgroups are small. Let p be a prime and $A\in S(\mathbb{Z}^{(\mathbb{N})})$ generated by px_1 and $px_{n+1}-x_n$ for every ≥1. Then $\mathbb{Z}^{(\mathbb{N})}/A\cong\mathbb{Z}(p^\infty)$. Let G be a group, let $f:G\to\mathbb{Z}^{(\mathbb{N})}$ be a homomorphism and we consider the canonical projection $\pi:\mathbb{Z}^{(\mathbb{N})}\to\mathbb{Z}^{(\mathbb{N})}/A$. Assume that $\overline{\pi}\circ\overline{f}=\overline{0}$. Then $(\pi\circ f)(G)$ is a finite subgroup of $\mathbb{Z}^{(\mathbb{N})}/A$. It is easily seen that $\overline{\pi}$ does not have a kernel $\overline{\alpha}$, where $\alpha:G\to\mathbb{Z}^{(\mathbb{N})}$ a group morphism. Furthermore, since the Prüfer groups $\mathbb{Z}(p^\infty)$ are a set of cogenerators for the category of all groups, there exits a non-zero group morphism $\xi:G\to\mathbb{Z}(p^\infty)$ and $\xi(G)$ is a group of dual Goldie dimension 1.

2. The lifting condition for a quotient group and fully invariant lifting groups

We start with the following proposition.

Proposition 2.1 *Let* G *be a lifting group and* $X\in S(G)$. *If, for every direct summand* K *of* G, $(X+K)/X$ *is a direct summand of* G/X *then* G/X *is a lifting group.*

Proof. Let $A/X\leq G/X$. Since G is lifting, there exists a direct summand D of G such that $D\subseteq A$ and A/D is small in G/D. By hypothesis, $(D+X)/X$ is a direct summand of G/X. Clearly, $(D+X)/X\subseteq A/X$. Now we show that $A/(D+X)$ is small in $G/(D+X)$. Let $G/(D+X)=A/(D+X)+L/(D+X)$ for some subgroup $L/(D+X)$ of $G/(D+X)$. Then $G=A+L$ implies that $G/D=A/D+L/D$. Since A/D is small in G/D, we have $G=L$. Therefore $A/(D+X)$ is small in $G/(D+X)$. Thus G/X is lifting. □

Let G be a group (not necessary finite or abelian). A subgroup A of G is called *fully invariant* if for every $f:G\to G$, $f(X)\subseteq X$ (see [3, page 8]).

Let $F(G)$ denote the set of all fully invariant subgroups of G. We give by stating the next lemma.

Lemma 2.2 *Let G be a group.*

(1) *The set $F(G)$ is closed under taking sums and intersections.*
(2) *Let $G = A \oplus B$ and $C \in F(G)$. Then $C = (C \cap A) \oplus (C \cap B)$.*
(3) *If $X \leq Y \leq G$ such that $Y \in F(G)$ and $X \in F(Y)$, then $X \in F(G)$.*
(4) *If $G = \oplus_{i \in I} X_i$ and $S \in F(G)$, then $S = \oplus_{i \in I} \pi_i(S) = \oplus_{i \in I}(X_i \cap S)$, where π is the $i-th$ projection homomorphism of G.*
(5) *If $X \leq Y \leq G$ such that $X \in F(G)$ and $Y/X \in F(G/X)$, then $Y \in F(G)$.*

Proof. (2) See [3, Lemma 9.3].
(5) Let $f : G \to G$ be any group homomorphism. Then $f(X) \subseteq X$. Now, consider the homomorphism $h : G/X \to G/X$ defined by $h(g + X) = f(g) + X, \quad (g \in G)$. Then $h(Y/X) \subseteq Y/X$. Clearly, $h(Y/X) = (f(Y) + X)/X$. Therefore $f(Y) \subseteq Y$.
The proof of the other properties can be seen similarly. □

By [2, Proposition 2.1], $S(G)$ with respect to inclusion is a modular self dual lattice. Recall that a group G is called *distributive* if its lattice of subgroups is a distributive lattice.

Theorem 2.3 *Let G be a lifting (finite) group.*

(1) *If G is distributive, then G/A is lifting for every $A \in S(G)$.*
(2) *Let $A \in S(G)$ and $e(A) \subseteq A$ for all $e^2 = e : G \to G$. Then G/A is lifting. In particular, for every fully invariant subgroup B of G, G/B is lifting.*

Proof. (1) We know that if a finite group G is distributive and lifting then every primary component of G is $\mathbb{Z}_{p^n}$, so by [2, Theorem 2.3] the factor group G/A is lifting.
(2) Let D be a direct summand of G. Then $G = D \oplus D'$ for some $D' \in S(G)$, and so $|G| = |G : D|.|G : D'|$ and $D \cap D' = 0$ by [2, Proposition 3.1]. Consider the projection map $e : G \to D$. Then $e^2 = e : G \to G$. By hypothesis, $e(A) \subseteq A$ and hence $e(A) = A \cap D$. Since $(A \cap D) \cap (A \cap D') = 0$ and $|A| = |A : A \cap D|.|A : A \cap D'|$, we have $A = (A \cap D) \oplus (A \cap D')$ by [2, Proposition 3.1]. It is also easy to see that $D \cap (A \cap D') = 0$ and $|D + A| = |(D + A) : D|.|(D + A) : (A \cap D')|$, and $D' \cap (A \cap D) = 0$ and $|D' + A| = |(D' + A) : D'|.|(D' + A) : (A \cap D)|$, therefore $(D + A)/A = (D \oplus (A \cap D'))/A$ and $(D' + A)/A = (D' \oplus (A \cap D))/A$. Hence

$$G = D \oplus D' = D + A + D' + A = [D \oplus (A \cap D')] + D' + A$$

implies that

$$G/A = (D \oplus (A \cap D'))/A + (D' + A)/A.$$

Since $[D \oplus (A \cap D')] \cap (D' + A) = (A \cap D') \oplus (A \cap D)$, we have $G/A = (D \oplus (A \cap D'))/A \oplus (D' + A)/A$. Thus by Proposition 2.1, G/A is lifting. □

Let G be a lifting finite group. In Theorem 2.3, we proved that, for every fully invariant subgroup B of G, G/B is lifting. Now, we determine a generalization of the lifting groups. We say the group G is *FI-lifting* if for every $A \in F(G)$, there exists a direct summand B of G such that $B \subseteq A$ and A/B small in G/B. Clearly, G is FI-lifting if and only if for every $A \in F(G)$, there is a decomposition $G = G_1 \oplus G_2$ such that $G_1 \leq A$ and $G_2 \cap A \ll G_2$.

Theorem 2.4 *Let G be a finite group and $A \in F(G)$. If G is FI-lifting then G/A is FI-lifting.*

Proof. Let $B \in S(G)$ with $A \subseteq B$ and assume that B/A is a fully invariant subgroup of G/A. By Lemma 2.2, $Y \in F(G)$. Since G is FI-lifting, there exists a direct summand D of G such that $D \leq B$ and $B/D \ll G/D$. Assume $G = D \oplus D'$ for some $D' \in S(G)$. Then $G = D + D'$, $D \cap D' = 0$ and $|G| = |G : D|.|G : D'|$. Let π be the projection with the kernel D and $i : D' \to G$ the inclusion map. Now, $\alpha = i\pi : G \to G$ be a homomorphism of G. Since $A, B \in F(G)$, we have $\alpha(A) \subseteq A$ and $\alpha(B) \subseteq B$. It is easy to see that $B = \alpha^{-1}(B)$. Now, $\alpha^{-1}(A) \subseteq B = \alpha^{-1}(B)$. Let $K \in S(G)$ with $\alpha^{-1}(A) \subseteq K$ and $G/\alpha^{-1}(A) = (B/\alpha^{-1}(A)) + (K/\alpha^{-1}(A))$. Then $G = B + K$ and since B/D is small in G/D, $G = K$. Therefore $(B/A)/(\alpha^{-1}(A)/A) \ll (G/A)/(\alpha^{-1}(A)/A)$. Now, we want to show that $\alpha^{-1}(A)/A$ is a direct summand of G/A. Since $G = D \oplus D'$, then $G = \alpha^{-1}(A) + D'$. Therefore $G/A = (\alpha^{-1}(A)/A) + (D' + A)/A$. Since $\alpha^{-1}(A) \cap (D' + A) = A + (\alpha^{-1}(A) \cap D') = A$, we have $(\alpha^{-1}(A)/A) \cap (D' + A)/A = 0$. Finally, $|G/A| = |G/A : (\alpha^{-1}(A)/A|.|G/A : (D' + A)/A|$. Hence $\alpha^{-1}(A)/A$ is a direct summand of G/A by [2, Proposition 3.1]. □

In [2, Page 68-69], the authors pointed out, any finite direct sum of lifting groups need not be again a lifting group.

Let $G \neq 0$. If every proper subgroup of G is small in G, then G is called *hollow*. Finite hollow groups are lifting (see [2]).

Corollary 2.5 *Let $G \neq 0$ be a finite group. Then*

(1) *If G is a finite direct sum of lifting (or hollow) groups, then G is FI-lifting.*
(2) *Assume that G is an FI-lifting group. If G has no proper direct summands then every proper fully invariant subgroup of G is small in G.*

Proof. Clear. □

Theorem 2.6 *Let G be an FI-lifting group. Then every fully invariant subgroup of the quotient group $G/\mathrm{rad}(G)$ is a direct summand.*

Proof. Let $A/\mathrm{rad}(G)$ be any fully invariant subgroup of $G/\mathrm{rad}(G)$. Then $A \in F(G)$ by Lemma 2.2. By hypothesis, there exists a decomposition $G = G_1 \oplus G_2$ such that $G_1 \leq A$ and $A \cap G_2 \ll G_2$. Since $A \cap G_2$ is also small in G, then $A \cap G_2 \leq \mathrm{rad}(G)$. Thus $(A/\mathrm{rad}(G)) \cap ((G_2 + \mathrm{rad}(G))/Rad(G)) = 0$. It follows that $G/\mathrm{rad}(G) = (A/\mathrm{rad}(G)) \oplus ((G_2 + \mathrm{rad}(G))/Rad(G))$, as required. □

3. Projection invariant lifting groups

Proposition 3.1 *Let G be a group with $F(G) = S(G)$ and let $G = A \oplus B$.*

(1) *Then $F(A) = S(A)$ and $F(B) = S(B)$.*
(2) *There is no non-zero homomorphism from A to B and from B to A.*
(3) *Let $\pi_A : G \to A$ and $\pi_B : G \to B$. Then the natural projections π_A and π_B are central idempotents of the endomorphism group of G.*

Proof. (1) and (2) Let $X \in S(A)$ and $f : A \to B$. Then f can be extended to a $g : G \to G$, so that $g(B) = 0$. By [3, Lemma 9.5], we have $f(X) = g(X) \leq X$. The rest is clear.
(3) Let $G = A \oplus B$ and π denote the projection of G with kernel B. Let $f : G \to G$ and

$g = a + b'$ where $a \in A$, $b \in B$. By (1), we have $F(A) = S(A)$ and $F(B) = S(B)$, so that $f = f' + f''$ where $f' : A \to A$, $f'' : B \to B$. Now

$$\begin{aligned} \pi(g) &= a \\ f(g) &= f'(a) + f''(b) \\ f(a) &= f'(a) \\ f(b) &= f''(b). \end{aligned}$$

Then

$$f(\pi(g)) = f(a) = f'(a) = \pi(f(g))$$

for all $g \in G$. This follows that $f\pi = \pi f$ for all endomorphism f of G. □

Let G be a group. We say that G has *summand intersection property* if the intersection of any two direct summands of G is a direct summand of G and we denote it by SIP (see [5],[6],[12]).

Theorem 3.2 *Let G be a finite group with $F(G) = S(G)$. Then G has the SIP.*

Proof. Let G be a finite group with $F(G) = S(G)$. Let A and B be two direct summands of G with $F(A) = S(A)$ and $F(B) = S(B)$. Then, there exists $A', B' \in S(G)$ such that $G = A \oplus A' = B \oplus B'$. Note that $A \cap A' = 0$ and $|G| = |G : A|.|G : A'|$, and $B \cap B' = 0$ and $|G| = |G : B|.|G : B'|$. Let α and β be the natural projections such as $G = A \oplus A' \xrightarrow{\alpha} A$ and $G = B \oplus B' \xrightarrow{\beta} B$ with $A = \alpha(G)$ and $B = \beta(G)$. Since $\alpha(G) \cap A' = 0$, $\beta(G) \cap B' = 0$, $|G| = |G : \alpha(G)|.|G : A'|$ and $|G| = |G : \alpha(G)|.|G : B'|$, we have $G = \alpha(G) \oplus A'$ and $G = \beta(G) \oplus B'$ by [2, Proposition 3.1]. By hypothesis and Lemma 2.2(2), we have $\alpha(G) = \alpha(G) \cap \beta(G) \oplus \alpha(G) \cap B'$. It follows that $A \cap B = \alpha(G) \cap \beta(G)$ is a direct summand of G. Hence G has the SIP. □

Let G be a group. A subgroup $A \in S(G)$ is called a *projection invariant subgroup* if every projection π of G onto a direct summand maps A into itself, i.e. A is invariant under any projection of G (see [3, Page 50]). Let $P(G)$ denote the set of all projection invariant subgroups of G. We list below some of the basic properties of projection invariant subgroups that will be needed in this paper.

Lemma 3.3 *Let G be a group and $A \in S(G)$. Then;*

(1) *$A \in P(G)$ if and only if $\pi(A) = A \cap \pi(G)$ for every projection π of G.*
(2) *$A \in P(G)$ if and only if A is an intersection of projection invariant subgroups G.*
(3) *The set $P(G)$ is closed under taking any sums and intersections.*
(4) *If $A \in P(G)$ with direct summand of G, then $A \in F(G)$.*
(5) *Let $G = G_1 \oplus G_2$ be a decomposition and $A \in P(G)$. Then $A = (A \cap G_1) \oplus (A \cap G_2)$.*
(6) *If $G = \oplus_{i \in I} G_i$ and $A \in P(G)$, then $A = \oplus_{i \in I} \pi_i(A) = \oplus_{i \in I}(G_i \cap A)$, where π_i is the i-th projection homomorphism of G along G_i.*

Proof. See [3, Page 50]. □

Theorem 3.4 *Let G be a group with $P(G) = S(G)$. Then;*

(1) *If A is a direct summand of G, then $P(A) = S(A)$ and $F(A) = S(A)$.*
(2) *G has the SIP.*

Proof. (1) Assume that $P(G) = S(G)$ and $G = A \oplus B$ with $A, B \in S(G)$. Let $\pi_A : G \to A$ be the canonical projection and let $X \in S(A)$. We consider the canonical projection $\pi : A = A \oplus (0) \to A$. Then $p = \pi\pi_A$ is a projection of G and $\pi(X) = p(X)$ which is contained in X because $P(G) = S(G)$. It follows that $X \in P(A)$.
(2) It follows from Lemma 3.3 and Theorem 3.2 □

A group G is called PI-lifting if for every $A \in P(G)$, there exists a direct summand B of G such that $B \subseteq A$ and $A/B \ll G/B$.

Lemma 3.5 *Let G be a finite group.*

1. *G is PI-lifting.*
2. *For every $A \in P(G)$, there is a decomposition $A = D \oplus S$ with D a direct summand of G and S small in G.*

Proof. (1) $\Rightarrow$ (2) Let $A \in P(G)$. Since G is PI-lifting, there exists a decomposition $G = G_1 \oplus G_2$ such that $G_1 \leq A$ and $G_2 \cap A \ll G_2$. Then $G_1 \cap G_2 = 0$ and $|G| = |G : G_1|.|G : G_2|$ by [2, Proposition 3.1]. Note that $A = G_1 + (A \cap G_2)$, $G_1 \cap (A \cap G_2) = 0$, and $|A| = |A : G_1|.|A : (A \cap G_2)|$. Therefore $A = G_1 \oplus (A \cap G_2)$, as required.
(2) $\Rightarrow$ (1) Assume that every element of $P(G)$ has the stated decomposition. Let $A \in P(G)$. By hypothesis, there exists a direct summand D of G and a small subgroup S of G such that $A = D \oplus S$. Then $A = D + S$, $D \cap S = 0$ and $|A| = |A : D|.|A : S|$ by [2, Proposition 3.1]. Since D is a direct summand of G, by [2, Proposition 3.1], we have $G = D \oplus D'$ for some $D' \in S(G)$. Consider the natural epimorphism $\pi : G \longrightarrow G/D$. Then $\pi(S) = (S + D)/D = A/D$ small in G/D. Therefore G is PI-lifting. □

Theorem 3.6 *Let $G = \oplus_{i=1}^{n} G_i$. If each G_i is PI-lifting, then G is PI-lifting.*

Proof. Let $A \in P(G)$. It is easy to see that for every $1 \leq i \leq n$, $A \cap G_i$ is projection invariant in G_i. Since G_i is PI-lifting for every i, there exists a direct summand D_i of G_i such that $D_i \leq A \cap G_i$ and $(A \cap G_i)/D_i$ is small in G_i/D_i for every i. Note that $D = \oplus_{i=1}^{n} D_i$ is a direct summand of G and $D \subseteq \oplus_{i=1}^{n}(A \cap G_i)$. By Lemma 3.3, $\oplus_{i=1}^{n}(A \cap G_i) = A$.
Now consider the homomorphism $\beta : \oplus_{i=1}^{n}(G_i/D_i) \to (\oplus_{i=1}^{n} G_i)/D$ with $(g_1 + D_1, ..., g_n + D_n) \to (\Sigma_{i=1}^{n} g_i) + D_i$, where $g_i \in G_i$ for $1 \leq i \leq n$. Then $\beta(\oplus_{i=1}^{n}((A \cap G_i)/D_i)) = (\oplus_{i=1}^{n}(A \cap G_i))/D$.
Since any finite sum of small subgroups again a small subgroup, $\oplus_{i=1}^{n}((A \cap G_i)/D_i) \ll \oplus_{i=1}^{n}(G_i/D_i)$. Then $(\oplus_{i=1}^{n}(A \cap G_i))/D \ll G/D$. □

Proposition 3.7 *Let G be a finite group and $X \in P(G)$. Assume that $X'/X \in P(G/X)$ where $X \subseteq X' \subseteq G$. Then $X' \in P(G)$. If G is a PI-lifting group then G/X is PI-lifting.*

Proof. Let $Y \in S(G)$ with $X \subseteq Y$ and let $Y/X \in P(G/X)$. By assumption, Y is a projection invariant subgroup of G, i.e., $Y \in P(G)$. Since G is a PI-lifting group, there exists a direct summand D of G such that $D \leq Y$ and $Y/D \ll G/D$. Assume $G = D \oplus D'$ for some $D' \in S(G)$. Then $D \cap D' = 0$ and $|G| = |G : D|.|G : D'|$ by [2, Proposition 3.1]. Let π be the projection with the kernel D and $i : D' \to G$ the inclusion map. Now, $\alpha = i\pi : G \to G$ be a homomorphism of G. Since $X, Y \in P(G)$, then $\alpha(X) \subseteq X$ and $\alpha(Y) \subseteq Y$. It is easy to see that $Y = \alpha^{-1}(Y)$. Now, $\alpha^{-1}(X) \subseteq Y = \alpha^{-1}(Y)$.

Let $K \in S(G)$ with $\alpha^{-1}(X) \subseteq K$ and $G/\alpha^{-1}(X) = (Y/\alpha^{-1}(X)) + (K/\alpha^{-1}(X))$. Then $G = Y + K$ and, since Y/D is small in G/D, $G = K$. Therefore $Y/\alpha^{-1}(X)$ is small in $G/\alpha^{-1}(X)$, namely $(Y/X)/(\alpha^{-1}(X)/X) \ll (G/X)/(\alpha^{-1}(X)/X)$. Now, we want to show that $\alpha^{-1}(X)/X$ is a direct summand of G/X. Since $G = D \oplus D'$, then $G = \alpha^{-1}(X) + D'$.

Therefore $G/X = (\alpha^{-1}(X)/X) + (D'+X)/X$. Since $\alpha^{-1}(X)\cap(D'+X) = X+(\alpha^{-1}(X)\cap D') = X$ and $|G| = |G : \alpha^{-1}(X)|.|G : D'|$, then $\alpha^{-1}(X)/X$ is a direct summand of G/X by [2, Proposition 3.1]. Hence G/X is a PI-lifting group. □

Theorem 3.8 Let G be a finite group with $P(G) = S(G)$ and G has a decomposition $G = G_1 \oplus G_2$. Then G is a PI-lifting group if and only if G_1 and G_2 are PI-lifting groups.

Proof. By Theorem 3.6 and Proposition 3.7. □

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