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A new characterization of symmetric groups for some n

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Abstract

Let G be a finite group and let $\pi_e(G)$ be the set of element orders G. Let $k \in \pi_e(G)$ and let m_k be the number of elements of order k in G. Set nse(G):=$\{m_k | k \in \pi_e(G)\}$. In this paper, we prove that if G is a group such that nse(G)=nse(S_n) where $n \in \{3, 4, 5, 6, 7\}$, then $G \cong S_n$.

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1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Set $m_i = m_i(G) = |\{g \in G|$ the order of g is $i\}|$ and nse(G):=$\{m_i|\ i \in \pi_e(G)\}$.

Let $L_t(G) := \{g \in G|\ g^t = 1\}$. Then G_1 and G_2 are of the same order type if and only if $|L_t(G_1)| = |L_t(G_2)|$, $t = 1, 2, ...$ The idea of this paper springs from Thompson's Problem as follows:

Thompson's Problem. Suppose that G_1 and G_2 are of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Unfortunately, as so far, no one can prove it completely, or even give a counterexample. However, if groups G_1 and G_2 are of the same order type, we see clearly that $|G_1| = |G_2|$

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and $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$. So it is natural to investigate the Thompson's Problem by $|G|$ and $\mathrm{nse}(G)$.

In [13], it is proved that all simple K_4-groups can be uniquely determined by $\mathrm{nse}(G)$ and $|G|$. Further, it is claimed that some simple groups could be characterized by exactly the set nse without considering group order. For instance, in [3, 12], it is proved that the alternating groups A_n where $n \in \{4, 5, 6, 7, 8\}$ are uniquely determined by $\mathrm{nse}(G)$. Also in [10], it is proved that $L_2(q)$ where $q \in \{7, 8, 11, 13\}$ are uniquely determined by $\mathrm{nse}(G)$. Analogously, for infinite simple groups, there are also some interesting results: In [1], the author prove that $G \cong \mathrm{PGL}_2(p)$ if and only the two conditions hold: (1) $p \in \pi(G)$ but $p^2 \nmid |G|$; (2) $\mathrm{nse}(G) = \mathrm{nse}(\mathrm{PGL}_2(p))$, where $p > 3$ is a prime. In [2], the authors proved that all sporadic groups characterizable by $\mathrm{nse}(G)$ and $|G|$.

In this paper we show that the symmetric group S_n is characterizable by $\mathrm{nse}(G)$ for $n \in \{3, 4, 5, 6, 7\}$. In fact the main theorem of our paper is as follows:

Main Theorem: Let G be a group such that $\mathrm{nse}(G)$=$\mathrm{nse}(S_n)$ where $n \in \{3, 4, 5, 6, 7\}$. Then $G \cong S_n$.

Note that not all groups can be characterized by $\mathrm{nse}(G)$ and $|G|$. For instance, in 1987, Thompson gave an example as follows: Let G_1= $(C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be the maximal subgroups of M_{23}, where M_{23} is the Mathieu group of degree 23. Although $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$, we still have $G_1 \ncong G_2$.

Throughout this paper, we denote by ϕ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow $q-$subgroup of G and $n_q(G)$ is the number of Sylow $q-$subgroup of G, that is, $n_q(G) = |\mathrm{Syl}_q(G)|$. All other notations are standard and we refer to [5], for example.

2. Some lemmas

In this section we collect some preliminary lemmas used in the proof of the main theorem.

Lemma 2.1. [6] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | \ g^m = 1\}$, then $m \mid |L_m(G)|$.

Lemma 2.2. [12] Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k | \ k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Lemma 2.3. [11] Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow $p-$subgroup of G and $n = p^s m$ where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.

Lemma 2.4. [8] Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and let h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1} ... q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, ..., s\}$:

(1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.

(2) The order of some chief factor of G is divisible by $q_i^{\beta_i}$.

Lemma 2.5. [13] Let G be a finite group, $P \in \mathrm{Syl}_p(G)$ where $p \in \pi(G)$. Let G have a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \nmid |K|$, then the following hold:

(1) $N_{G/K}(PK/K) = N_G(P)K/K$;

(2) $|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;

(3) $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t =$

$n_p(G) = n_p(L)$ for some positive integer t, and $|N_K(P)|t = |K|$.

Lemma 2.6. [9] If G is a simple K_3- group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.

Lemma 2.7. [14] Let G be a simple group of order $2^a \cdot 3^b \cdot 5 \cdot p^c$ where $p \notin \{2, 3, 5\}$ is a prime and $abc \neq 0$. Then G is isomorphic to one of the following groups: A_7, A_8, A_9; M_{11}, M_{12}; $L_2(q)$, $q = 11, 16, 19, 31, 81$; $L_3(4)$, $L_4(3)$, $S_6(2)$, $U_4(3)$ or $U_5(2)$. In particular, if $p = 11$, then $G \cong M_{11}$, M_{12}, $L_2(11)$ or $U_5(2)$; if $p = 7$, then $G \cong A_7$, A_8, A_9, A_{10}, $L_2(49)$, $L_3(4)$, $S_4(7)$, $S_6(2)$, $U_3(5)$, $U_4(3)$, J_2, or $O_8^+(2)$.

Let G be a group such that nse(G)=nse(S_n) where $n \in \{3, 4, 5, 6, 7\}$. By Lemma 2.2, we can assume that G is finite. Let m_n be the number of elements of order n. We note that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \mid |G|$, then by Lemma 2.1 and the above notation, we have

$$\begin{cases} \phi(n) \mid m_n \\ n \mid \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we often apply $(*)$ and the above comments.

3. Proof of the Main Theorem.

Case 1. Let G be a group such that nse(G)=nse(S_3)=$\{1, 2, 3\}$. First we prove that $\pi(G) \subseteq \{2, 3\}$. Since $3 \in$nse(G), it follows that by $(*)$, $2 \in \pi(G)$ and $m_2 = 3$. Let $2 \neq p \in \pi(G)$. By $(*)$, $p \mid (1 + m_p)$ and $(p - 1) \mid m_p$, which implies that $p = 3$. Thus $\pi(G) \subseteq \{2, 3\}$. If $3 \in \pi(G)$, then $m_3 = 2$. If $6 \in \pi_e(G)$, then by $(*)$, $m_6 = 2$ and $6 \mid (1 + m_2 + m_3 + m_6) = 8$, a contradiction. If $2^i \in \pi_e(G)$ for some $i \geq 2$, then $2^{i-1} = \phi(2^i) \mid m_{2^i} = 2$. So $i = 2$. If $3^j \in \pi_e(G)$ for some $j \geq 2$, then $2 \times 3^{j-1} = \phi(3^j) \mid m_{3^j} = 2$, a contradiction. Therefore, $\pi_e(G) \subseteq \{1, 2, 3, 4\}$ and $|G| = 6 + 2k = 2^m \times 3^n$ where k, m and n are non-negative integers. Now we consider the following subcases:

Subcase (a). If $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 4\}$. Since $|\pi_e(G)| \leq 3$, $|G| = 2^m = 6+2k$ where $k = 0$, a contradiction.

Subcase (b). If $\pi(G) = \{2, 3\}$, then since $|G| = 6 + 2k = 2^m \times 3^n$ and $|\pi_e(G)| \leq 4$, $0 \leq k \leq 1$. It easy to check that the only solution of the equation is $(k, m, n) = (0, 1, 1)$. Thus $|G| = 6$, $\pi_e(G) = \{1, 2, 3\}$. Therefore, $G \cong S_3$.

Case 2. Let G be a group such that nse(G)=nse(S_4)=$\{1, 6, 8, 9\}$. First we prove that $\pi(G) \subseteq \{2, 3\}$. Since $9 \in$nse(G), it follows that by $(*)$, $2 \in \pi(G)$ and $m_2 = 9$. Let $2 \neq p \in \pi(G)$. By $(*)$, $p \in \{3, 7\}$. Thus $\pi(G) \subseteq \{2, 3, 7\}$. If $7 \in \pi(G)$, then $m_7 = 6$.

We prove that $14 \notin \pi_e(G)$. If $14 \in \pi_e(G)$, then $m_{14} = 6$, by $(*)$. On the other hand, $14 \mid (1 + m_2 + m_7 + m_{14}) = 22$, a contradiction. Therefore, the group P_7 acts fixed point freely on the set of elements of order 2. Hence $|P_7| \mid m_2 = 9$, a contradiction. Hence $\pi(G) \subseteq \{2, 3\}$. If $3 \in \pi(G)$, then by $(*)$, $m_3 = 8$. It is clear that by $(*)$, G does not contain any elements of order 6 and 9.

If $2^i \in \pi(G)$ for some $i \geq 2$, then $2^{i-1} \mid m_{2^i}$ where $m_{2^i} \in \{6, 8\}$. So $2 \leq i \leq 4$. Also by $(*)$, $m_4 = 6$, $m_8 = 8$ and $m_{16} = 8$. Therefore $\pi_e(G) \subseteq \{1, 2, 3, 4, 8, 16\}$ and

$|G| = 24 + 6k_1 + 8k_2 = 2^m \times 3^n$ where k_1, k_2, m and n are non-negative integers. Now we consider the following subcases:

Subcase (a). If $\pi(G) = \{2\}$, then since $|G| = 24 + 6k_1 + 8k_2 = 2^m$ and $|\pi_e(G)| \leq 6$, $0 \leq k_1 + k_2 \leq 1$. It easy to check that the only solution is $(k_1, k_2, m) = (0, 1, 5)$. Thus $|G| = 2^5$ and $\pi_e(G) = \{1, 2, 4, 8, 16\}$. But all groups of order 32 with element of order 16 are known, in particular, there are only four such non-Abelian groups by [7] (see, Chapter 5, Theorem 4.4). Since nse of such non-Abelian groups are not equal to nse(G), it is impossible.

Subcase (b). Suppose that $\pi(G) = \{2, 3\}$. By assumption $|G| = 24+6k_1+8k_2 = 2^m \times 3^n$. Since $|\pi_e(G)| \leq 6$, $0 \leq k_1 + k_2 \leq 2$. Hence $12 + 3k_1 + 4k_2 = 2^{m-1} \times 3^n$. Since $3 \mid k_2$, it follows that $k_2 = 0$. It easy to check that the only solutions of equation are $(k_1, k_2, m, n) = (0, 0, 3, 1)$ or $(2, 0, 2, 2)$.

If $(k_1, k_2, m, n) = (2, 0, 2, 2)$, then $|G| = 36$ and $|\pi_e(G)| = 6$. On the other hand, $|P_2| = 4$ so $\pi_e(G) = \{1, 2, 3, 4\}$, a contradiction.

Therefore $(k_1, k_2, m, n) = (0, 0, 3, 1)$, $|G| = 24$ and $\pi_e(G) = \{1, 2, 3, 4\}$, which implies that $G \cong S_4$.

Case 3. Let G be a group such that nse(G)=nse(S_5)=$\{1, 20, 24, 25, 30\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5\}$. Since $25 \in$nse(G), it follows that $2 \in \pi(G)$ and $m_2 = 25$. Let $2 \neq p \in \pi(G)$. By $(*)$, $p \in \{3, 5, 31\}$.

If $p = 31$, then $m_{31} = 30$. On the other hand, if $62 \in \pi_e(G)$, then $m_{62} = 30$ and $62 \mid 1 + m_2 + m_{31} + m_{62} = 86$, a contradiction. So $62 \notin \pi_e(G)$. Then the group P_{31} acts fixed point freely on the set of elements of order 2. Thus $|P_{31}| \mid m_2$, a contradiction.

Therefore, $\pi(G) \subseteq \{2, 3, 5\}$. If $3, 5 \in \pi(G)$, then $m_3 = 20$ and $m_5 = 24$. It is clear that by $(*)$, G does not contain any elements of order 15, 16, 18 or 25. If $4, 16 \in \pi_e(G)$, then $m_4 = 30$ and $m_8 = 24$. If $2^i \in \pi_e(G)$ for some $i \geq 2$, then $2^{i-1} \mid m_{2^i}$ where $m_{2^i} \in \{20, 24, 30\}$. Hence $2 \leq i \leq 3$. If $3^j \in \pi_e(G)$ for some $j \geq 2$, then $2 \times 3^{j-1} \mid m_{3^j}$ where $m_{3^j} \in \{24, 30\}$. Thus $j = 2$. If $5^k \in \pi_e(G)$ for some $k \geq 2$, then $4 \times 5^{k-1} \mid m_{5^k}$ where $m_{5^k} \in \{20, 24\}$. Thus $k = 2$. Since $25 \notin \pi_e(G)$,we get a contradiction.

Therefore $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 24\}$ and $|G| = 100+20k_1+24k_2+30k_3 = 2^m \times 3^n \times 5^r$ where k_1, k_2, k_2, m, n and r are non-negative integers. Now we consider the following subcases:

Subcase (a). Suppose that $\pi(G) = \{2\}$. Then $|\pi_e(G)| \leq 4$. Since nse(G) have five elements and $|\pi_e(G)| \leq 4$, we get a contradiction.

Subcase (b). Suppose that $\pi(G) = \{2, 5\}$. Then $|G| = 100+20k_1+24k_2+30k_3 = 2^m \times 5^n$ and by $|\pi_e(G)| \leq 6$, we have $0 \leq k_1 + k_2 + k_3 \leq 1$. Hence $5 \mid k_2$, which implies that $k_2 = 0$, and so $50 + 10k_1 + 15k_3 = 2^{m-1} \times 5^n$. Hence $2 \mid k_3$, which implies that $k_3 = 0$. It is easy to check that the only solution of the equation is $(k_1, k_2, k_3, m, n) = (0, 0, 0, 2, 2)$. Thus $|G| = 2^2 \times 5^2$. It is clear that $\pi_e(G) = \{1, 2, 4, 5, 10\}$, and so $\exp(P_2) = 4$. Then P_2 is cyclic. Thus $n_2 = m_4/\phi(4) = 15$. Since every cyclic Sylow 2-subgroup has one element of order 2, $m_2 \leq 15$, a contradiction.

Subcase (c). Suppose that $\pi(G) = \{2, 3\}$. Since $27 \notin \pi_e(G)$, $\exp(P_3) = 3$ or 9. If $\exp(P_3) = 3$, then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6, 8, 12, 24\}$. By Lemma 2.1, $|P_3| \mid (1+m_3) = 21$. Hence $|P_3| = 3$. Therefore, $100+20k_1+24k_2+30k_3 = 2^m \times 3 = |G|$ and $0 \leq k_1+k_2+k_3 \leq$

3. It is clear that $100 \leq 2^m \times 3 \leq 190$. Hence $m = 6$ and $20k_1 + 24k_2 + 30k_3 = 92$. It is easy to check that the equation has no solution.

If $\exp(P_3) = 9$, then since $m_9 = 24$, $|P_3| \mid (1 + m_3 + m_9) = 45$. Hence $|P_3| = 9$ and $n_3 = m_9/\phi(9) = 4$. Since a cyclic group of order 9 have two elements of order 3, $m_3 \leq 4 \times 2 = 8$, a contradiction.

Subcase (d). Suppose that $\pi(G) = \{2, 3, 5\}$. Since G has no element of order 15, the group P_5 acts fixed point freely on the set of elements of order 3. Thus $|P_5| \mid m_3 = 20$, which implies that $r = 1$. Similarly, the group P_3 acts fixed point freely on the set of elements of order 5. Thus $|P_3| \mid m_5 = 24$, which implies that $n = 1$.

We will show $10 \notin \pi_e(G)$. Suppose that $10 \in \pi_e(G)$. We know that if P and Q are Sylow 5−subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate. Therefore $m_{10} = \phi(10) \cdot n_5 \cdot k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_5)$. Since $n_5 = m_5/\phi(5) = 6$, $24 \mid m_{10}$. Hence $m_{10} = 24$. By Lemma 2.1, $10 \mid (1 + m_2 + m_5 + m_{10}) = 74$, a contradiction.

Therefore the group P_2 acts fixed point freely on the set of elements of order 5. Then $|P_2| \mid m_5 = 24$, which implies that $|P_2| \mid 8$. Thus $100 + 20k_1 + 24k_2 + 30k_3 = 2^m \times 3 \times 5$, where $0 \leq k_1 + k_2 + k_3 \leq 4$ and $m \leq 3$. It is clear that $100 \leq 2^m \times 3 \times 5 \leq 190$, hence $m = 3$. It is easy to check that the only solution of the equation is $(k_1, k_2, k_3) = (1, 0, 0)$. Thus $|G| = 2^3 \times 3 \times 5$, $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$, and by the main result of [4], $G \cong S_5$.

Case 4. Let G be a group such that nse(G)=nse(S_6)=$\{1, 75, 80, 180, 144, 240\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5\}$. Since 75 ∈nse($G$), it follows that $2 \in \pi(G)$ and $m_2 = 75$. Let $2 \neq p \in \pi(G)$. By $(*)$, $p \in \{3, 5, 181, 241\}$. If $181 \in \pi(G)$, then by $(*)$, $m_{181} = 180$. If $282 \in \pi_e(G)$, then we conclude that $m_{282} = 180$, but by $(*)$, we get a contradiction. Therefore $282 \notin \pi_e(G)$.

Since $282 \notin \pi_e(G)$, the group P_{181} acts fixed point freely on the set of elements of order 2. Then $|P_{181}| \mid m_2$, a contradiction. Similarly, if $241 \in \pi(G)$, we get a contradiction. Hence $\pi(G) \subseteq \{2, 3, 5\}$.

If $3, 5 \in \pi_e(G)$, then $m_3 = 80$ and $m_5 = 144$, by $(*)$. If $2^i \in \pi_e(G)$ for some $i \geq 2$, then $2^{i-1} = \phi(2^i) \mid m_{2^i}$. Thus $2 \leq i \leq 5$. If $3^j \in \pi_e(G)$ for some $j \geq 2$, then $2 \times 3^{j-1} = \phi(3^j) \mid m_{3^j}$. Then $2 \leq j \leq 3$. If $5^k \in \pi_e(G)$ for some $k \geq 2$, then $4 \times 5^{k-1} \mid m_{5^k}$ and so $k = 2$. If $2^a \times 3^b \in \pi_e(G)$ for some $a, b > 0$, then $1 \leq a \leq 4$ and $1 \leq b \leq 3$. If $2^a \times 5^b \in \pi_e(G)$ for some $a, b > 0$, then $1 \leq a \leq 5$ and $1 \leq b \leq 2$. If $3^a \times 5^b \in \pi_e(G)$ for some $a, b > 0$, then $1 \leq a \leq 2$ and $1 \leq b \leq 2$. If $2^a \times 3^b \times 5^c \in \pi_e(G)$ for some $a, b, c > 0$, then $1 \leq a \leq 3$, $1 \leq b \leq 2$ and $1 \leq c \leq 2$.

Therefore $\pi_e(G) \subseteq \{ 1, 2, 2^2, 2^3, 2^4, 2^5, 3, 3^2, 3^2, 3^3, 5, 5^2\} \bigcup \{2^a \times 3^b \mid 1 \leq a \leq 4, 1 \leq b \leq 3 \} \bigcup \{2^a \times 5^b \mid 1 \leq a \leq 3, 1 \leq b \leq 2 \} \bigcup \{ 3^a \times 5^b \mid 1 \leq a \leq 2, 1 \leq b \leq 2\} \bigcup \{ 2^a \times 3^b \times 5^c \mid 1 \leq a \leq 3, 1 \leq b \leq 2, 1 \leq c \leq 2\}$.

Hence $|G| = 2^m \times 3^n \times 5^r = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4$ where k_1, k_2, k_2, m, n and r are non-negative integers. Now we consider the following subcases:

Subcase (a). Suppose that $\pi(G) = \{2\}$. Then $360 + 40k_1 + 72k_2 + 90k_3 + 120k_4 = 2^{m-1}$. Since $|\pi_e(G)| \leq 6$, $k_1+k_2+k_3+k_4 = 0$. It is easy to see that this equation has no solution.

Subcase (b). Suppose that $\pi(G) = \{2, 5\}$. Since $5^3 \notin \pi_e(G)$, $\exp(P_5) = 5$ or 25. Let $\exp(P_5) = 5$, then by Lemma 2.1, $|P_5| \mid (1 + m_5) = 145$. Hence $|P_5| = 5$. On the other hand, $|\pi_e(G)| \leq 10$. Therefore $720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^m \times 5$, where $0 \leq k_1 + k_2 + k_3 + k_4 \leq 4$. Hence $5 \mid k_2$, then $k_2 = 0$. It is easy to see that this equation has no solution.

If $\exp(P_5) = 25$, then by Lemma 2.1, $|P_5| \mid (1 + m_5 + m_{25})$. Hence $|P_5| = 25$ and P_5 is cyclic. Thus $n_5 = m_{25}/\phi(25)$. Since $m_{25} \in \{80, 180\}$, $n_5 = 4$ or 9, a contradiction.

Subcase (c). Suppose that $\pi(G) = \{2, 3\}$. Since $3^4 \notin \pi_e(G)$, $\exp(P_3) = 3$, 9 or 27. Let $\exp(P_3) = 3$. Then $|P_3| \mid (1 + m_3) = 81$, by Lemma 2.1. If $|P_3| = 3$, then $n_3 = m_3/\phi(3) = 40 \mid |G|$, we get a contradiction by $5 \notin \pi(G)$.

If $|P_3| = 9$, then $|G| = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^m \times 9$. Since $\pi_e(G) \subseteq \{ 1, 2, 2^2, 2^3, 2^4, 2^5, 3, 3 \times 2, 3 \times 2^2, 3 \times 2^3, 3 \times 2^4\}$, $0 \leq k_1 + k_2 + k_3 + k_4 \leq 5$. As $720 \leq 2^m \times 9 \leq 1920$, $m = 7$. Therefore, $432 = 80k_1 + 144k_2 + 180k_3 + 240k_4$. The only solution of this equation is $(k_1, k_2, k_3, k_4) = (0, 3, 0, 0)$. Then $|\pi_e(G)| = 9$, it is clear that $\exp(P_2) = 16$ or 32.

If $\exp(P_2) = 16$, then $\pi_e(G) = \{1, 2, 3, 4, 6, 8, 12, 16, 24\}$. Since $48 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 16. Hence $|P_3| \mid m_{16}$. We have $m_{16} \in \{144, 240\}$. If $m_{16} = 240$, we get a contradiction by $|P_3| \mid m_{16}$. If $m_{16} = 144$, then by $(*)$, $m_{24} = 240$. If $m_8 = 144$, then $m_4 = 180$ and if $m_8 = 180$, then $m_4 = 144$. By Lemma 2.1, $|P_2| \mid (1 + m_2 + m_4 + m_8 + m_{16}) = 544$. Because $|P_2| = 2^7$, we get a contradiction.

If $\exp(P_2) = 32$, then $\pi_e(G) = \{1, 2, 3, 4, 6, 8, 12, 16, 32\}$. Since 24, 48, $96 \notin \pi_e(G)$, then the group P_3 acts fixed point freely on the set of elements of order 8, 16 or 32. Hence $|P_3| \mid m_8$, m_{16} or m_{32}. We know that $m_8 \in \{144, 180, 240\}$, if $m_8 = 144$, then by $(*)$, $m_4 = 180$. Therefore m_{16} or $m_{32} \neq 144$. Since $|P_3| \mid m_{16}$ or m_{32}, we get a contradiction. If $m_8 = 180$, then $m_4 = 144$. Thus m_{16} or $m_{32} \neq 144$. Since $|P_3| \mid m_{16}$ or m_{32}, a contradiction. Similarly, if $m_8 = 240$, we get a contradiction by $|P_3| \mid m_8$.

Suppose that $|P_3| = 27$. Then $|G| = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^m \times 27$ where $0 \leq k_1 + k_2 + k_3 + k_4 \leq 5$. Hence $720 \leq 2^m \times 27 \leq 1920$. Thus $m = 5$ or 6.

If $m = 5$, then $144 = 80k_1 + 144k_2 + 180k_3 + 240k_4$. The only solution of this equation is $(k_1, k_2, k_3, k_4) = (0, 1, 0, 0)$. Thus $|\pi_e(G)| = 7$, it is clear that $\exp(P_2) = 8$, 16 or 32.

If $\exp(P_2) = 8$, then $\pi_e(G) = \{1, 2, 3, 4, 6, 8, 12\}$. Since $24 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 8. Hence $|P_3| \mid m_8$, by $m_8 \in \{144, 180, 240\}$, we get a contradiction.

If $\exp(P_2) = 16$, then $\pi_e(G) = \{1, 2, 3, 4, 6, 8, 16\}$. Since $12 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 4. Hence $|P_3| \mid m_4$. By $m_4 \in \{144, 80, 240\}$, we get a contradiction.

If $\exp(P_2) = 32$, then $\pi_e(G) = \{1, 2, 3, 4, 8, 16, 32\}$. Since $6 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 2. Hence $|P_3| \mid m_2$. This is a contradiction because $|P_3| = 9$.

If $m = 6$, then by arguing as above we can rule out this case. Also by arguing as above we can rule out the case $|P_3| = 81$.

Suppose that $\exp(P_3) = 9$. By $(*)$, $m_9 \in \{144, 180\}$. Then $|P_3| \mid (1 + m_3 + m_9) = 225$ or 261. Hence $|P_3| = 9$ and $n_3 = m_9/\phi(9) \in \{24, 30\}$, a contradiction.

If $\exp(P_3) = 27$, then by $(*)$, $m_{27} \in \{144, 180\}$. If P_3 be a cyclic group, then since $\exp(P_3) = 27$, $n_3 = m_{27}/\phi(27) \in \{8, 10\}$. If $n_3 = 8$, then we get a contradiction by Sylow theorem and if $n_3 = 10$, then since a cyclic group of order 27 have two elements of order 3, $m_3 \leq 10 \times 2 = 20$, a contradiction. Therefore P_3 is not cyclic. By Lemma 2.3, $27 \mid m_{27}$, a contradiction.

Subcase (d). Suppose that $\pi(G) = \{2, 3, 5\}$. We know that $\exp(P_5) = 5$ and $|P_5| = 5$. Suppose that $15 \in \pi_e(G)$, then $m_{15} = \phi(15) \cdot n_5 \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_5)$. Since $n_5 = m_5/\phi(5) = 36$, $288 \mid m_{15}$, a contradiction. Thus $15 \notin \pi_e(G)$. Similarly, $10 \notin \pi_e(G)$.

Since $15 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 5. Hence $|P_3| \mid m_5 = 144$. Then $|P_3| = 3$ or 9. Since $10 \notin \pi_e(G)$, the group P_2 acts fixed point freely on the set of elements of order 5. Hence $|P_2| \mid m_5 = 144$. Then $|P_2| = 2^m$, where $1 \leq m \leq 4$. Therefore $|G| = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^m \times 3^n \times 5$ where $1 \leq m \leq 4$ and $1 \leq n \leq 2$. The only solution of this equation is $(k_1, k_2, k_3, k_4, m, n) = (0, 0, 0, 0, 4, 2)$. Thus $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$, $\{1, 2, 3, 4, 5, 8\}$, $\{1, 2, 3, 4, 5, 9\}$ or $\{1, 2, 3, 5, 6, 9\}$.

If $\pi_e(G) = \{1, 2, 3, 4, 5, 9\}$ or $\{1, 2, 3, 5, 6, 9\}$, then $\exp(P_3) = 9$ and $|P_3| = 9$. Because $m_9 \in \{144, 180\}$, $n_3 = m_9/\phi(9) \in \{24, 30\}$, we get a contradiction.

If $\pi_e(G) = \{1, 2, 3, 4, 5, 8\}$, then $6 \notin \pi_e(G)$. Thus the group P_3 acts fixed point freely on the set of elements of order 2. So, $|P_3| \mid m_2 = 75$. Since $|P_3| = 9$, we get a contradiction.

Therefore $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$. Now by the main result of [4], $G \cong S_6$.

Case 5. Let G be a group such that nse(G)=nse(S_7)=$\{1, 231, 350, 420, 504, 720, 840, 1470\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5, 7\}$. Since $231 \in$nse(G), it follows that $2 \in \pi(G)$ and $m_2 = 231$. Let $2 \neq p \in \pi(G)$. By $(*)$, $p \in \{3, 5, 29, 421, 1471\}$. If $29 \in \pi(G)$, then $m_{29} = 840$. If $58 \in \pi_e(G)$, then $m_{58} \in \{504, 420, 840\}$, but by $(*)$, we get a contradiction. Thus $58 \notin \pi_e(G)$.

Since $58 \notin \pi_e(G)$, the group P_{29} acts fixed point freely on the set of elements of order 2. Then $|P_{29}| \mid m_2$ and this is a contradiction. Similarly, if 421 and $1471 \in \pi(G)$, we get a contradiction. Hence $\pi(G) \subseteq \{2, 3, 5, 7\}$. If $3, 5, 7 \in \pi_e(G)$, then $m_3 = 350$, $m_5 = 504$ and $m_7 = 720$. It is clear that G does not contain any elements of order 64, 81, 125 and 343.

Let $25 \in \pi_e(G)$. Then $m_{25} = 420$ or 720 by $(*)$. By Lemma 2.1, $|P_5| \mid (1+m_5+m_{25}) = 920$ or 1225. Hence $|P_5| = 25$ and $n_5 = m_{25}/\phi(25) = 21$ or 36. Since in a cyclic group of order 25, there are four elements of order 5, so $m_5 \leq 21 \times 4 = 84$ or $m_5 \leq 36 \times 4 = 144$, a contradiction. Therefore $25 \notin \pi_e(G)$.

Let $49 \in \pi_e(G)$. Then $m_{49} = 504$. By Lemma 2.1, $|P_7| \mid (1+m_7+m_{49}) = 1225$. Then $|P_7| = 49$, and so $n_7 = m_{49}/\phi(49) = 12$. By Sylow's theorem $n_7 = 1 + 7k$ for some k, as $n_7 = 12$, we get a contradiction. So $49 \notin \pi_e(G)$.

Therefore if $5, 7 \in \pi(G)$, then $\exp(P_5) = 5$ and $\exp(P_7) = 7$, and by Lemma 2.1, $|P_5| = 5$ and $|P_7| = 7$. Hence $n_5 = m_5/\phi(5) = 2 \times 9 \times 7$ and $n_7 = m_7/\phi(7) = 8 \times 3 \times 5$. We conclude that if $5 \in \pi(G)$, then $3, 7 \in \pi(G)$, and if $7 \in \pi_e(G)$, then $3, 5 \in \pi_e(G)$. In follows, we show that $\pi(G)$ could not be the sets $\{2\}$, $\{2, 3\}$, and so $\pi(G)$ must be equal to $\{2, 3, 5, 7\}$. Now we consider the following subcases:

Subcase (a). Suppose that $\pi(G) = \{2\}$. Since $64 \notin \pi_e(G)$, $\pi_e(G) \subseteq \{1, 2, 4, 8, 16, 32\}$. Therefore $|G| = 2^m = 4536 + 350k_1 + 504k_2 + 420k_3 + 720k_4 + 840k_5 + 1470k_6$ where $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 0$. It is easy to see that this equation has no solution.

Subcase (b). Suppose that $\pi(G) = \{2, 3\}$. Since $81 \notin \pi_e(G)$, $\exp(P_3) = 3$, 9 or 27. Let $\exp(P_3) = 3$. Then $|P_3| \mid (1+m_3) = 351$, by Lemma 2.1. Hence $|P_3| \mid 27$. If $|P_3| = 3$, then $n_3 = m_3/\phi(3) = 175 \mid |G|$, because $5 \notin \pi(G)$, we get a contradiction.

If $|P_3| = 9$, then since $\exp(P_3) = 3$ and $64, 96 \notin \pi_e(G)$, $|\pi_e(G)| \leq 11$. Therefore $|G| = 2^m \times 9 = 4536 + 350k_1 + 504k_2 + 420k_3 + 720k_4 + 840k_5 + 1470k_6$, where k_1, k_2, k_3, k_4, k_5, k_6 and m are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leq 3$.

We know that $4536 \leq 2^m \times 9 \leq 4536 + 1470 \times 3$, so $m = 10$. Then $4680 = 4536 + 350k_1 + 504k_2 + 420k_3 + 720k_4 + 840k_5 + 1470k_6$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leq 3$. By an easy computer calculation, it is easy to see this equation has no solution.

Similarly, we can rule out the case $|P_3| = 27$.

Let $\exp(P_3) = 9$. By $(*)$, $m_9 \in \{504, 720\}$. Hence by Lemma 2.1, $|P_3| = 9$. Therefore $n_3 = m_9/\phi(9) \in \{84,120\}$. Because $5, 7 \notin \pi(G)$, we get a contradiction.

If $\exp(P_3) = 27$, then $m_{27} \in \{504, 720\}$. If P_3 be a cyclic group, then since $\exp(P_3) = 27$, $n_3 = m_{27}/\phi(27) \in \{28, 40\}$. Because $5, 7 \notin \pi(G)$, we get a contradiction. Thus P_3 is not cyclic. By Lemma 2.3, $27 \mid m_{27}$, a contradiction.

Therefore $\pi(G) = \{2, 3, 5, 7\}$. We prove that $21 \notin \pi_e(G)$. Suppose that $21 \in \pi_e(G)$. Then $m_{21} = \phi(21) \cdot n_7 \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_7)$. Since $n_7 = m_7/\phi(7) = 120$, $720 \mid m_{21}$, a contradiction. Thus $21 \notin \pi_e(G)$. Similarly, $14 \notin \pi_e(G)$.

Since $21 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 7. Hence $|P_3| \mid m_7 = 720$. Then $|P_3| = 3$ or 9. Also since $14 \notin \pi_e(G)$, the group P_2 acts fixed point freely on the set of elements of order 7. Hence $|P_2| \mid m_7 = 720$. Then $|P_2| \mid 16$. On the other hand, $4536 \leq |G|$, thus $|G| = 2^4 \times 3^2 \times 5 \times 7 = |S_7|$.

Now we claim that G is non-solvable group. Suppose that G is solvable. Since $n_7 = 120$ by Lemma 2.4, $3 \equiv 1 \pmod 7$, a contradiction. Hence G is non-solvable group and $p \| |G|$, where $p \in \{5, 7\}$. Therefore G has a normal series

$$1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$$

such that N is a maximal solvable normal subgroup of G and H/N is an non-solvable minimal normal subgroup of G/N. Then H/N is a non-Abelian simple K_3-group or simple K_4-group. If H/N be simple K_3- group, then by Lemma 2.6, H/N is isomorphic to one of the groups: A_5 , A_6, $L_2(7)$ or $L_2(8)$.

Suppose that $H/N \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, $n_5(H/N)t = n_5(G)$ for some positive integer t and $5 \nmid t$, by Lemma 2.5. Since $n_5(A_5) = 6$, $n_5(G) = 6t$. Thus $m_5 = n_5(G) \times 4 = 24t = 504$ and so $t = 21$. By Lemma 2.5, $21 \times |N_N(P_5)| = |N|$. Since $|N| \mid 2^3 \times 3 \times 7$, then $n_7(N) = 1$or 8. So $m_7 = 6$ or 48, a contradiction.

Suppose that $H/N \cong A_6$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, $n_5(H/N)t = n_5(G)$ for some positive integer t and $5 \nmid t$, by Lemma 2.5. Since $n_5(A_6) = 36$, $n_5(G) = 36t$ and $m_5 = n_5(G) \times 4 = 144t = 504$, a contradiction.

Suppose that $H/N \cong L_2(7)$. If $P_7 \in \mathrm{Syl}_7(G)$, then $P_7N/N \in \mathrm{Syl}_7(H/N)$, $n_7(H/N)t = n_7(G)$ for some positive integer t and $7 \nmid t$, by Lemma 2.5. Since $n_7(L_2(7)) = 8$, $n_7(G) = 8t$ and $m_7 = n_7(G) \times 6 = 48t = 720$. Hence $t = 15$. By Lemma 2.5, $15 \times |N_N(P_7)| = |N|$. Since $|N| \mid 2 \times 3^2 \times 5$, $n_5(N) = 1$ or 6. So $m_5 = 4$ or 24, a contradiction.

Similarly, if $H/N \cong L_2(8)$, we get a contradiction. Hence H/N is simple K_4-group. Then by Lemma 2.7, H/N is isomorphic to A_7. Now set $\overline{H} := H/N \cong A_7$ and $\overline{G} := G/N$. We have

$$A_7 \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $A_7 \leq G/K \leq \mathrm{Aut}(A_7)$. Then $G/K \cong A_7$ or $G/K \cong S_7$. If $G/K \cong A_7$, then $|K| = 2$. We have $N \leq K$ and N is a maximal solvable normal subgroup of G, then $N = K$. Now we know that $G/N \cong A_7$ where $|N| = 2$, so G has a normal subgroup G of order 2, generated by a central involution z. Let x be an element of order 7 in G. Since $xz = zx$ and $(o(x), o(z)) = 1$, $o(xz) = 14$.

Hence $14 \in \pi_e(G)$. We know $14 \notin \pi_e(G)$, a contradiction.

If $G/K \cong S_7$, then $|K| = 1$ and $G \cong S_7$. Now the proof of the main theorem is complete.

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