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On operators of A_p and A_p^* class

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Abstract

In this paper we study the properties of bounded linear operators namely A_p-class and A_p^*-class operators that satisfy $T^*T \leq (T^{*p}T^p)^{\frac{1}{p}}$ and $TT^* \leq (T^{*p}T^p)^{\frac{1}{p}}$ respectively. We use some known operator inequalities and we show that if $T \in \mathcal{B}(\mathscr{H})$ is an A_p-class or an A_p^*-class operator, then $r(T) = \| T \|$.

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1. Introduction

In this paper, we denote the set of all bounded linear operators on $\mathscr{H}$ by $\mathcal{B}(\mathscr{H})$, where $\mathscr{H}$ is a complex Hilbert space. Let $T \in \mathcal{B}(\mathscr{H})$, we denote the spectrum of T by $\sigma(T)$ and the spectral radius of T by $r(T)$ where

$$r(T) = \sup\{|\lambda|, \lambda \in \sigma(T)\}\,.$$

We say that $T \in \mathcal{B}(\mathscr{H})$ is an A_-class operator if $|T|^2 \geq |T^2|$. In [1, 2] the properties of A_- class operators are studied and it is shown that operators in this class satisfy the Weyl theorem.

Our main purpose in this paper is to introduce the A_p^*-class and A_p-class operators and to denote the properties of this class of operators. In the main section we prove that operators in A_p or A_p^* class satisfy $r(T) = \| T \|$. These classes of operators present some important classes of operators that probably fall in one of this A_p classes. We are trying to answer the following question; are there operators which belong to A_p^*-class operators but do not belong to A_p-class operators? We give an example of operator T such that

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$TT^* \leq (T^{*2}T^2)^{\frac{1}{2}}$ and $T^*T \nleq (T^{*2}T^2)^{\frac{1}{2}}$, this says that there is an operator T which is A_p^*-class operator but T is not A_p- class operator. If $T \in \mathcal{B}(\mathscr{H})$, we write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null space and the range space of T respectively. An operator $T \in \mathcal{B}(\mathscr{H})$ is called Fredholm if it has closed range and

$$\dim \mathcal{N}(T) < \infty \quad , \quad \dim \mathcal{R}(T)^{\perp} < \infty \, .$$

If $T \in \mathcal{B}(\mathscr{H})$ is Fredholm then we denote the index of T by ind(T) which is given by

$$\mathrm{ind(T)} = \dim(\mathcal{N}(\mathrm{T})) - \dim(\mathcal{R}(\mathrm{T})^{\perp}) \, .$$

An operator $T \in \mathcal{B}(\mathscr{H})$ is called a Weyl operator if it is Fredholm of index zero. Also, let $\pi_{00}(T)$ be the set of isolated eigenvalues with finite multiplicity and $\omega(T)$ be the Weyl spectrum of T. We have

$$\pi_{00}(T) = \{\lambda \in \mathrm{iso}(\sigma(\mathrm{T})); \; 0 < \dim \mathcal{N}(\mathrm{T} - \lambda \mathrm{I}) < \infty\}$$

$$\omega(T) = \{\lambda \in \mathbb{C}; \; T - \lambda I \text{ is not weyl}\}$$

where iso(A) is the isolated point of the set A.

2. main results

We give some lemmas that we use in the next.

2.1. Lemma. [5] *Let T be a self-adjoint operator on the Hilbert space $\mathscr{H}$, we have*

$$\| T \| = \sup\{|\langle Tx, x\rangle|, \| x \| = 1\}$$

2.2. Lemma. [3] *(Hansen inequality). If $A, B \in \mathcal{B}(\mathscr{H})$ satisfy $A \geq 0$ and $\| B \| \leq 1$, then*

$$(B^*AB)^{\delta} \geq B^*A^{\delta}B$$

for all $0 < \delta \leq 1$.

2.3. Lemma. [4] *(Hölder-McCarthy inequality). If $A \in \mathcal{B}(\mathscr{H})$ is a positive operator and $x \in \mathscr{H}$, then*

(1) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r \| x \|^{2(1-r)}$ *for $r > 1$,*
(2) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \| x \|^{2(1-r)}$ *for $0 \leq r \leq 1$.*

2.4. Definition. The operator $T \in \mathcal{B}(\mathscr{H})$ is called an A_p-class operator if

$$T^*T \leq (T^{*p}T^p)^{\frac{1}{p}} \, .$$

2.5. Definition. The operator $T \in \mathcal{B}(\mathscr{H})$ is called an A_p^*-class operator if

$$TT^* \leq (T^{*p}T^p)^{\frac{1}{p}} \, . \tag{2.1}$$

If in Definition 2.5 we put $p = 2$, we get the definition of $(A, *)-$class operator where it has been introduced in [3]. Thus A_p^*-class operators are generalization of $(A, *)-$class operators.
In the next, we show that A_p^*-class operators are normaloid in the sense that $r(T) = \| T \|$, we give a useful lemma as follows,

2.6. Lemma. *If $T \in \mathcal{B}(\mathscr{H})$ is an A_p^*-class operator then*

$$\left\|T^{*np-p-1}|T|^4 T^{np-p-1}\right\| \geq \left\|T^{*np-p}T^{np-p}\right\|^2 \left\|T^{np-p-1}\right\|^{-2} \, , \tag{2.2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$

Proof. Suppose that $x \in \mathscr{H}$, $\|x\| = 1$ and $T^{np-p-1}x \neq 0$ then by the Lemma 2.3 we get

$$\begin{aligned}
&\langle T^{*np-p-1}|T|^4 T^{np-p-1}x, x\rangle \left\|T^{np-p-1}x\right\|^2 \\
&= \langle |T|^4 T^{np-p-1}x, T^{np-p-1}x\rangle \left\|T^{np-p-1}x\right\|^2 \\
&\geq \langle |T|^2 T^{np-p-1}x, T^{np-p-1}x\rangle^2 \\
&= \langle T^{*np-p-1}|T|^2 T^{np-p-1}x, x\rangle^2 \\
&= \langle T^{*np-p}T^{np-p}x, x\rangle^2 . \qquad (2.3)
\end{aligned}$$

therefore

$$\langle T^{*np-p-1}|T|^4 T^{np-p-1}x, x\rangle \left\|T^{np-p-1}x\right\|^2 \geq \langle T^{*np-p}T^{np-p}x, x\rangle^2 . \qquad (2.4)$$

If $T^{np-p-1}x = 0$ then the inequality (2.4) is valid, too.
Now by taking the supremum over all $x \in \mathscr{H}$ such that $\|x\| = 1$ in the inequality (2.4) and by use of the Lemma 2.1 we get

$$\left\|T^{*np-p-1}|T|^4 T^{np-p-1}\right\| \left\|T^{np-p-1}\right\|^2 \geq \left\|T^{*np-p}T^{np-p}\right\|^2 , \qquad (2.5)$$

this completes the proof. □

2.7. Theorem. *Assume that T is an A_p^*-class or an A_p-class operator (p is an integer), then $r(T) = \| T \|$.*

Proof. Since λT is A_p^*- class (A_p-class) operator for all $\lambda \in \mathbb{C}$, where T is an A_p^*- class (A_p-class) operator, without loss of generality we assume that $\|T\| = 1$. Firstly, suppose that T is an A_p^*- class operator then

$$\begin{aligned}
(T^{*np}T^{np})^{\frac{1}{p}} &= \left(T^* T^{*np-1}T^{np-1}T\right)^{\frac{1}{p}} \\
&\geq T^* \left(T^{*np-1}T^{np-1}\right)^{\frac{1}{p}} T \qquad \text{(by lemma 2.2)} \\
&\;\;\vdots \\
&\geq T^{*np-p} \left(T^{*p}T^p\right)^{\frac{1}{p}} T^{np-p} . \qquad (2.6)
\end{aligned}$$

By (2.1) and (2.6) we get

$$\begin{aligned}
(T^{*np}T^{np})^{\frac{1}{p}} &\geq T^{*np-p} \left(T^{*p}T^p\right)^{\frac{1}{p}} T^{np-p} \\
&\geq T^{*np-p} \left(TT^*\right) T^{np-p} \\
&= T^{*np-p-1}|T|^4 T^{np-p-1} .
\end{aligned}$$

Therefore

$$\| \left(T^{*np}T^{np}\right)^{\frac{1}{p}} \| \geq \|T^{*np-p-1}|T|^4 T^{np-p-1}\| , \qquad (2.7)$$

by the Lemma 2.6 we have

$$\begin{aligned}
\| T^{*np-p-1}|T|^4 T^{np-p-1} \| &\geq \| T^{*np-p}T^{np-p} \|^2 \| T^{np-p-1} \|^{-2} \\
&\geq \| T^{*np-p}T^{np-p} \|^2 .
\end{aligned}$$

By using (2.7) we get,

$$\left\|\left(T^{*np}T^{np}\right)^{\frac{1}{p}}\right\| \geq \left\|\left(T^{*np-p}T^{np-p}\right)^{\frac{1}{p}}\right\|^{2p} . \qquad (2.8)$$

We repeat the inequality (2.8) to get

$$\begin{aligned}\| \left(T^{*np}T^{np}\right)^{\frac{1}{p}} \| &\geq \| \left(T^{*p}T^{p}\right)^{\frac{1}{p}} \|^{(2p)^{n-1}} \\ &\geq \| TT^{*} \|^{(2p)^{n-1}} \\ &= \| T \|^{2(2p)^{n-1}} = 1\,.\end{aligned}$$

On the other hand $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$, this yields that

$$\begin{aligned}[r(T)]^2 = r(T^*)r(T) &= \lim_{n\to\infty} \|T^{*np}\|^{\frac{1}{np}} \| T^{np} \|^{\frac{1}{np}} \\ &\geq \lim_{n\to\infty} \| T^{*np}T^{np} \|^{\frac{1}{np}} \geq 1. \qquad (2.9)\end{aligned}$$

Therefore $r(T) = 1$.
secondly, if T is an A_p-class operator then by the Definition 2.4 and the Lemma 2.2 we have

$$\begin{aligned}\left(T^{*n(p-1)}T^{n(p-1)}\right)^{\frac{1}{p}} &\geq \left(T^{*}T^{*n(p-1)-1}T^{n(p-1)-1}T\right)^{\frac{1}{p}} \\ &\geq T^{*}\left(T^{*n(p-1)-1}T^{n(p-1)-1}\right)^{\frac{1}{p}} T \\ &\vdots \\ &\geq T^{*(n-1)(p-1)-1}\left(T^{*p}T^{p}\right)^{\frac{1}{p}} T^{(n-1)(p-1)-1} \\ &\geq T^{*(n-1)(p-1)-1}\left(T^{*}T\right) T^{(n-1)(p-1)-1} \\ &= T^{*(n-1)(p-1)}T^{(n-1)(p-1)}\,, \qquad (2.10)\end{aligned}$$

therefore

$$\left\|\left(T^{*n(p-1)}T^{n(p-1)}\right)^{\frac{1}{p}}\right\| \geq \left\|\left(T^{*(n-1)(p-1)}T^{(n-1)(p-1)}\right)^{\frac{1}{p}}\right\|^{p}\,, \qquad (2.11)$$

if we repeat the inequality (2.11) we get

$$\begin{aligned}\left\|\left(T^{*n(p-1)}T^{n(p-1)}\right)^{\frac{1}{p}}\right\| &\geq \left\|\left(T^{*(p-1)}T^{p-1}\right)^{\frac{1}{p}}\right\|^{p^{n-1}} \\ &= \|T^{*(p-1)}T^{p-1}\|^{p^{n-2}} \\ &= \left(\|T^*\| \left\|T^{*(p-1)}T^{p-1}\right\| \|T\|\right)^{p^{n-2}} \\ &\geq \|T^{*p}T^{p}\|^{p^{n-2}} \\ &\geq \|T^{*}T\|^{p^{n-2}} = 1\,.\end{aligned}$$

According to the (2.9) it can be concluded that $r(T) \geq 1$, thus in this case $r(T) = 1$, too. □

2.8. Corollary. *If T is an A_p^*-class or an A_p-class operator and also T is quasinilpotent operator, then $T = 0$.*

Proof. By Theorem 2.7, T is normaloid, on the other hand, zero is single operator such that it is normaloid and quasinilpotent, hence $T = 0$. □

2.9. Theorem. *Suppose that $T \in \mathcal{B}(\mathscr{H})$ and $\lambda \in \mathbb{C}$, if T is an A_p^*-class operator and $(T-\lambda)(x) = 0$ then $(T-\lambda)^*(x) = 0$.*

Proof. Without lose of generality we take $\| x \|= 1$, so by use of the Lemma 2.3 we have

$$\begin{aligned}\| (T-\lambda)^* x \|^2 &= \langle (T-\lambda)^* x, (T-\lambda)^* x\rangle \\ &= \langle (T-\lambda)\left(T^*-\bar{\lambda}\right) x, x\rangle \\ &= \langle TT^* x, x\rangle - \langle \lambda T^* x, x\rangle - \langle \bar{\lambda} T x, x\rangle + |\lambda|^2 \langle x, x\rangle \\ &= \langle |T^*|^2 x, x\rangle - \lambda\langle x, Tx\rangle - \bar{\lambda}\langle Tx, x\rangle + |\lambda|^2\langle x, x\rangle \\ &\leq \langle (T^{*p}T^p)^{\frac{1}{p}} x, x\rangle - |\lambda|^2\langle x, x\rangle \leq \langle (T^{*p}T^p) x, x\rangle^{\frac{1}{p}} - |\lambda|^2\langle x, x\rangle = 0\,.\end{aligned}$$

Thus $(T-\lambda)^* x = 0$.

□

2.10. Theorem. *If T is an A_p^*- class operator and $0 \in \sigma(T)\backslash\omega(T)$, then $0 \in \pi_{00}(T)$.*

Proof. Let $0 \in \sigma(T)\backslash\omega(T)$, then T is a Weyl operator. Hence T is a Fredholm operator with $\mathrm{indT} = 0$. This yields that $\mathcal{R}(T)$ is close and $\dim\mathcal{N}(\mathrm{T}) = \dim\mathcal{R}(\mathrm{T})^{\perp}$. We show that zero is the isolated eigenvalue of T. If zero is not eigenvalue of T, then $\dim\mathcal{N}(\mathrm{T}) = 0$. Therefore

$$\dim\mathcal{R}(\mathrm{T})^{\perp} = 0 \quad , \quad \mathcal{R}(\mathrm{T}) = \mathscr{H}$$

so T is an invertible operator, this is a contradiction. Now let $0 \notin \mathrm{iso}(\sigma(\mathrm{T}))$, we have

$$\mathcal{N}(T) \subseteq \mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}\,,$$

since $\dim\mathcal{N}(\mathrm{T}) = \dim\mathcal{R}(\mathrm{T})^{\perp}$, so $\mathcal{N}(T) = \mathcal{R}(T)^{\perp}$. We consider $\mathscr{H}$ and T as the following,

$$\mathscr{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^{\perp} \qquad T = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

because of the operator $B : \mathcal{N}(T)^{\perp} \to \mathcal{N}(T)^{\perp}$ is one to one and $\mathcal{N}(T)^{\perp} = \overline{\mathcal{R}(T)} = \mathcal{R}(T)$, we have

$$\mathcal{R}(B) = \mathcal{R}(T) = \mathcal{N}(T)^{\perp}$$

therefore B is a surjective operator, hence B is an invertible operator.
On the other hand $0 \notin \mathrm{iso}\sigma(T)$ and $\sigma(T) = \{0\} \cup \sigma(B)$, so there exists a sequence like $\{\lambda_n\}$ of $\sigma(B)$ such that $\lambda_n \to 0$, therefore $0 \in \sigma(B)$. This is a contradiction and the proof is complete. □

In the next, we give an example that it clarifies the set of A_p^*- class operators are different from the set of A_p- class operators.

2.11. Example. We consider the infinite matrix operators T as follows,

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

Hence we get $TT^* \leq (T^{*2}T^2)^{\frac{1}{2}}$ and $T^*T \nleq (T^{*2}T^2)^{\frac{1}{2}}$. Thus T is an A_2^*- class operator but T is not A_2- class operator.

2.12. Remark. Note that if $T \in \mathcal{B}(\mathscr{H})$ is an invertible and T^*, T^{-1} are A_2^*-class operator, then by the Definition 2.5 we have

$$T^{-1}T^{*-1} \leq (T^{*-2}T^{-2})^{\frac{1}{2}} \tag{2.12}$$

and $T^*T \leq (T^2T^{*2})^{\frac{1}{2}}$, by taking the inverse of (2.12) we get $T^*T \geq (T^2T^{*2})^{\frac{1}{2}}$ this inequalities showed $T^*T = (T^2T^{*2})^{\frac{1}{2}}$, moreover

$$|T|^2 = |T^{*2}|.$$

2.13. Remark. If T is an A_p^*-class operator then by the Theorem 2.9 we have $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. If T^* is an A_p^*-class operator then $\mathcal{N}(T^*) \subset \mathcal{N}(T)$. Therefore if T and T^* are A_p^*-class operators then we have $\mathcal{N}(T) = \mathcal{N}(T^*)$ and we get

$$\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}.$$

2.14. Remark. Note that if T is an A_p^*-class operator and T^p is A_q-class operator then T is A_{pq}^*-class operator, also if T^* is A_p-class operator and T^p is A_q^*-class operator then T is A_{pq}^*-class operator.

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