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An order UV obtained by uninorm and nullnorm

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Abstract: Uninorms and nullnorms are hot topics to study nowadays. In this paper, a new order definition on bounded lattice is given using the uninorm and nullnorm on sub-interval of bounded lattice order defined on. Some interesting properties of the order are investigated. It is posed that even if L chain with order defined on, L may not be chain with the order UV.

Keywords: Uninorm, nullnorm, order.

1 Introduction

In the last decade, researchers have intensely studied on aggregation functions ([11,12]). aggregation functions can be found in [9] in detail. These studies also had some special aggregation functions such as uninorms, nullnorms, t-norms and t-conorms.

Uninorms are highly interested in having structures that generalize the notions of t-norms and t-conorms by researchers [4,5,6,7]. Uninorms were first defined by Yager and Rybalov on the unit interval $[0,1]$[16]. As mentioned nullnorms that are originally spesial aggregation operators were first presented in [3,15]. Later on, these operators worked on clusters because these constructions are quite general [10].

Recently, the order generating problem has been seen as a special study area [13,14]. This studies that started with t-norms continued in uninorms and nullnorms [1,6,13]. Main motivation of this paper is to define new order obtained from the orders induced by uninorms and nullnorms.

This paper consist of four parts. In the first part, the past of the subject was briefly mentioned, and in the second part some necessary definitions and theorems were expressed. In the third part, new order definition was given and some features were studied. In the last part, the works in the study were shortly summarized.

2 Notations, definitions and a review of previous results

A bounded lattice $(L,\leqslant)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, i.e., there exist two elements $1,0\in L$ such that $0\leqslant x\leqslant 1$, for all $x\in L$.

Definition 1. [2] *Given a bounded lattice $(L,\leq,0,1)$, and $a,b\in L$, if a and b are incomparable, in this case we use the notation $a\|b$.*

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Definition 2. [2] *Given a bounded lattice* $(L,\leq,0,1)$, *and* $a,b\in L$, $a\leq b$, *a subinterval* $[a,b]$ *of* L *is a sublattice of* L *defined as*

$$[a,b]=\{x\in L\mid a\leq x\leq\ b\}.$$

Similarly, $(a,b]=\{x\in L\mid a<x\leq\ b\}$, $[a,b)=\{x\in L\mid a\leq x<b\}$ *and* $(a,b)=\{x\in L\mid a<x<b\}$.

Definition 3. [12] *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $U:L^2\to L$ *is called a uninorm on* L, *if it is commutative, associative, increasing with respect to the both variables and has a neutral element* $e\in L$.

In this study, the notation $\mathscr{U}(e)$ *will be used for the set of all uninorms on* L *with neutral element* $e\in L$.

Definition 4. [5] *An operation* T *(*S*) on a bounded lattice* L *is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element* 1 *(*0*).*

Let $(L,\leq,0,1)$ *be a bounded lattice,* $U\in\mathscr{U}(e)$ *and* $e\in L$. *It is known that if it is* $e=1$, *uninorm* U *coincides t-norm and if it is* $e=0$, *uninorm* U *coincides t-conorm on* L.

Proposition 1. [12] *Let* $(L,\leq,0,1)$ *be a bounded lattice,* $U\in\mathscr{U}(e)$. *Then*

(i) $T_U:=U\downarrow_{[0,e]^2}:[0,e]^2\to[0,e]$ *is a t-norm on* $[0,e]$.
(ii) $S_U:=U\downarrow_{[e,1]^2}:[e,1]^2\to[e,1]$ *is a t-norm on* $[e,1]$.

Definition 5. [13] *A t-norm* T *(or a t-conorm* S*) on a bounded lattice* L *is divisible if the following condition holds. For all* $x,y\in L$ *with* $x\leq y$ *there is* $z\in L$ *such that* $x=T(y,z)$ *(or* $y=S(x,z)$*).*

Definition 6. [11] *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $V:L^2\to L$ *is called a nullnorm on* L, *if it is commutative, associative, increasing with respect to the both variables and there is an element* $a\in L$ *such that* $V(x,0)=x$ *for all* $x\leq a$, $V(x,1)=x$ *for all* $x\geq a$.

It can be easily obtained that $V(x,a)=a$ *for all* $x\in L$. *So, the element* $a\in L$ *that provide* $V(x,a)=a$ *for all* $x\in L$ *is called (absorbing) zero element for operator* V *on* L. *In this study, the notation* $\mathscr{V}(a)$ *will be used for the set of all nullnorms on* L *with zero element* $a\in L$.

Proposition 2. [11] *Let* $(L,\leq,0,1)$ *be a bounded lattice,* $V\in\mathscr{V}(a)$. *Then*

(i) $S_V:=V\downarrow_{[0,a]^2}:[0,a]^2\to[0,a]$ *is a t-conorm on* $[0,a]$.
(ii) $T_V:=V\downarrow_{[a,1]^2}:[a,1]^2\to[a,1]$ *is a t-norm on* $[a,1]$.

Definition 7. [13,14] *A t-norm* T *(or a t-conorm* S*) on a bounded lattice* L *is divisible if the following condition holds. For all* $x,y\in L$ *with* $x\leq y$ *there is* $z\in L$ *such that* $x=T(y,z)$ *(or* $y=S(x,z)$*).*

Definition 8. [13] *Let* L *be a bounded lattice,* T *be a t-norm on* L. *The order defined by*

$$x\preceq_T y :\Leftrightarrow T(\ell,y)=x \text{ for some } \ell\in L$$

is called a $T-$ *partial order (triangular order) for t-norm* T.

Similarly, the notion $S-$ *partial order* can be defined as follows:

Definition 9. *Let* L *be a bounded lattice,* S *be a t-conorm on* L. *The order defined by is called a* $S-$ *partial order for t-conorm* S.

$$x\preceq_S y :\Leftrightarrow S(\ell,x)=y \text{ for some } \ell\in L$$

is called a $S-$ *partial order for t-conorm* S.

Note that many properties satisfied for *$T-$ partial order* are also satisfied for *$S-$ partial order*.

Definition 10. [6] *Let $(L,\leq,0,1)$ be a bounded lattice and $U\in\mathscr{U}(e)$. Define the following relation,for $x,y\in L$, as*

$$x\preceq_U y:\Leftrightarrow\begin{cases} \text{if} \quad x,y\in[0,e] \quad \text{and there exist} \quad k\in[0,e] \quad \text{such that} \quad U(k,y)=x \quad \text{or}, \\ \text{if} \quad x,y\in[e,1] \quad \text{and there exist} \quad \ell\in[e,1] \quad \text{such that} \quad U(x,\ell)=y \quad \text{or}, \\ \text{if} \quad (x,y)\in L^* \quad \text{and} \quad x\leq y, \end{cases} \tag{1}$$

where $I_e=\{x\in L\mid x\|e\}$ and $L^*=[0,e]\times[e,1]\cup[0,e]\times I_e\cup[e,1]\times[0,e]\cup[e,1]\times I_e\cup I_e\times[0,e]\cup I_e\times[e,1]\cup I_e\times I_e$.

Here, note that the notation $x||y$ denotes that x and y are incomparable.

Definition 11. [1] *Let $(L,\leq,0,1)$ be a bounded lattice and $V\in\mathscr{V}(a)$. Define the following relation,for $x,y\in L$, as*

$$x\preceq_V y:\Leftrightarrow\begin{cases} \text{if} \quad x,y\in[0,a] \quad \text{and there exist} \quad k\in[0,a] \quad \text{such that} \quad V(k,x)=y \quad \text{or}, \\ \text{if} \quad x,y\in[a,1] \quad \text{and there exist} \quad \ell\in[a,1] \quad \text{such that} \quad V(y,\ell)=x \quad \text{or}, \\ \text{if} \quad (x,y)\in L^* \quad \text{and} \quad x\leq y, \end{cases} \tag{2}$$

where $I_e=\{x\in L\mid x\|e\}$ and $L^*=[0,e]\times[e,1]\cup[0,e]\times I_e\cup[e,1]\times[0,e]\cup[e,1]\times I_e\cup I_e\times[0,e]\cup I_e\times[e,1]\cup I_e\times I_e$.

3 The order UV obtained by uninorm and nullnorm

Definition 12. *Let $(L,\leq,0,1)$ be a bounded lattice, $U:[0,a]^2\to[0,a]$ be an uninorm with neutral element e and $V:[e,1]^2\to[e,1]$ be a nullnorm with zero element a such that $0\leq e\leq a\leq 1$ and $U\downarrow_{[e,a]^2}=V\downarrow_{[e,a]^2}$. Define the following relation: For every $x,y\in L$,*

$$UV(x,y)=\begin{cases} x\preceq_U y, & \text{if } (x,y)\in[0,a]^2 \\ x\preceq_V y, & \text{if } (x,y)\in[e,1]^2 \\ x\leq y, & \text{otherwise.} \end{cases} \tag{3}$$

Proposition 3. *The relation $\preceq_{UV}$ defined in (3) is a partial order on bounded lattice L.*

Proof. (i) If $x\in[0,a]$, it is obtained that $x\preceq_{UV}x$ since $x\preceq_U x$. If $x\in[e,1]$, it is obtained that $x\preceq_{UV}x$ since $x\preceq_V x$. Otherwise, since $x\leq x$, it is obtained that $x\preceq_{UV}x$. Then, reflexivity property is hold.

(ii) Let $x\preceq_{UV}y$ and $y\preceq_{UV}x$ for elements $x,y\in L$.

1. $x\in[0,a]$.

1.1. $y\in[0,a]$. For this case, $x\preceq_{UV}y$ and $y\preceq_{UV}x$ imply that $x\preceq_U y$ and $y\preceq_U x$, respectively. It follows $x=y$ from $\preceq_U$ is a partial order on $[0,a]$.

1.2. $y\notin[0,a]$. For this case, $x\preceq_{UV}y$ and $y\preceq_{UV}x$ imply that $x\leq y$ and $y\leq x$, respectively. Then, it is obtained that $x=y$.

2. $x\in[e,1]$.

2.1. $y\in[e,1]$. For this case, $x\preceq_{UV}y$ and $y\preceq_{UV}x$ imply that $x\preceq_V y$ and $y\preceq_V x$, respectively. It follows $x=y$ from $\preceq_V$ is a partial order on $[e,1]$.

2.2. $y\notin[e,1]$. For this case, $x\preceq_{UV}y$ and $y\preceq_{UV}x$ imply that $x\leq y$ and $y\leq x$, respectively. Then, it is obtained that $x=y$.

3. $x\in L\setminus\{[0,a]\cup[e,1]\}$. In this case, it is obtained that $x\leq y$ and $y\leq x$ for all $y\in L$. Thus, $x=y$.

So the anti-symmetry property holds.

(iii) Let $x\preceq_{UV}y$ and $y\preceq_{UV}z$ for elements $x,y,z\in L$.

Possible cases are as follows:

3.1. $x \in [0,a]$.

3.1.1. $y \in [0,a]$.

3.1.1.1. $z \in [0,a]$.

Since $x \preceq_{UV} y$ and $y \preceq_{UV} z$, it is obtained that $x \preceq_U y$ and $y \preceq_U z$, respectively. Then, it follows $x \preceq_U z$ from transitivity of $\preceq_U$. Since $x,z \in [0,a]$ and $x \preceq_U z$, it is obtained that $x \preceq_{UV} z$.

3.1.1.2. $z \notin [0,a]$.

$x \preceq_{UV} y$ implies that $x \preceq_U y$. Since $x \preceq_U y$, $x \leq y$. On the other hand, $y \preceq_{UV} z$ implies that $y \leq z$. Since $x \leq y$ and $y \leq z$, it is obtained that $x \leq z$. It follows $x \preceq_{UV} z$ from that $x \leq z$.

3.1.2. $y \notin [0,a]$.

Since $y \notin [0,a]$, $x \preceq_{UV} y$ implies that $x \leq y$. If $z \in [0,a]$, $y \preceq_{UV} z$ implies that $y \leq z$. It is a contradiction. Thus, it must be that $z \notin [0,a]$.

Possible cases are as follows for this case:

(i) $y,z \in [e,1]$,

(ii) $y \in [e,1]$ and $z \notin [e,1]$,

(iii) $y \notin [e,1]$ and $z \in [e,1]$,

(iv) $y,z \notin [e,1]$.

If $y,z \in [e,1]$, $y \preceq_{UV} z$ implies that $y \preceq_V z$, namely $y \leq z$. Then, it follows $x \preceq_{UV} z$ from $x \leq z$.

For other all cases, $y \preceq_{UV} z$ implies that $y \leq z$. Since $x \leq y$ and $y \leq z$, it is obtained that $x \leq z$. Thus, $x \preceq_{UV} z$.

3.2. $x \in [e,1]$.

Let $y \notin [e,1]$. It is obtained that $x \leq y$ since $x \preceq_{UV} y$, it is a contradiction. Then, $y \in [e,1]$.

3.2.1. $y \in [e,1]$.

Let $z \notin [e,1]$. It is obtained that $y \leq z$ since $y \preceq_{UV} z$, it is a contradiction. Then, $z \in [e,1]$.

3.2.1.1. $z \in [e,1]$.

Since $x \preceq_{UV} y$ and $y \preceq_{UV} z$, it is obtained that $x \preceq_V y$ and $y \preceq_V z$, respectively. Then, it follows $x \preceq_V z$ from transitivity of $\preceq_V$. Since $x,z \in [e,1]$ and $x \preceq_V z$, it is obtained that $x \preceq_{UV} z$.

3.3. $x \in L \setminus \{[0,a] \cup [e,1]\}$.

In this case, $x \preceq_{UV} y$ implies that $x \leq y$.

If $y,z \in [0,a]$. $y \preceq_{UV} z$ implies that $y \preceq_U z$, namely $y \leq z$. Since $x \leq y$ and $y \leq z$, it is obtained that $x \leq z$. It is obtained that $x \preceq_{UV} z$, since $x \in L \setminus \{[0,a] \cup [e,1]\}$.

If $y,z \in [e,1]$. $y \preceq_{UV} z$ implies that $y \preceq_V z$, namely $y \leq z$. Since $x \leq y$ and $y \leq z$, it is obtained that $x \leq z$. It is obtained that $x \preceq_{UV} z$, since $x \in L \setminus \{[0,a] \cup [e,1]\}$. So the transitivity holds.

Remark. (i) If $a = 1$ is taken in (3) in Definition 12, then $\preceq_{UV} = \preceq_U$.

(ii) If $e = 0$ is taken in (3) in Definition 12, then $\preceq_{UV} = \preceq_V$.

Proposition 4. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $U : [0,a]^2 \to [0,a]$ *be an uninorm with neutral element* e *and* $V : [e,1]^2 \to [e,1]$ *be a nullnorm with zero element* a *such that* $0 \leq e \leq a \leq 1$ *and* $U \downarrow_{[e,a]^2} = V \downarrow_{[e,a]^2}$. *Then,* $(L, \preceq_{UV}, 0, 1)$ *is bounded partially ordered set.*

Proof. It follows $(L, \preceq_{UV})$ is a partially ordered set from Proposition 3.

Let $x \in [0,a]$. Since $U(0,x) \leq U(0,e) = 0$, we have that $U(0,x) = 0$. It is obtained that $0 \preceq_{UV} x$.

Let $x \notin [0,e]$. Then, it follows $0 \preceq_{UV} x$ from $0 \leq x$. So, for any $x \in L$, $0 \preceq_{UV} x$.

In similar way, one can show that 1 is the greatest element with respect to $\preceq_{UV}$.

Proposition 5. *Let $(L,\leq,0,1)$ be a bounded lattice, $U:[0,a]^2\to[0,a]$ be an uninorm with neutral element e and $V:[e,1]^2\to[e,1]$ be a nullnorm with zero element a such that $0\leq e\leq a\leq 1$ and $U\downarrow_{[e,a]^2}=V\downarrow_{[e,a]^2}$. If $x\preceq_{UV} y$ for any $x,y\in L$, then $x\leq y$.*

Proof. Let $x\preceq_{UV} y$ for $x,y\in L$.
If $x,y\in[0,a]$, then it is obtained that $x\preceq_U y$. Since $\preceq_U$ is partial order on $[0,a]$, it is obtained that $x\leq y$.
If $x,y\in[e,1]$, then it is obtained that $x\preceq_V y$. Since $\preceq_V$ is partial order on $[e,1]$, it is obtained that $x\leq y$.
Otherwise, since $x\preceq_{UV} y$, it is clear that $x\leq y$.

Remark. The converse of Proposition 5 may not be satisfied. For example, consider the lattice $(L=\{0,a,b,c,d,e,f,g,1\},\leq,0,1)$ such that $0<a<b<c<d<e<f<g<1$ and define the functions U on $[0,a]$ and V on $[e,1]$ as follows

U	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	a	a	a	a
b	0	a	b	c	d	e
c	0	a	c	e	e	e
d	0	a	d	e	e	e
e	0	a	e	e	e	e

Table 1: U on $[0,a]$

V	b	c	d	e	f	g	1
b	b	c	d	e	e	e	e
c	c	e	e	e	e	e	e
d	d	e	e	e	e	e	e
e	e	e	e	e	e	e	e
f	e	e	e	e	e	e	f
g	e	e	e	e	e	g	g
1	e	e	e	e	f	g	1

Table 2: V on $[e,1]$

One can easily show that U is an uninorm on $[0,e]$ with neutral element b and V is an nullnorm on $[b,1]$ with zero element e and it is satisfied that $0\leq b\leq e\leq 1$ and $U\downarrow_{[b,e]^2}=V\downarrow_{[b,e]^2}$.

Diagram of $\preceq_{UV}$ are as follows

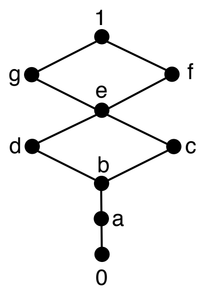

Fig. 1: $\preceq_{UV}$

Remark. Taking into account the example given in Remark 3, it is easily seen that even if $(L,\leq,0,1)$ is a chain, $(L,\preceq_{UV},0,1)$ may not be chain.

Corollary 1. *$(L,\leq,0,1)$ be a bounded lattice, $U:[0,a]^2\to[0,a]$ be an uninorm with neutral element e and $V:[e,1]^2\to[e,1]$ be a nullnorm with zero element a such that $0\leq e\leq a\leq 1$ and $U\downarrow_{[e,a]^2}=V\downarrow_{[e,a]^2}$. Then, $T_U,S_U=S_V,T_V$ are divisible if and only if $\preceq_{UV}=\leq$.*

4 Conclusion

In this paper, an order relation on bounded lattice is proposed via uninorm and nullnorm on sub-intervals of bounded lattice. Some properties of the order are studied. The order is compared with the orders obtained from uninorm and nullnorm.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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