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THE MEAN REMAINING STRENGTH OF PARALLEL SYSTEMS IN A STRESS-STRENGTH MODEL BASED ON EXPONENTIAL DISTRIBUTION

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Abstract. The mean remaining strength of any coherent system is one of the important characteristics in stress-strength reliability. It shows that the system on the average how long can be safe under the stress. In this paper, we consider the mean remaining strength of the parallel systems in the stress-strength model. We assume that the strength and stress components constitute parallel systems separately. The mean remaining strength and its estimations are obtained when the all components follow the exponential distribution. The likelihood ratio order between the remaining strength of the parallel systems is presented for two-component case. The simulation study is performed to compare the derived estimates and their results are presented.

1. Introduction

In the reliability theory, the stress-strength model describes the reliability of a component or system in terms of random variables. The reliability is defined as $R = P(X > Y)$ where Y is the random stress experienced by the system and X is the random strength of the system available to overcome the stress. The system fails if the stress exceeds the strength. This main idea was introduced by Birnbaum [1] and developed by Birnbaum and McCarty [2]. The last few decades, the problem of estimating R has been considerable investigated by many authors for the different data types and the distributional assumptions on X and Y. Examples of such results and references can be found in Kotz et al. [3], Kundu and Gupta [4], Basirat et al. [5, 6], Asgharzadeh et al. [7]. However, some results in the multicomponent stress-strength models can be found in Bhattacharyya and Johnson [8, 9], Eryilmaz [10, 11], Pakdaman and Ahmadi [12, 13], Hassan [14], Kızılaslan [15].

Let X and Y be two independent random variables. It is assumed that X is the strength to failure of a component subject to a random stress Y and the component

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works if its strength is greater than the applied stress, that is $X > Y$. Then, we may estimate the component's survival function under the stress Y. We may also wish to learn for how long, on average, the component can still be safe under the stress. The mean remaining strength (MRS) of the component can be defined as the expected remaining strength under the stress Y, i.e. $\Phi = E\left(X - Y \,|X > Y\right)$.

The MRS of the systems has been presented by Gurler [16] for the simple stress-strength model, k-out-of-$n : F$ system, series and parallel systems under the common stress. When the component is alive at the strength level t under the applied stress Y, the MRS of the component was defined as $\Phi(t) = E\left(X - Y - t\,|X - Y > t\right)$ for $t > 0$ by Bairamov et al. [17]. They obtained that the MRS of the k-out-of-$n : F$ system, series and parallel systems for the exchangeable strength components under the common stress. The MRS of the two-component parallel and series systems were considered by Gurler et al. [18] for the dependent strength components which are subject to a common stress.

In this study, the parallel stress and strength systems are considered. It is assumed that $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ are independent and identical strength and stress random variables follow the exponential distribution with parameters λ_1 and λ_2, respectively. Stochastic comparison of the remaining strength of two-component parallel strength and stress systems are studied. Maximum likelihood (ML) and Bayesian estimations of the MRS of this system are obtained. Bayesian estimates are derived by using Lindley's approximation and Markov Chain Monte Carlo (MCMC) method due to the lack of explicit forms. In Section 2, we introduce preliminaries for our system and obtain some distributional properties and stochastic ordering results. In Section 3, we derive ML and Bayesian estimations of the MRS of our system. Moreover, the asymptotic confidence and the highest probability density (HPD) credible intervals of the MRS are constructed. In Section 4, we present a simulation study to compare the proposed estimates of the MRS.

2. Model description

Let X be a random variable with exponential distribution with parameter λ and mean $1/\lambda$. Then, it is known that the cdf and pdf of X are given by

$$F_X(x) = 1 - e^{-\lambda x},\; f_X(x) = \lambda e^{-\lambda x},\; x > 0,\; \lambda > 0,$$

respectively and denoted by $X \sim Exp(\lambda)$.

For our case, it is assumed that $X_1, ..., X_{n_1}$ strength and $Y_1, ..., Y_{n_2}$ stress variables follow the exponential distribution with parameters λ_1 and λ_2. It is known that the distribution of the parallel system (or its maximum) is generalized exponential (GE) or exponentiated exponential distribution when the components are independent and identical exponential distribution. The GE distribution was introduced by Gupta and Kundu [19]. This distribution has been studied extensively in the literature since then.

If we assume that $X_{1:n_1} \leq X_{2:n_1} \leq \ldots \leq X_{n_1:n_1}$ are the ordered strength of the components, then $X_{1:n_1}$ and $X_{n_1:n_1}$ are the weakest and strongest components. It is clear that the strength and stress of the parallel systems are $\max_{1\leq i\leq n_1}(X_i) = X_{n_1:n_1}$ and $\max_{1\leq i\leq n_2}(Y_i) = Y_{n_2:n_2}$. The cdfs and pdfs of $X_{n_1:n_1}$ and $Y_{n_2:n_2}$ are

$$F_{X_{n_1:n_1}}(x) = (1-e^{-\lambda_1 x})^{n_1},\ f_{X_{n_1:n_1}}(x) = n_1\lambda_1 e^{-\lambda_1 x}(1-e^{-\lambda_1 x})^{n_1-1},$$

and

$$F_{Y_{n_2:n_2}}(y) = (1-e^{-\lambda_2 y})^{n_2},\ f_{Y_{n_2:n_2}}(y) = n_2\lambda_2 e^{-\lambda_2 y}(1-e^{-\lambda_2 y})^{n_2-1},$$

that is $X_{n_1:n_1} \sim GE(n_1, \lambda_1)$ and $Y_{n_2:n_2} \sim GE(n_2, \lambda_2)$ where n_i and λ_i $i = 1, 2$ are the shape and scale parameters.

In this case, the reliability for the strength and stress of the parallel systems is given by

$$\begin{aligned} R_{n_1,n_2} &= P(X_{n_1:n_1} > Y_{n_2:n_2}) = \int_0^\infty F_{Y_{n_2:n_2}}(x) f_{X_{n_1:n_1}}(x)dx \\ &= n_1 \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\binom{n_2}{j}\binom{n_1-1}{i}(-1)^{i+j}\frac{\lambda_1}{\lambda_1(i+1)+\lambda_2 j}. \end{aligned} \tag{1}$$

It is also obtained by Pakdaman and Ahmadi [12, 13] (see Equations 2.8 and 9, respectively).

Our system works if the strength is greater than the applied stress, that is $X_{n_1:n_1} > Y_{n_2:n_2}$. It is important to learn this system on the average how long can be safe under the stress. Hence, we want to estimate the mean remaining strength (MRS) of this system when the stress $Y_{n_2:n_2}$ is applied. The MRS of our parallel systems are the expected remaining strength under the stress $Y_{n_2:n_2}$ and given by

$$\Phi_{n_1,n_2} = E\left(X_{n_1:n_1} - Y_{n_2:n_2}\,|X_{n_1:n_1} > Y_{n_2:n_2}\right). \tag{2}$$

The cdf of the conditional random variable $\psi \equiv (X_{n_1:n_1} - Y_{n_2:n_2}\,|X_{n_1:n_1} > Y_{n_2:n_2})$ is

$$\begin{aligned} F_\psi(x) &= P(X_{n_1:n_1} - Y_{n_2:n_2} \leq x\,|X_{n_1:n_1} > Y_{n_2:n_2}) \\ &= \frac{P(X_{n_1:n_1} \leq Y_{n_2:n_2} + x,\ X_{n_1:n_1} > Y_{n_2:n_2})}{P(X_{n_1:n_1} > Y_{n_2:n_2})} \\ &= \frac{P(X_{n_1:n_1} \leq Y_{n_2:n_2} + x,\ X_{n_1:n_1} > Y_{n_2:n_2})}{R_{n_1,n_2}}. \end{aligned}$$

Then, conditioning on $Y_{n_2:n_2} = y$,

$$\begin{aligned} P(X_{n_1:n_1} &\leq Y_{n_2:n_2} + x,\ X_{n_1:n_1} > Y_{n_2:n_2}) = \int_0^\infty P(y < X_{n_1:n_1} \leq y + x) dF_{Y_{n_2:n_2}}(y) \\ &= \int_0^\infty \left(F_{X_{n_1:n_1}}(y+x) - F_{X_{n_1:n_1}}(y)\right) dF_{Y_{n_2:n_2}}(y) \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty F_{X_{n_1:n_1}}(y+x)dF_{Y_{n_2:n_2}}(y) - \int_0^\infty F_{X_{n_1:n_1}}(y)dF_{Y_{n_2:n_2}}(y) \\ &\equiv I_1 - I_2 \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_0^\infty F_{X_{n_1:n_1}}(y+x)dF_{Y_{n_2:n_2}}(y) \\ &= \int_0^\infty (1-e^{-\lambda_1(y+x)})^{n_1} n_2\lambda_2 e^{-\lambda_2 y}(1-e^{-\lambda_2 y})^{n_2-1}dy \\ &= n_2\lambda_2 \sum_{i=0}^{n_1}\sum_{j=0}^{n_2-1}\binom{n_2-1}{j}\binom{n_1}{i}(-1)^{i+j}\frac{e^{-\lambda_1 ix}}{\lambda_1 i+\lambda_2(j+1)}, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^\infty F_{X_{n_1:n_1}}(y)dF_{Y_{n_2:n_2}}(y) \\ &= \int_0^\infty (1-e^{-\lambda_1 y})^{n_1} n_2\lambda_2 e^{-\lambda_2 y}(1-e^{-\lambda_2 y})^{n_2-1}dy \\ &= n_2\lambda_2 \sum_{i=0}^{n_1}\sum_{j=0}^{n_2-1}\binom{n_2-1}{j}\binom{n_1}{i}(-1)^{i+j}\frac{1}{\lambda_1 i+\lambda_2(j+1)}. \end{aligned}$$

Hence,

$$F_\psi(x) = \frac{n_2\lambda_2}{R_{n_1,n_2}}\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\binom{n_2-1}{j}\binom{n_1}{i}(-1)^{i+j}\frac{(e^{-\lambda_1 ix}-1)}{\lambda_1 i+\lambda_2(j+1)} \tag{3}$$

and

$$f_\psi(x) = \frac{dF_\psi(x)}{dx} = \frac{n_2\lambda_2}{R_{n_1,n_2}}\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\binom{n_2-1}{j}\binom{n_1}{i}(-1)^{i+j+1}\frac{\lambda_1 i\ e^{-\lambda_1 ix}}{\lambda_1 i+\lambda_2(j+1)}. \tag{4}$$

Then,

$$\begin{aligned} \Phi_{n_1,n_2} &= E\left(X_{n_1:n_1} - Y_{n_2:n_2}\,|X_{n_1:n_1} > Y_{n_2:n_2}\right) \\ &= E_\psi(x) = \int_0^\infty x f_\psi(x)dx \\ &= \frac{n_2}{R_{n_1,n_2}}\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\binom{n_2-1}{j}\binom{n_1}{i}(-1)^{i+j+1}\frac{\lambda_2}{\lambda_1 i\left(\lambda_1 i+\lambda_2(j+1)\right)}. \end{aligned} \tag{5}$$

It can be also rewritten as

$$\Phi_{n_1,n_2} = \frac{R^*_{n_1,n_2}}{R_{n_1,n_2}}, \tag{6}$$

where

$$R^*_{n_1,n_2} = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \binom{n_2-1}{j}\binom{n_1}{i}(-1)^{i+j+1}\frac{\lambda_2}{\lambda_1 i\left(\lambda_1 i+\lambda_2(j+1)\right)}. \tag{7}$$

In Figure 1, some plots of Φ_{n_1,n_2} with respect to the parameters λ_1 and λ_2 are presented. It is observed that Φ_{n_1,n_2} is a decreasing function of λ_1 for fixed value of λ_2 and increasing function of λ_2 for fixed value of λ_1.

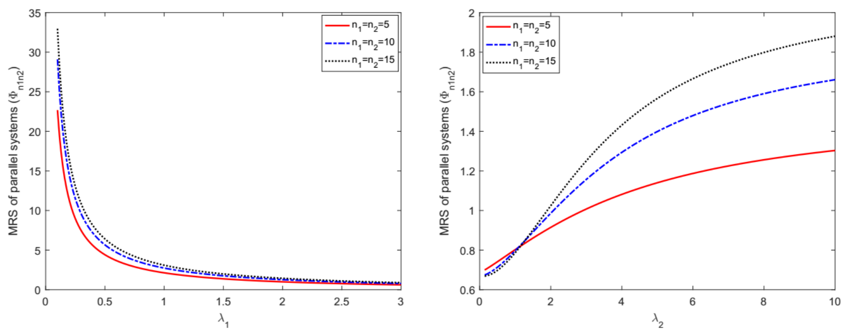


FIGURE 1. Plots of Φ_{n_1,n_2} with respect to the parameters λ_1 and λ_2.

2.1. **Stochastic ordering results.** In this section, we present the likelihood ratio ordering result associated with the remaining strength of parallel systems i.e. the conditional random variable $\psi \equiv X_{n_1:n_1} - Y_{n_2:n_2}\,|X_{n_1:n_1} > Y_{n_2:n_2}$. This random variable is a special case of the residual life of a random variable X at random time Θ which is defined as $X_\Theta = X - \Theta | X > \Theta$ (see Dewan and Khaledi [20] and Misra and Naqvi [21, 22].

Let X and Y be two lifetime random variables with pdfs $f(x)$ and $g(x)$, respectively. X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x for all x for which this ratio is well defined. It is known that the likelihood ratio order implies other stochastic orders. Hence the likelihood ratio order is the most interesting order in stochastic comparison. For more details on stochastic comparisons, see Shaked and Shanthikumar [23].

The coefficients of the cdf and pdf of $\psi \equiv X_{n_1:n_1} - Y_{n_2:n_2}\,|X_{n_1:n_1} > Y_{n_2:n_2}$ in equations (3) and (4) can be negative or positive. That is why general stochastic comparisons is not possible for ψ random variable. As a special case we consider two-component parallel systems (i.e. $n_1 = n_2 = 2$). In this case, we have

$$f_\psi(x) = \frac{4\lambda_1\lambda_2^2}{R_{2,2}\,(\lambda_1+\lambda_2)}\left[\frac{e^{-\lambda_1 x}}{\lambda_1+2\lambda_2} - \frac{e^{-2\lambda_1 x}}{2(2\lambda_1+\lambda_2)}\right],$$

where

$$R_{2,2} = 1 - \frac{5\lambda_1}{\lambda_1+\lambda_2} + \frac{2\lambda_1}{\lambda_1+2\lambda_2} + \frac{4\lambda_1}{2\lambda_1+\lambda_2},$$

from equations (4) and (1).

Theorem 1. *Suppose X_i, X_i^* $i=1,2$ are the strength and Y_i, Y_i^* $i=1,2$ are the stress variables with $X_i \sim Exp(\lambda_1)$, $X_i^* \sim Exp(\lambda_1^*)$, $Y_i \sim Exp(\lambda_2)$ and $Y_i^* \sim Exp(\lambda_2^*)$, $i=1,2$. If $\lambda_1^* < \lambda_1$ and $\lambda_2 < \lambda_2^*$, then we have $\psi \leq_{lr} \psi^*$ where $\psi = X_{2:2} - Y_{2:2}\,|X_{2:2} > Y_{2:2}$ and $\psi^* = X_{2:2}^* - Y_{2:2}^*\,|X_{2:2}^* > Y_{2:2}^*$.*

Proof. If we show that $f_{\psi^*}(x)/f_\psi(x)$ is increasing function in x, it completes the proof. Then, we have

$$\Lambda(x) = \frac{f_\psi(x)}{f_{\psi^*}(x)} = \frac{\lambda_1\lambda_2^2}{\lambda_1^*\lambda_2^{*2}}\;\frac{R_{2,2}^*\;(\lambda_1^*+\lambda_2^*)\;D_1^*\;D_2^*}{R_{2,2}\;(\lambda_1+\lambda_2)\;D_1\;D_2}\;\frac{2D_1\;e^{-\lambda_1 x} - D_2\;e^{-2\lambda_1 x}}{2D_1^*\;e^{-\lambda_1^* x} - D_2^*\;e^{-2\lambda_1^* x}},$$

where $D_1 = 2\lambda_1+\lambda_2$, $D_1^* = 2\lambda_1^*+\lambda_2^*$, $D_2 = \lambda_1+2\lambda_2$ and $D_2^* = \lambda_1^*+2\lambda_2^*$. After some computations

$$\begin{aligned}
&\Lambda^{'}(x) \overset{sign}{=} 4D_1\;D_1^*\;e^{-\lambda_1 x}e^{-\lambda_1^* x}(\lambda_1^*-\lambda_1) + 2D_2\;D_2^*\;e^{-2\lambda_1 x}e^{-2\lambda_1^* x}(\lambda_1^*-\lambda_1)\\
&+2D_1\;D_2^*\;e^{-\lambda_1 x}e^{-2\lambda_1^* x}(\lambda_1-2\lambda_1^*) + 2D_1^*\;D_2\;e^{-2\lambda_1 x}e^{-\lambda_1^* x}(2\lambda_1-\lambda_1^*)\\
<\;\;& 2D_1\,[2\;D_1^*(\lambda_1^*-\lambda_1) + D_2^*\;(\lambda_1-2\lambda_1^*)] + 2\;D_2\,[D_2^*(\lambda_1^*-\lambda_1) + D_1^*\;(2\lambda_1-\lambda_1^*)]\\
=\;\;& 6\lambda_1\lambda_1^*\,[(\lambda_1^*-\lambda_1)+(\lambda_2-\lambda_2^*)]\,,
\end{aligned}$$

where $a \overset{sign}{=} b$ means that a and b have the same sign. The last inequality implies that $\Lambda(x)$ is a decreasing function in x for $\lambda_1^* < \lambda_1$ and $\lambda_2 < \lambda_2^*$. Hence, it completes the proof. □

Example 2. *Theorem 1 results are observed in Figure 2. When the theorem conditions are not satisfied in Figure 3 (A) and (B), the graphic of $f_{\psi^*}(x)/f_\psi(x)$ can be concave or convex. However, it is observed that all these results are also valid for $n_1, n_2 > 2$.*

3. Estimation of Φ_{n_1,n_2}

In this section, we consider the estimation problem of MRS. Although the estimation of the stress-strength reliability of different systems has been considered extensively, the similar problem for MRS has not been studied in the literature except for Gurler et al. [18]. In our case, ML and Bayes estimations of the MRS are studied.

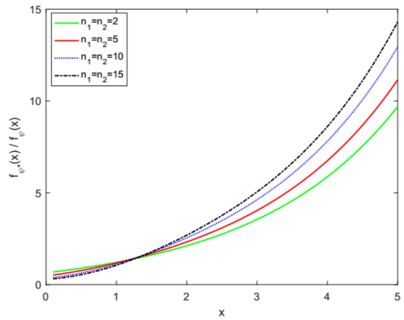
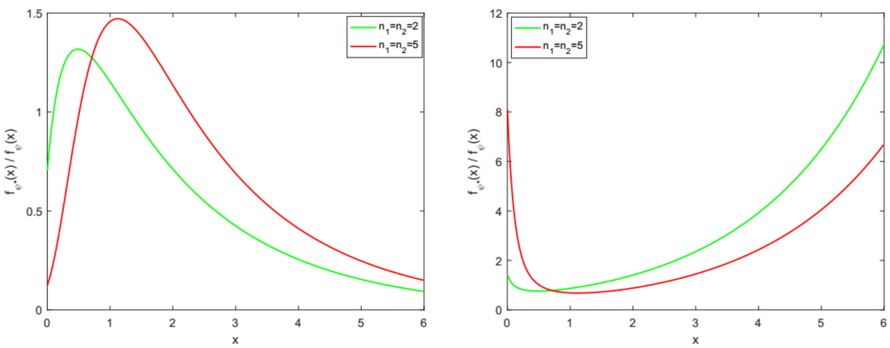


FIGURE 2. Plot of $f_{\psi^*}(x)/f_{\psi}(x)$ for $(\lambda_1, \lambda_2) = (2, 3)$ and $(\lambda_1^*, \lambda_2^*) = (1.5, 3.5)$.



(A) $(\lambda_1, \lambda_2) = (1, 1)$ and $(\lambda_1^*, \lambda_2^*) = (1.5, 10)$

(B) $(\lambda_1, \lambda_2) = (1.5, 10)$ and $(\lambda_1^*, \lambda_2^*) = (1, 1)$

FIGURE 3. Plots of $f_{\psi^*}(x)/f_{\psi}(x)$ for different parameters.

3.1. **MLE case.** The random strength and stress of the parallel systems are denoted by $V = \max_{1 \le i \le n_1}(X_i)$ and $W = \max_{1 \le j \le n_2}(Y_j)$. It is known that $V \sim GE(n_1, \lambda_1)$ and $W \sim GE(n_2, \lambda_2)$ when X_i $i = 1, ..., n_1$ and Y_j, $j = 1, ..., n_2$ are exponential distributions with parameters λ_1 and λ_2. Let $V_1, ..., V_n$ be a random sample of size n from $GE(n_1, \lambda_1)$ and $W_1, ..., W_m$ be a random sample of size m from $GE(n_2, \lambda_2)$. Then, the likelihood function based on the observed sample $\{\mathbf{v} = (v_1, ..., v_n),\ \mathbf{w} = (w_1, ..., w_m)\}$ is given by

$$L(\lambda_1, \lambda_2 \,|\mathbf{v}, \mathbf{w}) = \prod_{i=1}^{n} \prod_{j=1}^{m} f_{V_i}(v_i) f_{W_j}(w_j)$$

$$= \quad n_1^n \lambda_1^n n_2^m \lambda_2^m \exp\left(-\lambda_1 \sum_{i=1}^{n} v_i + (n_1-1)\sum_{i=1}^{n} \ln(1-e^{-\lambda_1 v_i})\right)$$
$$\exp\left(-\lambda_2 \sum_{j=1}^{m} w_j + (n_2-1)\sum_{j=1}^{m} \ln(1-e^{-\lambda_2 w_j})\right).$$

Hence, the MLEs of λ_1 and λ_2, say $\widehat{\lambda}_1$and $\widehat{\lambda}_2$, are the solution of the following nonlinear equations:

$$\frac{n}{\lambda_1} - \sum_{i=1}^{n} \frac{v_i}{1-e^{-\lambda_1 v_i}} + n_1 \sum_{i=1}^{n} \frac{v_i e^{-\lambda_1 v_i}}{1-e^{-\lambda_1 v_i}} = 0,$$
$$\frac{m}{\lambda_2} - \sum_{j=1}^{m} \frac{w_j}{1-e^{-\lambda_2 w_j}} + n_2 \sum_{j=1}^{m} \frac{w_j e^{-\lambda_2 w_j}}{1-e^{-\lambda_2 w_j}} = 0.$$

$\widehat{\lambda}_1$and $\widehat{\lambda}_2$ can be obtained by using the fixed point method or Newton-Raphson method or other numerical methods. Ghitany et al. [24] proved that if at least one observation is different minimum of the all observations, then this type nonlinear equations have unique solution. When we obtain $\widehat{\lambda}_1$and $\widehat{\lambda}_2$, the MLE of Φ_{n_1,n_2}, say $\widehat{\Phi}_{n_1,n_2}^{MLE}$, is obtained from (5) by using the invariance property of MLEs.

Moreover, an asymptotic confidence interval of Φ_{n_1,n_2} can be constructed based on the MLEs. The Fisher information matrix of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is

$$I(\boldsymbol{\lambda}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \lambda_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1}\right) & E\left(\frac{\partial^2 l}{\partial \lambda_2^2}\right) \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}.$$

where $l = \ln(L(\lambda_1, \lambda_2 \,|v, \mathbf{w}))$. The elements of the matrix are obtained as $I_{12} = I_{21} = 0$,

$$I_{11} = \frac{n}{\lambda_1^2} + \frac{n_1 n}{\lambda_1^2 (n_1-2)} \left\{ (\psi(2) - \psi(n_1))^2 + \psi^{'}(2) - \psi^{'}(n_1) \right\}$$

and

$$I_{22} = \frac{m}{\lambda_2^2} + \frac{n_2 m}{\lambda_2^2 (n_2-2)} \left\{ (\psi(2) - \psi(n_2))^2 + \psi^{'}(2) - \psi^{'}(n_2) \right\},$$

for $n_1 > 2$ and $n_2 > 2$ by using the formula 4.261(17) in Gradshteyn and Ryzhik [25] where $\psi(x) = d \ln \Gamma(x)/dx$ is a Psi function. $\widehat{\Phi}_{n_1,n_2}^{MLE}$ is asymptotically normal with mean Φ_{n_1,n_2} and asymptotic variance

$$\sigma^2_{\Phi_{n_1,n_2}} = \sum_{j=1}^{2} \sum_{i=1}^{2} \frac{\partial \Phi_{n_1,n_2}}{\partial \lambda_i} \frac{\partial \Phi_{n_1,n_2}}{\partial \lambda_j} I_{ij}^{-1},$$

where I_{ij}^{-1} is the (i,j)th element of the inverse of the $I(\boldsymbol{\lambda})$; see Rao [26]. Then,

$$\sigma^2_{\Phi_{n_1,n_2}} = \left(\frac{\partial \Phi_{n_1,n_2}}{\partial \lambda_1}\right)^2 \frac{1}{I_{11}} + \left(\frac{\partial \Phi_{n_1,n_2}}{\partial \lambda_2}\right)^2 \frac{1}{I_{22}}.$$

The partial derivatives of R_{n_1,n_2} and $R^*_{n_1,n_2}$ with respect to λ_1 and λ_2 are given by

$$\frac{\partial R_{n_1,n_2}}{\partial \lambda_1} = n_1 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \binom{n_2}{j} \binom{n_1-1}{i} (-1)^{i+j} \frac{\lambda_2 j}{\left(\lambda_1(i+1)+\lambda_2 j\right)^2},$$

$$\frac{\partial R_{n_1,n_2}}{\partial \lambda_2} = n_1 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \binom{n_2}{j} \binom{n_1-1}{i} (-1)^{i+j+1} \frac{\lambda_1 j}{\left(\lambda_1(i+1)+\lambda_2 j\right)^2},$$

$$\frac{\partial R^*_{n_1,n_2}}{\partial \lambda_1} = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \binom{n_2-1}{j} \binom{n_1}{i} (-1)^{i+j} \frac{\lambda_2(2\lambda_1 i+\lambda_2(j+1))}{\lambda_1^2 i\left(\lambda_1 i+\lambda_2(j+1)\right)^2},$$

$$\frac{\partial R^*_{n_1,n_2}}{\partial \lambda_2} = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \binom{n_2-1}{j} \binom{n_1}{i} (-1)^{i+j+1} \frac{1}{\left(\lambda_1 i+\lambda_2(j+1)\right)^2}.$$

Then, $\partial \Phi_{n_1,n_2}/\partial \lambda_1$ and $\partial \Phi_{n_1,n_2}/\partial \lambda_2$ are evaluated by using these partial derivatives. Therefore, an asymptotic $100(1-\gamma)\%$ confidence interval of Φ_{n_1,n_2} is given by

$$\Phi_{n_1,n_2} \in \left(\widehat{\Phi}^{MLE}_{n_1,n_2} \pm z_{\gamma/2}\widehat{\sigma}_{\Phi_{n_1,n_2}}\right),$$

where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\widehat{\sigma}_{\Phi_{n_1,n_2}}$ is the value of $\sigma_{\Phi_{n_1,n_2}}$ at the MLE of the parameters.

3.2. **Bayesian case.** In this section, we assume that the parameters λ_1 and λ_2 are random variables and have statistically independent gamma prior distributions with parameters (a_i, b_i), $i = 1, 2$, respectively. The pdf of a gamma random variable X with parameters (a_i, b_i) is

$$f(x) = \frac{b_i^{a_i}}{\Gamma(a_i)} x^{a_i-1} e^{-xb_i}, \ x > 0, \ a_i, b_i > 0$$

where $a_i, b_i > 0$, $i = 1, 2$. Then, the joint posterior density function of λ_1 and λ_2 is

$$\begin{aligned}\pi(\lambda_1, \lambda_2 \,|\mathbf{v}, \mathbf{w}) \propto \lambda_1^{n+a_1-1} \exp\left(-\lambda_1\left(b_1 + \sum_{i=1}^{n} v_i\right) + (n_1-1)\sum_{i=1}^{n} \ln(1-e^{-\lambda_1 v_i})\right) \\ \lambda_2^{m+a_2-1} \exp\left(-\lambda_2\left(b_2 + \sum_{j=1}^{m} w_j\right) + (n_2-1)\sum_{j=1}^{m} \ln(1-e^{-\lambda_2 w_j})\right).\end{aligned} \tag{8}$$

The Bayes estimator of Φ_{n_1,n_2} under the SE loss function is given by

$$\widehat{\Phi}^{Bayes}_{n_1,n_2} = \int_0^\infty \int_0^\infty \Phi_{n_1,n_2} \pi(\lambda_1, \lambda_2 \,|\mathbf{v}, \mathbf{w}) d\lambda_1 d\lambda_2. \tag{9}$$

Since the integrals given in (9) is not computed analytically, Lindley's approximation and MCMC methods can be applied to approximate (9).

3.2.1. *Lindley's approximation.* Lindley [27] introduced an approximate procedure for the computation of the ratio of two integrals. This procedure, applied to the posterior expectation of the function $U(\lambda)$ for a given $\mathbf{x}$, is

$$E(u(\theta)\,|\mathbf{x}) = \frac{\int u(\theta)e^{Q(\theta)}d\theta}{\int e^{Q(\theta)}d\theta},$$

where $Q(\theta) = l(\theta) + \rho(\theta)$, $l(\theta)$ is the logarithm of the likelihood function and $\rho(\theta)$ is the logarithm of the prior density of θ. Using Lindley's approximation, $E(u(\theta)\,|\mathbf{x})$ is approximately estimated by

$$\begin{aligned} E(u(\theta)\,|\mathbf{x}) &= \left[u + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l\right]_{\widehat{\lambda}} \\ &\quad + \text{terms of order } n^{-2} \text{ or smaller,} \end{aligned}$$

where $\theta = (\theta_1, \theta_2, ..., \theta_m)$, $i, j, k, l = 1, ..., m$, $\widehat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $L_{ijk} = \partial^3 l/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$ and $\sigma_{ij} = (i,j)$th element in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters.

For the two parameter case $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, Lindley's approximation leads to

$$\widehat{u}_{Lin} = u(\boldsymbol{\lambda}) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right],$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} u_{ij}\tau_{ij}$, $Q_{ij} = \partial Q^{i+j}/\partial^i\theta\lambda_1\partial^j\lambda_2$ for $i, j = 0, 1, 2, 3$, $i + j = 3$, $u_i = \partial u/\partial\lambda_i$, $u_{ij} = \partial^2 U/\partial\lambda_i\partial\lambda_j$ for $i, j = 1, 2$ and $B_{ij} = (u_i\tau_{ii} + u_j\tau_{ij})\tau_{ii}$, $C_{ij} = 3u_i\tau_{ii}\tau_{ij} + u_j(\tau_{ii}\tau_{ij} + 2\tau_{ij}^2)\tau_{ij}$ for $i \neq j$. τ_{ij} is the (i,j)th element in the inverse of matrix $Q^* = (-Q_{ij}^*)$, $i, j = 1, 2$ such that $Q_{ij}^* = \partial Q^2/\partial\lambda_i\partial\lambda_j$. The approximate Bayes estimate $\widehat{u}_{Lin}$ is evaluated at $(\widetilde{\lambda}_1, \widetilde{\lambda}_2)$ which is the mode of the posterior density.

In our case, $u(\boldsymbol{\lambda}) = \Phi_{n_1,n_2}$,

$$\begin{aligned} Q &= \ln\pi(\lambda_1, \lambda_2\,|\mathbf{v}, \mathbf{w}) \propto (n + a_1 - 1)\ln\lambda_1 + (m + a_2 - 1)\ln\lambda_2 - \lambda_1\left(b_1 + \sum_{i=1}^{n} v_i\right) \\ &\quad -\lambda_2\left(b_2 + \sum_{j=1}^{m} w_j\right) + (n_1 - 1)\sum_{i=1}^{n}\ln(1 - e^{-\lambda_1 v_i}) + (n_2 - 1)\sum_{j=1}^{m}\ln(1 - e^{-\lambda_2 w_j}). \end{aligned}$$

The posterior mode of λ_1 and λ_2, say $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$, are the solution of the following nonlinear equations from Q

$$\frac{n + a_1 - 1}{\lambda_1} - \left(b_1 + \sum_{i=1}^{n} v_i\right) + (n_1 - 1)\sum_{i=1}^{n}\frac{v_i e^{-\lambda_1 v_i}}{1 - e^{-\lambda_1 v_i}} = 0,$$

$$\frac{m+a_2-1}{\lambda_2}-\left(b_2+\sum_{j=1}^{m}w_j\right)+(n_2-1)\sum_{j=1}^{m}\frac{w_je^{-\lambda_2 w_j}}{1-e^{-\lambda_2 w_j}}=0.$$

Moreover, it is obtained that

$$\tau_{11}=\left[\frac{n+a_1-1}{\lambda_1^2}+(n_1-1)\sum_{i=1}^{n}\frac{v_i^2e^{-\lambda_1 v_i}}{\left(1-e^{-\lambda_1 v_i}\right)^2}\right]^{-1},$$

$$\tau_{22}=\left[\frac{m+a_2-1}{\lambda_2^2}+(n_2-1)\sum_{j=1}^{m}\frac{w_j^2e^{-\lambda_2 w_j}}{\left(1-e^{-\lambda_2 w_j}\right)^2}\right]^{-1},$$

$\tau_{12}=\tau_{21}=0$, $Q_{12}=Q_{21}=0$, $Q_{03}=2(m+a_2-1)/\lambda_2^3$, $Q_{30}=2(n+a_1-1)/\lambda_1^3$, $B_{12}=u_1\tau_{11}^2$, $B_{21}=u_2\tau_{22}^2$, $B=u_{11}\tau_{11}+u_{22}\tau_{22}$,

$$\begin{aligned}u_{11} &= \frac{\partial^2\Phi_{n_1,n_2}}{\partial\lambda_1^2}=\frac{1}{\left(R_{n_1,n_2}\right)^2}\left(R_{n_1,n_2}\frac{\partial^2R^*_{n_1,n_2}}{\partial\lambda_1^2}-R^*_{n_1,n_2}\frac{\partial^2R_{n_1,n_2}}{\partial\lambda_1^2}\right)\\ &\quad -\frac{2}{\left(R_{n_1,n_2}\right)^3}\frac{\partial R_{n_1,n_2}}{\partial\lambda_1}\left(R_{n_1,n_2}\frac{\partial R^*_{n_1,n_2}}{\partial\lambda_1}-R^*_{n_1,n_2}\frac{\partial R_{n_1,n_2}}{\partial\lambda_1}\right)\end{aligned}$$

and

$$\begin{aligned}u_{22} &= \frac{\partial^2\Phi_{n_1,n_2}}{\partial\lambda_2^2}=\frac{1}{\left(R_{n_1,n_2}\right)^2}\left(R_{n_1,n_2}\frac{\partial^2R^*_{n_1,n_2}}{\partial\lambda_2^2}-R^*_{n_1,n_2}\frac{\partial^2R_{n_1,n_2}}{\partial\lambda_2^2}\right)\\ &\quad -\frac{2}{\left(R_{n_1,n_2}\right)^3}\frac{\partial R_{n_1,n_2}}{\partial\lambda_2}\left(R_{n_1,n_2}\frac{\partial R^*_{n_1,n_2}}{\partial\lambda_2}-R^*_{n_1,n_2}\frac{\partial R_{n_1,n_2}}{\partial\lambda_2}\right).\end{aligned}$$

u_{11} and u_{22} are evaluated by using the second partial derivatives of R_{n_1,n_2} and $R^*_{n_1,n_2}$ with respect to λ_1 and λ_2. Therefore, the approximate Bayes estimate of Φ_{n_1,n_2} is

$$\widehat{\Phi}^{Lin}_{n_1,n_2}=\Phi_{n_1,n_2}+\frac{1}{2}\left[B+Q_{30}B_{12}+Q_{03}B_{21}\right]_{(\lambda_1,\lambda_2)=(\widetilde{\lambda}_1,\widetilde{\lambda}_2)}. \tag{10}$$

3.2.2. *MCMC method.* The joint posterior density function of λ_1 and λ_2 is given in (8). The marginal posterior density functions of λ_1 and λ_2 are given respectively as

$$\pi_1(\lambda_1\,|\lambda_2,\mathbf{v},\mathbf{w})\propto\lambda_1^{n+a_1-1}\exp\left(-\lambda_1\left(b_1+\sum_{i=1}^{n}v_i\right)+(n_1-1)\sum_{i=1}^{n}\ln(1-e^{-\lambda_1 v_i})\right),$$

and

$$\pi_2(\lambda_2\,|\lambda_1,\mathbf{v},\mathbf{w})\propto\lambda_2^{m+a_2-1}\exp\left(-\lambda_2\left(b_2+\sum_{j=1}^{m}w_j\right)+(n_2-1)\sum_{j=1}^{m}\ln(1-e^{-\lambda_2 w_j})\right).$$

Since these density functions are not well-known distribution, it is not possible to sample directly by standard methods. If the posterior density function is unimodal and roughly symmetric, then it is often convenient to approximate it by a normal distribution (see Gelman et al., [28]). To see the marginal posterior densities are unimodal and roughly symmetric, we check whether the posterior densities have the log-concavity property. It is easily seen that the marginal posterior densities of λ_1 and λ_2 are log-concave. Therefore, we use the Metropolis-Hasting algorithm with the normal proposal distribution to generate a random sample from the posterior densities of λ_1 and λ_2 in our implementation. The following algorithm is used.

Step 1: Start with initial guess $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$.

Step 2: Set $i = 1$.

Step 3: Generate $\lambda_1^{(i)}$ from $\pi_1(\lambda_1 \,|\lambda_2, \mathbf{v}, \mathbf{w})$ using the Metropolis-Hastings algorithm with the proposal distribution $q_1(\lambda_1) \equiv N(\lambda_1^{(i-1)}, V_{\lambda_1})$ as follows.

a) Let $v = \lambda_1^{(i-1)}$.

b) Generate w from the proposal distribution q_1.

c) Let $p(v, w) = \min\left\{1, \dfrac{\pi_1(w \left| \lambda_2^{(i)}, \mathbf{v}, \mathbf{w}\right)\, q_1(v)}{\pi_1(v \left| \lambda_1^{(i)}, \mathbf{v}, \mathbf{w}\right)\, q_1(w)}\right\}$.

d) Generate u from $Uniform(0, 1)$. If $u \leq p(v, w)$, then accept the proposal and set $\lambda_1^{(i)} = w$; otherwise, set $\lambda_1^{(i)} = v$.

Step 4: Similarly, $\lambda_2^{(i)}$ is generated from $\pi_2(\lambda_2 \,|\lambda_1, \mathbf{v}, \mathbf{w})$ using the Metropolis-Hastings algorithm with the proposal distribution $q_2(\lambda_2) \equiv N(\lambda_2^{(i-1)}, V_{\lambda_2})$.

Step 5: Compute the $\Phi_{n_1,n_2}^{(i)}$ at $(\lambda_1^{(i)}, \lambda_2^{(i)})$.

Step 6: Set $i = i + 1$.

Step 7: Repeat Steps 2 through -7, N times and obtain the posterior sample $\Phi_{n_1,n_2}^{(i)}$, $i = 1, ..., N$.

This sample is used to compute the Bayes estimate and to construct the HPD credible interval for Φ_{n_1,n_2}. The Bayes estimate of $R_{s,k}$ under a SE loss function is given by

$$\widehat{\Phi}_{n_1,n_2}^{MCMC} = \frac{1}{N - M} \sum_{i=M+1}^{N-M} \Phi_{n_1,n_2}^{(i)}, \tag{11}$$

where M is the burn-in period.

The HPD $100(1 - \gamma)\%$ credible interval of $R_{s,k}$ is obtained by the method of Chen and Shao [29].

4. Simulation Study

In this section, some numerical results are presented to compare the performance of the ML and Bayes estimates of Φ_{n_1,n_2} for different parameters and sample sizes. The performances of the point estimators are compared by using mean squared error (MSE) and estimated risks (ERs). The performances of the asymptotic confidence and credible intervals are compared by using average confidence lengths and coverage probabilities (cps). The coverage probability of a confidence interval is the proportion of the time that the interval contains the true value of interest. The ER of θ, when θ is estimated by $\widehat{\theta}$, is given by

$$ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta_i\right)^2,$$

under the SE loss function. All of the computations are performed by using MATLAB. All the results are based on 2500 replications.

In Tables 1-4, strength and stress samples are generated for $(n_1, n_2) = (5, 5)$, $(10, 10)$, $(15, 10)$ and $(\lambda_1, \lambda_2) = (0.5, 10)$, $(1, 10)$, $(1.5, 10)$, $(2, 10)$ and different sample sizes n and $m = 10(10)50$. The hyperparameters are chosen that prior means are exactly equal to the true values of the parameters. For this reason $(a_1, b_1) = (5, 10)$, $(10, 10)$, $(15, 10)$, $(20, 10)$ and $(a_2, b_2) = (5, 1/2)$ are used for $(\lambda_1, \lambda_2) = (0.5, 10)$, $(1, 10)$, $(1.5, 10)$, $(2, 10)$, respectively. For these samples, estimations of Φ_{n_1,n_2} are listed based on the MLE and Bayesian estimates which are obtained by using Lindley's approximation and MCMC method. Moreover, 95% asymptotic confidence interval and HPD credible interval of Φ_{n_1,n_2} with its coverage probabilities (cps) are presented.

In the MCMC case, we run three MCMC chains with fairly different initial values and generate 5000 iterations for each chain. To diminish the effect of the starting distribution, a certain number of the first 2500 draws is discarded. This is known as the burn-in. In our case, we discard the first 2500 iterations of each sequence and focus on the other 2500 iterations. In order to break the dependence between draws in the Markov chain, it is suggested only to keep every dth draw of the chain. This is known as thinning. In our case, we calculate the Bayesian MCMC estimates by the means of every 5th sampled values after discarding the first 2500 iterations of the chains. To monitor convergence of MCMC simulations the scale reduction factor estimate is used. The estimate is given by $\sqrt{Var(\psi)/W}$, where ψ is the estimand of interest, $Var(\psi) = (n-1)W/n + B/n$ with the iteration number n for each chain, the between-sequence variance B and the within-sequence variance W, see Gelman et al. [28]. In our case, the scale factor values of the MCMC estimators are found to be below 1.1, which is an acceptable value for their convergence.

From Tables 1-4, it is observed that the average MSEs of ML estimates and ERs of the Bayes estimates of Φ_{n_1,n_2} decrease as the sample size increases in all cases, as expected. The Bayes estimates of Φ_{n_1,n_2} have smaller errors than that of MLEs. Moreover, the ERs of the Bayes estimates which are obtained from the MCMC method are smaller than those obtained from Lindley's approximation. The average lengths of the intervals decrease as the sample size increases. The average lengths of the Bayesian credible intervals are smaller than those of the asymptotic

TABLE 1. Estimates and confidence interval of Φ_{n_1,n_2}.

						$\lambda_1 = 0.5$ and $\lambda_2 = 10$			
n_1	n_2	n	m	Φ_{n_1,n_2}	$\widehat{\Phi}^{MLE}_{n_1,n_2}$	$\widehat{\Phi}^{Lindley}_{n_1,n_2}$	$\widehat{\Phi}^{MCMC}_{n_1,n_2}$	ACI of $\Phi_{n_1:n_2}$	$HPDCI$ of $\Phi_{n_1:n_2}$
5	5	10	10	4.3387	4.3587	4.5344	4.4223	(2.8714,5.8460)	(3.0964,5.8736)
					0.5969	0.5405	0.4843	2.9746/0.9332	2.7773/0.9524
		20	20		4.3463	4.4411	4.3779	(3.2975,5.3951)	(3.4074,5.4112)
					0.2838	0.2704	0.2543	2.0976/0.9428	2.0037/0.9496
		30	30		4.3452	4.4100	4.3660	(3.4891,5.2013)	(3.5637,5.2104)
					0.1908	0.1849	0.1772	1.7122/0.9488	1.6467/0.9484
		40	40		4.3493	4.3984	4.3651	(3.6073,5.0913)	(3.6655,5.0973)
					0.1434	0.1407	0.1363	1.4840/0.9564	1.4318/0.9504
		50	50		4.3476	4.3871	4.3599	(3.6841,5.0110)	(3.7324,5.0131)
					0.1138	0.1121	0.1091	1.3270/0.9460	1.2807/0.9484
10	10	10	10	5.6550	5.6204	5.7751	5.6556	(4.1033,7.1374)	(4.2848,7.1228)
					0.6173	0.5974	0.5391	3.0341/0.9444	2.8380/0.9488
		20	20		5.5861	5.6682	5.6031	(4.5199,6.6522)	(4.6182,6.6403)
					0.2945	0.2895	0.2743	2.1323/0.9460	2.0221/0.9464
		30	30		5.5887	5.6443	5.5994	(4.7178,6.4596)	(4.7880,6.4430)
					0.2111	0.2095	0.2025	1.7418/0.9512	1.6550/0.9424
		40	40		5.5788	5.6210	5.5865	(4.8258,6.3317)	(4.8867,6.3141)
					0.1480	0.1470	0.1436	1.5059/0.9480	1.4274/0.9344
		50	50		5.5667	5.6008	5.5738	(4.8947,6.2387)	(4.9488,6.2203)
					0.1218	0.1204	0.1185	1.3440/0.9460	1.2715/0.9320
15	10	10	10	6.3436	6.3792	6.5217	6.4019	(4.8683,7.8901)	(5.0233,7.8625)
					0.5917	0.5764	0.5284	3.0218/0.9464	2.8392/0.9468
		20	20		6.3653	6.4397	6.3746	(5.2994,7.4313)	(5.3864,7.4065)
					0.2979	0.2948	0.2818	2.1319/0.9528	2.0201/0.9408
		30	30		6.3483	6.3988	6.3551	(5.4802,7.2165)	(5.5465,7.1931)
					0.1912	0.1890	0.1845	1.7363/0.9536	1.6466/0.9472
		40	40		6.3619	6.3999	6.3670	(5.6086,7.1153)	(5.6660,7.0888)
					0.1465	0.1463	0.1429	1.5067/0.9568	1.4227/0.9392
		50	50		6.3434	6.3741	6.3479	(5.6715,7.0154)	(5.7204,6.9906)
					0.1186	0.1176	0.1160	1.3439/0.9472	1.2702/0.9312

Notes: 1st row represents the average estimates, 95% confidence interval and 2nd row represents corresponding MSE or ERs, interval lengths and cps for the point and interval estimates, respectively.

TABLE 2. Estimates and confidence interval of Φ_{n_1,n_2}.

						$\lambda_1 = 1$ and $\lambda_2 = 10$			
n_1	n_2	n	m	Φ_{n_1,n_2}	$\widehat{\Phi}^{MLE}_{n_1,n_2}$	$\widehat{\Phi}^{Lindley}_{n_1,n_2}$	$\widehat{\Phi}^{MCMC}_{n_1,n_2}$	ACI of Φ_{n_1,n_2}	$HPDCI$ of $\Phi_{n_1:n_2}$
5	5	10	10	2.0583	2.0612	2.1311	2.0919	(1.3207,2.8018)	(1.4553,2.7841)
					0.1466	0.1027	0.0938	1.4811/0.9392	1.3288/0.9676
		20	20		2.0723	2.1120	2.0858	(1.5459,2.5986)	(1.6075,2.5936)
					0.0715	0.0604	0.0566	1.0527/0.9492	0.9862/0.9632
		30	30		2.0638	2.0924	2.0731	(1.6357,2.4919)	(1.6754,2.4909)
					0.0481	0.0428	0.0409	0.8561/0.9460	0.8156/0.9564
		40	40		2.0628	2.0850	2.0697	(1.6922,2.4334)	(1.7215,2.4332)
					0.0359	0.0329	0.0318	0.7412/0.9492	0.7117/0.9548
		50	50		2.0607	2.0788	2.0664	(1.7295,2.3919)	(1.7533,2.3919)
					0.0270	0.0252	0.0245	0.6624/0.9552	0.6385/0.9596
10	10	10	10	2.6362	2.6484	2.7128	2.2660	(1.8914,3.4055)	(1.9922,3.3828)
					0.1532	0.1238	0.1131	1.5140/0.9428	1.3906/0.9596
		20	20		2.6503	2.6860	2.6575	(2.1145,3.1860)	(2.1624,3.1755)
					0.0749	0.0678	0.0641	1.0715/0.9504	1.0131/0.9584
		30	30		2.6433	2.6683	2.6481	(2.2069,3.0796)	(2.2397,3.0724)
					0.0520	0.0485	0.0468	0.8728/0.9412	0.8326/0.9468
		40	40		2.6395	2.6588	2.6429	(2.2622,3.0168)	(2.2880,3.0095)
					0.0381	0.0361	0.0351	0.7546/0.9504	0.7215/0.9476
		50	50		2.6392	2.6548	2.6420	(2.3017,2.9766)	(2.3231,2.9700)
					0.0292	0.0280	0.0274	0.6749/0.9440	0.6468/0.9432
15	10	10	10	3.0253	3.0237	3.0849	3.0377	(2.2690,3.7783)	(2.3546,3.7561)
					0.1396	0.1175	0.1095	1.5093/0.9568	1.4015/0.9688
		20	20		3.0367	3.0695	3.0412	(2.5015,3.5718)	(2.5431,3.5591)
					0.0766	0.0709	0.0678	1.0704/0.9456	1.0160/0.9464
		30	30		3.0331	3.0558	3.0358	(2.5966,3.4696)	(2.6249,3.4597)
					0.0509	0.0483	0.0469	0.8729/0.9488	0.8348/0.9496
		40	40		3.0283	3.0457	3.0305	(2.6509,3.4057)	(2.6734,3.3968)
					0.0366	0.0351	0.0345	0.7548/0.9524	0.7235/0.9520
		50	50		3.0280	3.0421	3.0295	(2.6905,3.3655)	(2.7097,3.3565)
					0.0293	0.0284	0.0280	0.6749/0.9584	0.6468/0.9520

Notes: 1st row represents the average estimates, 95% confidence interval and 2nd row represents corresponding MSE or ERs, interval lengths and cps for the point and interval estimates, respectively.

TABLE 3. Estimates and confidence interval of Φ_{n_1,n_2}.

$\lambda_1 = 1.5$ and $\lambda_2 = 10$									
n_1	n_2	n	m	Φ_{n_1,n_2}	$\widehat{\Phi}^{MLE}_{n_1,n_2}$	$\widehat{\Phi}^{Lindley}_{n_1,n_2}$	$\widehat{\Phi}^{MCMC}_{n_1,n_2}$	ACI of Φ_{n_1,n_2}	$HPDCI$ of $\Phi_{n_1:n_2}$
5	5	10	10	1.3032	1.3086	1.3452	1.3306	(0.8180,1.7991)	(0.9194,1.7751)
					0.0614	0.0348	0.0323	0.9811/0.9444	0.8558/0.9832
		20	20		1.3063	1.3299	1.3175	(0.9596,1.6531)	(1.0064,1.6477)
					0.0324	0.0240	0.0228	0.6934/0.9444	0.6413/0.9704
		30	30		1.3058	1.3230	1.3131	(1.0227,1.5890)	(1.0521,1.5873)
					0.0214	0.0174	0.0168	0.5663/0.9452	0.5352/0.9628
		40	40		1.3057	1.3192	1.3112	(1.0605,1.5509)	(1.0822,1.5503)
					0.0151	0.0129	0.0126	0.4905/0.9476	0.4681/0.9608
		50	50		1.3029	1.3142	1.3073	(1.0839,1.5218)	(1.1010,1.5214)
					0.0125	0.0110	0.0107	0.4379/0.9492	0.4204/0.9576
10	10	10	10	1.6609	1.6686	1.7037	1.6837	(1.1628,2.1744)	(1.2394,2.1553)
					0.0680	0.0472	0.0440	1.0117/0.9452	0.9159/0.9676
		20	20		1.6700	1.6908	1.6760	(1.3118,2.0282)	(1.3469,2.0202)
					0.0336	0.0281	0.0267	0.7164/0.9516	0.6733/0.9640
		30	30		1.6661	1.6811	1.6701	(1.3744,1.9579)	(1.3975,1.9529)
					0.0218	0.0193	0.0187	0.5836/0.9496	0.5554/0.9528
		40	40		1.6690	1.6805	1.6719	(1.4160,1.9220)	(1.4336,1.9178)
					0.0163	0.0150	0.0146	0.5060/0.9544	0.4842/0.9532
		50	50		1.6622	1.6719	1.6648	(1.4366,1.8878)	(1.4509,1.8847)
					0.0131	0.0121	0.0119	0.4512/0.9512	0.4338/0.9528
15	10	10	10	1.9194	1.9289	1.9617	1.9401	(1.4209,2.4370)	(1.4829,2.4192)
					0.0690	0.0520	0.0486	1.0161/0.9400	0.9363/0.9692
		20	20		1.9290	1.9479	1.9330	(1.5698,2.8882)	(1.5982 ,2.2801)
					0.0339	0.0295	0.0283	0.7184/0.9532	0.6819/0.9588
		30	30		1.9234	1.9370	1.9260	(1.6310,2.2157)	(1.6503,2.2098)
					0.0222	0.0202	0.0196	0.5847/0.9472	0.5595/0.9520
		40	40		1.9178	1.9285	1.9197	(1.6652,2.1704)	(1.6800,2.1656)
					0.0163	0.0150	0.0148	0.5052/0.9532	0.4856/0.9524
		50	50		1.9242	1.9327	1.9255	(1.6977,2.1508)	(1.7100,2.1464)
					0.0139	0.0132	0.0129	0.4531/0.9444	0.4364/0.9428

Notes: 1st row represents the average estimates, 95% confidence interval and 2nd row represents corresponding MSE or ERs, interval lengths and cps for the point and interval estimates, respectively.

TABLE 4. Estimates and confidence interval of Φ_{n_1,n_2}.

$\lambda_1 = 2$ and $\lambda_2 = 10$									
n_1	n_2	n	m	Φ_{n_1,n_2}	$\widehat{\Phi}^{MLE}_{n_1,n_2}$	$\widehat{\Phi}^{Lindley}_{n_1,n_2}$	$\widehat{\Phi}^{MCMC}_{n_1,n_2}$	ACI of Φ_{n_1,n_2}	$HPDCI$ of $\Phi_{n_1:n_2}$
5	5	10	10	0.9311	0.9382	0.9592	0.9553	(0.5748,1.3015)	(0.6567,1.2788)
					0.0356	0.0167	0.0160	0.7267/0.9396	0.6221/0.9888
		20	20		0.9357	0.9507	0.9448	(0.6789,1.1925)	(0.7167,1.1871)
					0.0165	0.0110	0.0106	0.5136/0.9484	0.4704/0.9776
		30	30		0.9345	0.9460	0.9406	(0.7250,1.1441)	(0.7489,1.1425)
					0.0112	0.0085	0.0082	0.4191/0.9456	0.3936/0.9696
		40	40		0.9325	0.9419	0.9373	(0.7513,1.1137)	(0.7688,1.1138)
					0.0083	0.0067	0.0065	0.3625/0.9580	0.3449/0.9732
		50	50		0.9303	0.9383	0.9343	(0.7686,1.0920)	(0.7821,1.0925)
					0.0069	0.0058	0.0057	0.3234/0.9424	0.3103/0.9624
10	10	10	10	1.1756	1.1804	1.2022	1.1943	(0.8023,1.5585)	(0.8643,1.5436)
					0.0370	0.0224	0.0212	0.7561/0.9452	0.6793/0.9800
		20	20		1.1808	1.1947	1.1867	(0.9131,1.4484)	(0.9413,1.4433)
					0.0186	0.0144	0.0138	0.5353/0.9428	0.5020/0.9644
		30	30		1.1759	1.1864	1.1801	(0.9579,1.3938)	(0.9760,1.3913)
					0.0122	0.0102	0.0099	0.4359/0.9516	0.4154/0.9660
		40	40		1.1780	1.1861	1.1809	(0.9891,1.3670)	(1.0027,1.3649)
					0.0097	0.0085	0.0083	0.3780/0.9440	0.3622/0.9544
		50	50		1.1780	1.1846	1.1801	(1.0888,1.3471)	(1.0196,1.3451)
					0.0075	0.0068	0.0066	0.3383/0.9480	0.3254/0.9508
15	10	10	10	1.3671	1.3729	1.3933	1.3840	(0.9907,1.7552)	(1.0413,1.7431)
					0.0385	0.0258	0.0245	0.7645/0.9404	0.7018/0.9708
		20	20		1.3710	1.3836	1.3755	(1.1007,1.6413)	(1.1228,1.6366)
					0.0193	0.0157	0.0152	0.5407/0.9484	0.5137/0.9648
		30	30		1.3740	1.3828	1.3762	(1.1530,1.5950)	(1.1668,1.5914)
					0.0121	0.0106	0.0103	0.4420/0.9576	0.4245/0.9644
		40	40		1.3669	1.3742	1.3689	(1.1764,1.5575)	(1.1875,1.5549)
					0.0096	0.0086	0.0085	0.3811/0.9488	0.3674/0.9520
		50	50		1.3707	1.3765	1.3721	(1.1998,1.5415)	(1.2086,1.5390)
					0.0078	0.0072	0.0070	0.3417/0.9480	0.3304/0.9492

Notes: 1st row represents the average estimates, 95% confidence interval and 2nd row represents corresponding MSE or ERs, interval lengths and cps for the point and interval estimates, respectively.

confidence intervals. Their coverage probabilities are close to the nominal level 95%.

5. Conclusions

In this paper, we have studied the mean remaining strength of the parallel systems in the stress-strength model. We obtain the conditional random variable for the remaining strength of the parallel system under the applied parallel stress system. The likelihood ratio ordering between two systems is established for two-component case. Currently, we do not prove it is true in number of components are greater than two. The proof of this general case can be considered as a future work. Moreover, the maximum likelihood and Bayes estimates of the mean remaining strength of the system is derived and compared.

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