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# On star-K-Menger spaces

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**Abstract**

A space $X$ is *star-K-Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$ there exists a sequence $(K_n : n \in N)$ of compact subsets of $X$ such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of $X$. In this paper, we investigate the relationship between star-K-Menger spaces and related spaces, and study topological properties of star-K-Menger spaces.

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**Keywords:** Selection principles, star-Menger, strongly star-Menger, star-K-Menger, starcompact, star Lindelöf, strongly starcompact, strongly star Lindelöf, star-L-Lindelöf

## 1. Introduction

By a space, we mean a topological space. We give definitions of terms which are used in this paper. Let $\mathbb{N}$ denote the set of positive integers. Let $X$ be a space and $\mathcal{U}$ a collection of subsets of $X$. For $A \subseteq X$, let $St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space $X$. Then the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \}$ is an element of $\mathcal{B}$. The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [3,8]).

Kočinac [4,5] introduced star selection hypothesis similar to the previous ones. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space $X$. Then:

(A) The symbol $S^*_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.

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(B) The symbol $SS^*_{comp}(\mathcal{A},\mathcal{B})$ ($SS^*_{fin}(\mathcal{A},\mathcal{B})$) denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in N)$ of compact (resp., finite) subsets of $X$ such that $\{St(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $\mathcal{O}$ denote the collection of all open covers of $X$.

**1. Definition.** ([4,5]) A space $X$ is said to be *star-Menger* if it satisfies the selection hypothesis $S^*_{fin}(\mathcal{O},\mathcal{O})$.

**2. Definition.** ([4,5]) A space $X$ is said to be *star-K-Menger* (*strongly star-Menger*) if it satisfies the selection hypothesis $SS^*_{comp}(\mathcal{O},\mathcal{O})$ (resp., $SS^*_{fin}(\mathcal{O},\mathcal{O})$ ).

**3. Definition.** ([1,7]) A space $X$ is said to be *starcompact* (*star-Lindelöf*) if for every open cover $\mathcal{U}$ of $X$ there exists a finite (resp., countable, respectively) $\mathcal{V} \subseteq \mathcal{U}$ such that $St(\cup\mathcal{V},\mathcal{U}) = X$.

**4. Definition.** ([1,6,9]) A space $X$ is said to be *K-starcompact* (*strongly starcompact, strongly star-Lindelöf, star-L-Lindelöf*) if for every open cover $\mathcal{U}$ of $X$ there exists a compact (resp., finite, countable, Lindelöf) subset $F$ of $X$ such that $St(F,\mathcal{U}) = X$.

From the definitions, it is clear that every K-starcompact space is star-K-Menger, every strongly star-Menger space is star-K-Menger and every star-K-Menger space is star-Menger. Since every $\sigma$-sompact subset is Lindelöf, thus every star-K-Menger space is star-L-Lindelöf. But the converses do not hold (see Examples 2.1, 2.2, 2.3 and 2.4 below).

Kočinac [4,5] studied the star-Menger and related spaces. In this paper, our purpose is to investigate the relationship between star-K-Menger spaces and related spaces, and study topological properties of star-K-Menger spaces.

Throughout this paper, let $\omega$ denote the first infinite cardinal, $\omega_1$ the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal $\kappa$, let $\kappa^+$ be the smallest cardinal greater than $\kappa$. For each pair of ordinals $\alpha$, $\beta$ with $\alpha < \beta$, we write $[\alpha,\beta) = \{\gamma : \alpha \le \gamma < \beta\}$, $(\alpha,\beta] = \{\gamma : \alpha < \gamma \le \beta\}$, $(\alpha,\beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha,\beta] = \{\gamma : \alpha \le \gamma \le \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [2].

## 2. Star-K-Menger spaces and related spaces

In this section, we give some examples showing that the relationship between star-K-Menger spaces and other related spaces.

**2.1. Example.** There exists a Tychonoff star-K-Menger space $X$ which is not K-star-compact.

*Proof.* Let $X = \omega$ be the countably infinite discrete space. Clearly, $X$ is not K-starcompact. Since $X$ is countable, the singleton sets can serve as the compact sets which witness that $X$ is star-K-Menger, which completes the proof. □

**2.2. Example.** There exists a Tychonoff star-K-Menger space which is not strongly star-Menger.

*Proof.* Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $aD = D \cup \{d^*\}$ be one-point comactification of $D$. Let

$$X = (aD \times [0,\mathfrak{c}^+)) \cup (D \times \{\mathfrak{c}^+\})$$

be the subspace of the product space $aD \times [0,\mathfrak{c}^+]$. Clearly, $X$ is a Tychonoff space.

First we show that $X$ is star-K-Menger; we only show that $X$ is K-starcompact, since every K-starcompact space is star-K-Menger. To this end, let $\mathcal{U}$ be an open cover of $X$. For each $\alpha < \mathfrak{c}$, there exists $U_\alpha \in \mathcal{U}$ such that $\langle d_\alpha, \mathfrak{c}^+\rangle \in U_\alpha$. For each $\alpha < \mathfrak{c}$, we can find $\beta_\alpha < \mathfrak{c}^+$ such that $\{d_\alpha\} \times (\beta_\alpha, \mathfrak{c}^+] \subseteq U_\alpha$. Let $\beta = \sup\{\beta_\alpha : \alpha < \mathfrak{c}\}$. Then $\beta < \mathfrak{c}^+$. Let $K_1 = aD \times \{\beta\}$. Then $K_1$ is compact and $U_\alpha \cap K_1 \neq \emptyset$ for each $\alpha < \mathfrak{c}$. Hence

$$D \times \{\mathfrak{c}^+\} \subseteq St(K_1, \mathcal{U}).$$

On the other hand, since $aD \times [0, \mathfrak{c}^+)$ is countably compact and consequently $aD \times [0, \mathfrak{c}^+)$ is strongly starcompact (see [1,6]), hence there exists a finite subset $K_2$ of $aD \times [0, \mathfrak{c}^+)$ such that

$$aD \times [0, \mathfrak{c}^+) \subseteq St(K_2, \mathcal{U}).$$

If we put $K = K_1 \cup K_2$. Then $K$ is a compact subset of $X$ such that $X = St(K, \mathcal{U})$, which shows that $X$ is K-starcompact.

Next we show that $X$ is not strongly star-Menger. We only show that $X$ is not strongly star-Lindelöf, since every strongly star-Menger space is strongly star-Lindelöf. Let us consider the open cover

$$\mathcal{U} = \{\{d_\alpha\} \times [0, \mathfrak{c}^+] : \alpha < \mathfrak{c}\} \cup \{aD \times [0, \mathfrak{c}^+)\}$$

of $X$. It remains to show that $St(F, \mathcal{U}) \neq X$ for any countable subset $F$ of $X$. To show this, let $F$ any countable subset of $X$. Then there exists $\alpha_0 < \mathfrak{c}$ such that $F \cap (\{d_{\alpha_0}\} \times [0, \mathfrak{c}^+]) = \emptyset$. Hence $\langle d_{\alpha_0}, \mathfrak{c}^+\rangle \notin St(F, \mathcal{U})$, since $\{d_{\alpha_0}\} \times [0, \mathfrak{c}^+]$ is the only element of $\mathcal{U}$ containing the point $\langle d_{\alpha_0}, \mathfrak{c}^+\rangle$, which shows that $X$ is not strongly star-Lindelöf.

□

**2.3. Example.** There exists a Tychonoff star-L-Lindelöf space which is not star-K-Menger..

*Proof.* Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $bD = D \cup \{d^*\}$, where $d^* \notin D$. We topologize $bD$ as follows: for each $\alpha < \mathfrak{c}$, $\{d_\alpha\}$ is isolated and a set $U$ containing $d^*$ is open if and only if $bD \setminus U$ is countable. Then $bD$ is Lindelöf and every compact subset of $bD$ is finite. Let

$$X = (bD \times [0, \omega]) \setminus \{\langle d^*, \omega\rangle\}$$

be the subspace of the product space $bD \times [0, \omega]$. Then $X$ is star-L-Lindelöf, since $bD \times \omega$ is a Lindelöf dense subset of $X$.

Next we show that $X$ is not star-K-Menger. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times [0, \omega]$. For each $n \in \omega$, let $V_n = bD \times \{n - 1\}$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{V_n : n \in \mathbb{N}\}.$$

Then $\mathcal{U}_n$ is an open cover of $X$. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$. It suffices to show that $\bigcup_{n\in\mathbb{N}} St(K_n, \mathcal{U}_n) \neq X$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $X$. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of $X$. For each $n \in \mathbb{N}$, since $K_n$ is compact and $\{\langle d_\alpha, \omega\rangle : \alpha < \mathfrak{c}\}$ is a discrete closed subset of $X$, the set $K_n \cap \{\langle d_\alpha, \omega\rangle : \alpha < \mathfrak{c}\}$ is finite. Then there exists $\alpha_n < \mathfrak{c}$ such that

$$K_n \cap \{\langle d_\alpha, \omega\rangle : \alpha > \alpha_n\} = \emptyset.$$

Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha' < \mathfrak{c}$ and

$$(\bigcup_{n\in\mathbb{N}} K_n) \cap \{\langle d_\alpha, \omega\rangle : \alpha > \alpha'\} = \emptyset.$$

For each $n \in \mathbb{N}$, since $K_n \cap V_m$ is finite for each $m \in \mathbb{N}$, there exists $\alpha_{nm} < \mathfrak{c}$ such that

$$K_n \cap \{\langle d_\alpha, n\rangle : \alpha > \alpha_{nm}\} = \emptyset.$$

Let $\alpha'_n = \sup\{\alpha_{nm} : m \in \mathbb{N}\}$. Then $\alpha'_n < \mathfrak{c}$ and

$$K_n \cap \{\langle d_\alpha, m\rangle : \alpha > \alpha'_n, m \in \mathbb{N}\} = \emptyset.$$

Let $\alpha'' = \sup\{\alpha'_n : n \in \mathbb{N}\}$. Then $\alpha' < \mathfrak{c}$ and

$$(\bigcup_{n\in\mathbb{N}} K_n) \cap \{\langle d_\alpha, m\rangle : \alpha > \alpha'', m \in \mathbb{N}\} = \emptyset.$$

If we pick $\beta > \max\{\alpha', \alpha''\}$. Then $U_\beta \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Hence $\langle d_\beta, \omega\rangle \notin St(K_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since $U_\beta$ is the only element of $\mathcal{U}_n$ containing the point $\langle d_\beta, \omega\rangle$ for each $n \in \mathbb{N}$, which shows that $X$ is not star-K-Menger.

□

**2.4. Example.** There exists a $T_1$ star-Menger space which is not star-K-Menger.

*Proof.* Let $X = [0, \omega_1) \cup D$, where $D = \{d_\alpha : \alpha < \omega_1\}$ is a set of cardinality $\omega_1$. We topologize $X$ as follows: $[0, \omega_1)$ has the usual order topology and is an open subspace of $X$; a basic neighborhood of a point $d_\alpha \in D$ takes the form

$$O_\beta(d_\alpha) = \{d_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.$$

Then $X$ is a $T_1$ space.

First we show that $X$ is star-Menger. We only show that $X$ is starcompact, since every starcompact space is star-Menger. To this end, let $\mathcal{U}$ be an open cover of $X$. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open subsets of $X$. Thus it is sufficient to show that there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\bigcup\mathcal{V}, \mathcal{U}) = X$. Since $[0, \omega_1)$ is countably compact, it is strongly starcompact (see [1,6]), then we can find a finite subset $\mathcal{V}_1$ of $\mathcal{U}$ such that $[0, \omega_1) \subseteq St\bigcup\mathcal{V}_1, \mathcal{U})$. On the other hand, if we pick $\alpha_0 < \omega_1$, then there exists $U_{\alpha_0} \in \mathcal{U}$ such that $d_{\alpha_0} \in U_{\alpha_0}$. For each $\alpha < \omega_1$, there is $U_\alpha \in \mathcal{U}$ such that $d_\alpha \in U_\alpha$. Hence we have $U_{\alpha_0} \cap U_\alpha \neq \emptyset$ by the construction of the topology of $X$. Therefore $D \subseteq St(U_{\alpha_0}, \mathcal{U})$. If we put $\mathcal{V} = \mathcal{V}_1 \cup \{U_{\alpha_0}\}$, then $\mathcal{V}$ is a finite subset of $\mathcal{U}$ and $X = St(\bigcup\mathcal{V}, \mathcal{U})$, which shows that $X$ is starcompact.

Next we show that $X$ is not star-K-Menger. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{O_\alpha(d_\alpha) : \alpha < \omega_1\} \cup \{[0, \omega_1)\}.$$

Then $\mathcal{U}_n$ is an open cover of $X$. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$. It suffices to show that $\bigcup_{n\in\mathbb{N}} St(K_n, \mathcal{U}_n) \neq X$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $X$. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of $X$. For each $n \in \mathbb{N}$, the set $K_n \cap \{d_\alpha : \alpha < \omega_1\}$ is finite, since $K_n$ is compact and $\{d_\alpha : \alpha < \omega_1\}$ is a discrete closed subset of $X$. Then there exists $\alpha_n < \omega_1$ such that

$$K_n \cap \{d_\alpha : \alpha > \alpha_n\} = \emptyset.$$

Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha' < \omega_1$ and

$$(\bigcup_{n\in\mathbb{N}} K_n) \cap \{d_\alpha : \alpha > \alpha'\} = \emptyset.$$

For each $n \in \mathbb{N}$, the set $K_n$ is compact and $[0, \omega_1)$is countably compact. Hence $K_n \cap [0, \omega_1)$ is bounded in $[0, \omega_1)$. Thus there exists $\alpha'_n < \omega_1$ such that

$$K_n \cap (\alpha'_n, \omega_1) = \emptyset.$$

Let $\alpha'' = \sup\{\alpha'_n : n \in \mathbb{N}\}$. Then $\alpha'' < \omega_1$ and

$$(\bigcup_{n\in\mathbb{N}} K_n) \cap (\alpha'', \omega_1) = \emptyset.$$

If we pick $\beta > \max\{\alpha', \alpha''\}$. Then $O_\beta(d_\beta) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Hence $d_\beta \notin St(K_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since $O_\beta(d_\beta)$ is the only element of $\mathcal{U}_n$ containing the point $d_\beta$ for each $n \in \mathbb{N}$, which shows that $X$ is not star-K-Menger. □

**2.5. Remark.** The author does not know if there exists a Hausdorff (or Tychonoff) star-Menger space which is not star-K-Menger.

## 3. Properties of star-K-menger spaces

In this section, we study topological properties of star-K-Menger spaces. The space $X$ of the proof of Example 2.2 shows that a closed subset of a Tychonoff star-K-Menger space $X$ need not be star-K-Menger, since $D \times \{\mathfrak{c}^+\}$ is a discrete closed subset of cardinality $\mathfrak{c}$. Now we give an example showing that a regular-closed subset of a Tychonoff star-K-Menger space $X$ need not be star-K-Menger. Here a subset $A$ of a space $X$ is said to be *regular-closed* in $X$ if $cl_X int_X A = A$.

**3.1. Example.** There exists a Tychonoff star-K-Menger space having a regular-closed subspace which is not star-K-Menger.

*Proof.* Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $aD = D \cup \{d^*\}$ be one-point comactification of $D$.

Let $S_1$ be the same space $X$ in the proof of Example 2.2. Then $S_1$ is a Tychonoff star-K-Menger space.

Let
$$S_2 = (aD \times [0, \mathfrak{c})) \cup (D \times \{\mathfrak{c}\})$$
be the subspace of the product space $aD \times [0, \mathfrak{c}]$. To show that $S_2$ is not star-K-Menger. For each $\alpha < \mathfrak{c}$, let
$$U_\alpha = \{d_\alpha\} \times (\alpha, \mathfrak{c}] \text{ and } V_\alpha = aD \times [0, \alpha).$$
For each $n \in \mathbb{N}$, let
$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{V_\alpha : \alpha < \mathfrak{c}\}.$$
Then $\mathcal{U}_n$ is an open cover of $S_2$. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $S_2$. It suffices to show that $\bigcup_{n\in\mathbb{N}} St(K_n, \mathcal{U}_n) \neq X$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $X$. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of $X$. For each $n \in \mathbb{N}$, since $K_n$ is compact and $\{\langle d_\alpha, \mathfrak{c}\rangle : \alpha < \mathfrak{c}\}$ is a discrete closed subset of $S_2$, the set $A_n = \{\alpha : \langle d_\alpha, \mathfrak{c}\rangle \in K_n\}$ is finite. Let
$$K'_n = K_n \setminus \bigcup\{U_\alpha : \alpha \in A_n\}.$$
If $K'_n = \emptyset$. Then there exists $\alpha'_n < \mathfrak{c}$ such that
$$K_n \cap U_\alpha = \emptyset \text{ for each } \alpha > \alpha'_n.$$
If $K'_n \neq \emptyset$. Since $K'_n$ is closed in $K_n$, $K'_n$ is compact and $K'_n \subseteq aD \times [0, \mathfrak{c})$. Then $\pi(K'_n)$ is a compact subset of a countable compact space $[0, \mathfrak{c})$, where $\pi : aD \times [0, \mathfrak{c}) \to [0, \mathfrak{c})$ is the projection. Hence $\pi(K'_n)$ is bounded in $[0, \mathfrak{c})$, Thus there exists $\beta_n < \mathfrak{c}$ such that $\pi(K'_n) \cap (\beta_n, \mathfrak{c}) = \emptyset$. Choose $\alpha''_n > \max\{\alpha : \alpha \in A_n\} \cup \{\beta_n\}$. Then
$$U_\alpha \cap K_n = \emptyset \text{ for each } \alpha > \alpha''_n.$$
Hence, for each $n \in \mathbb{N}$ either $K'_n = \emptyset$ or $K'_n \neq \emptyset$, there exists $\alpha_n < \mathfrak{c}$ such that
$$U_\alpha \cap K_n = \emptyset \text{ for each } \alpha > \alpha_n.$$
Let $\beta_0 = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta_0 < \mathfrak{c}$ and
$$U_\alpha \cap K_n = \emptyset \text{ for each } \alpha > \beta_0 \text{ and each } n \in \mathbb{N}.$$

If we pick $\alpha' > \beta_0$. Then

$$U_{\alpha'} \cap K_n = \emptyset \text{ for each } n \in \mathbb{N}.$$

Hence

$$\langle d_{\alpha'}, \mathfrak{c} \rangle \notin St(K_n, \mathcal{U}_n) \text{ for each } n \in \mathbb{N},$$

since $U_{\alpha'}$ is the only element of $\mathcal{U}_n$ containing the point $\langle d_{\alpha'}, \mathfrak{c} \rangle$ for each $n \in \mathbb{N}$, which shows that $S_2$ is not star-K-Menger.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\mathfrak{c}^+\} \to D \times \{\mathfrak{c}\}$ be a bijection and let $X$ be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \mathfrak{c}^+ \rangle$ of $S_1$ with $\pi(\langle d_\alpha, \mathfrak{c}^+ \rangle\})$ of $S_2$ for every $\alpha < \mathfrak{c}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. It is clear that $\varphi(S_2)$ is a regular-closed subspace of $X$ which is not star-K-Menger, since it is homeomorphic to $S_2$.

Finally we show that $X$ is star-K-Menger; we only show that $X$ is K-starcompact, since every K-starcompact space is star-K-Menger. To this end, let $\mathcal{U}$ be an open cover of $X$. Since $\varphi(S_1)$ is homeomorphic to $S_1$ and consequently $\varphi(S_1)$ is K-starcompact. Thus there exists a compact subset $K_1$ of $\varphi(S_1)$ such that

$$\varphi(S_1) \subseteq St(K_1, \mathcal{U}).$$

Since $\varphi(aD \times [0, \mathfrak{c}))$ is homeomorphic to $aD \times [0, \mathfrak{c})$, the set $\varphi(aD \times [0, \mathfrak{c}))$ is countably compact, hence it is strongly starcompact (see [1,6]). Thus we can find a finite subset $K_2$ of $\varphi(aD \times [0, \mathfrak{c}))$ such that

$$\varphi(aD \times [0, \mathfrak{c})) \subseteq St(K_2, \mathcal{U}).$$

If we put $K = K_1 \cup K_2$. Then $K$ is a compact subset of $X$ such that $X = St(K, \mathcal{U})$, which shows that $X$ is K-starcompact.

□

Since a continuous image of a K-starcompact space is K-starcompact, it is not difficult to show the following result.

**3.2. Theorem.** *A continuous image of a star-K-Menger space is star-K-Menger.*

Next we turn to consider preimages. To show that the preimage of a star-K-Menger space under a closed 2-to-1 continuous map need not be star-K-Menger, we use the the Alexandorff duplicate $A(X)$ of a space $X$. The underlying set $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $\langle x, 0 \rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 0 \rangle\})$, where $U$ is a neighborhood of $x$ in $X$.

**3.3. Example.** There exists a closed 2-to-1 continuous map $f : X \to Y$ such that $Y$ is a star-K-Menger space, but $X$ is not star-K-Menger.

*Proof.* Let $Y$ be the same space $X$ in the proof of Example 2.2. As we proved in Example 2.2 above, $Y$ is star-K-Menger. Let $X$ be the Alexandorff duplicate $A(Y)$. Then $X$ is not star-K-Menger. In fact, let $A = \{\langle \langle d_\alpha, \mathfrak{c}^+ \rangle, 1 \rangle : \alpha < \mathfrak{c}\}$. Then $A$ is an open and closed subset of $X$ with $|A| = \mathfrak{c}$, and each point $\langle \langle d_\alpha, \mathfrak{c}^+ \rangle, 1 \rangle$ is isolated. Hence $A(X)$ is not star-K-Menger, since every open and closed subset of a star-K-Menger space is star-K-Menger and $A$ is not star-K-Menger. Let $f : X \to Y$ be the projection. Then $f$ is a closed 2-to-1 continuous map, which completes the proof. □

Now, we give a positive result:

**3.4. Theorem.** *Let $f$ be an open perfect map from a space $X$ to a star-K-Menger space $Y$. Then $X$ is star-K-Menger.*

*Proof.* Since $f(X)$ is open and closed in $Y$, we may assume that $f(X) = Y$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X$ and let $y \in Y$. For each $n \in \mathbb{N}$, since $f^{-1}(y)$ is compact, there exists a finite subcollection $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n_y}$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_{n_y}$. Pick an open neighborhood $V_{n_y}$of $y$ in $Y$ such that $f^{-1}(V_{n_y}) \subseteq \bigcup\{U : U \in \mathcal{U}_{n_y}\}$, then we can assume that

$$(3.1) \qquad V_{n_y} \subseteq \bigcap\{f(U) : U \in \mathcal{U}_{n_y}\},$$

because $f$ is open. For each $n \in \mathbb{N}$, taking such open set $V_{n_y}$ for each $y \in Y$, we have an open cover $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$ of $Y$. Thus $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of $Y$, there exists a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $Y$ such that $(St(K_n, \mathcal{V}_n) : n \in \mathbb{N})$ is an open cover of $Y$, since $Y$ is star-K-Menger. Since $f$ is perfect, the sequence $(f^{-1}(K_n) : n \in N)$ is the sequence of compact subsets of $X$. To show that $\{St(f^{-1}(K_n), \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of $X$. Let $x \in X$. Then there exists a $n \in \mathbb{N}$ and $y \in Y$ such that $f(x) \in V_{n_y}$ and $V_{n_y} \cap K_n \neq \emptyset$. Since

$$x \in f^{-1}(V_{n_y}) \subseteq \bigcup\{U : U \in \mathcal{U}_{n_y}\},$$

we can choose $U \in \mathcal{U}_{n_y}$ with $x \in U$. Then $V_{n_y} \subseteq f(U)$ by (3.1), and hence $U \cap f^{-1}(K_n) \neq \emptyset$. Therefore $x \in St(f^{-1}(K_n), \mathcal{U}_n)$. Consequently, we have $\{St(f^{-1}(K_n), \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of $X$, which shows that $X$ is star-K-Menger. □

By Theorem 3.4 we have the following corollary.

**3.5. Corollary.** *Let $X$ be a star-K-Menger space and $Y$ a compact space. Then $X \times Y$ is star-K-Menger.*

However, the product of two star-K-Menger spaces need not be star-K-Menger. In fact, the following well-known example showing that the product of two countably compact (and hence star-K-Menger) spaces need not be star-K-Menger. Here we give a rough proof for the sake of completeness. For a Tychonoff space $X$, let $\beta X$ denote the Čech-Stone compactification of $X$.

**3.6. Example.** There exists two countably compact spaces $X$ and $Y$ such that $X \times Y$ is not star-K-Menger.

*Proof.* Let $D$ be a discrete space of cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega_1} E_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where $E_\alpha$ and $F_\alpha$ are the subsets of $\beta D$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;

(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\beta| \leq \mathfrak{c}$;

(3) every infinite subset of $E_\alpha$ (resp., $F_\alpha$) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

These sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta D$ has cardinality at least $2^{\mathfrak{c}}$ (see [7]). Then $X \times Y$ is not star-K-Menger, because the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with cardinality $\mathfrak{c}$ and the open and closed subsets of star-K-Menger spaces are star-K-Menger. □

In [1, Example 3.3.3], van Douwen-Reed-Roscoe-Tree gave an example showing that there exist a countably compact space $X$ and a Lindelöf space $Y$ such that$X \times Y$ is not strongly star-Lindelöf. Now, we shall show that the product space $X \times Y$ is not star-K-Menger.

**3.7. Example.** There exist a countably compact (and hence star-K-Menger) space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not star-K-Menger.

*Proof.* Let $X = [0, \omega_1)$ with the usual order topology and $Y = \omega_1 + 1$ with the following topology: each point $\alpha$ with $\alpha < \omega_1$ is isolated and a set $U$ containing $\omega_1$ is open if and only if $Y \setminus U$ is countable. Then $X$ is countably compact and $Y$ is Lindelöf. Now, we show that $X \times Y$ is not star-K-Menger. For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha] \times [\alpha, \omega_1] \text{ and } V_\alpha = (\alpha, \omega_1) \times \{\alpha\}.$$

For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}.$$

Then $\mathcal{U}_n$ is an open cover of $X \times Y$. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of the open covers of $X \times Y$. It suffices to show that $\bigcup_{n \in \mathbb{N}} St(K_n, \mathcal{U}_n) \neq X \times Y$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $X \times Y$. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of $X \times Y$. For each $n \in \mathbb{N}$, since $K_n$ is compact, then $\pi(K_n)$ is a compact subset of $X$, where $\pi : X \times Y \to X$ is the projection. Thus there exists $\alpha_n < \omega_1$ such that

$$K_n \cap ((\alpha_n, \omega_1) \times Y) = \emptyset.$$

Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$ and

$$(\bigcup_{n \in \mathbb{N}} K_n) \cap ((\beta, \omega_1) \times Y) = \emptyset.$$

If we pick $\alpha > \beta$. Then $\langle \alpha + 1, \alpha \rangle \notin St(K_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since $V_\alpha$ is the only element of $\mathcal{U}_n$ containing the point $\langle \alpha + 1, \alpha \rangle$ for each $n \in \mathbb{N}$, which shows that $X \times Y$ is not star-K-Menger. □

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