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Oscillation results of higher order nonlinear neutral delay difference equations with a nonlinear neutral term

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Abstract

In this work, we shall consider higher order nonlinear neutral delay difference equation of the type

$$\Delta^m \left[x_n + p_n x_{n-k}^{\alpha}\right] + q_n x_{n-l}^{\beta} = 0 \quad , n = 0, 1, 2, ...$$

where $\{p_n\}$ is a sequence of real numbers, $\{q_n\}$ is a sequence of nonnegative real numbers, k and l are positive integers and $\alpha, \beta \in (0, \infty)$ are quotient of odd positive integers. We obtain sufficient conditions for the oscillations of all solutions of this equation.

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1. Introduction

In the present work, we consider the following higher order nonlinear neutral delay difference equation:

$$\Delta^m \left[x_n + p_n x_{n-k}^{\alpha}\right] + q_n x_{n-l}^{\beta} = 0 \quad , n = 0, 1, 2, ... \tag{1.1}$$

where Δ is the usual forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, k, l are positive integers, $\{p_n\}$ is a sequence of real numbers, $\{q_n\}$ is a sequence of nonnegative real numbers and $\alpha \in (0, \infty)$ and $\beta \in (0, \infty)$ are ratio of odd positive integers.

Recently, there have been a lot of studies concerning the behaviour of the oscillatory difference and differential equations, see [1-14] and the reference cited therein. In [3], Agarwal et al. and in [4], Agarwal and Grace studied behaviour of the oscillatory higher order nonlinear neutral difference equations with different form from equation (1.1). Later

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in [9], Tang and Liu investigated the oscillatory behaviour of the first order nonlinear delay difference equation of the form

$$\Delta x_n + q_n x_{n-l}^{\beta} = 0,$$

where $\{q_n\}$ is a sequence of nonnegative numbers, l is a positive integer and $\beta \in (0, \infty)$ is a quotient of odd positive integers. Later in [10], Thandapani et al. considered the neutral delay difference equation

$$\Delta \left[x_n + p_n x_{n-k}\right] + \delta q_n x_{n-l}^{\beta} = 0,$$

where $\delta = \pm 1$ and β is a ratio of odd positive integers and also $\{p_n\}, \{q_n\}$ are positive real sequences. Particularly, in [11], oscillation results were given for

$$\Delta^m \left[x_n + p_n x_{n-k}\right] + q_n x_{n-l}^{\beta} = 0,$$

where $\{p_n\}$ is a sequence of real numbers, $\{q_n\}$ is a sequence of nonnegative real numbers, k and l are positive integers and $\beta \in (0, \infty)$ is a quotient of odd positive integers. Note that (1.1) includes this equation with $\alpha = 1$.

In [5], X. Lin considered (1.1) with $m = 1$ which has the form

$$\Delta \left[x_n - p_n x_{n-k}^{\alpha}\right] + q_n x_{n-l}^{\beta} = 0 \quad , n = 0, 1, 2, ...$$

where $\alpha, \beta > 0$ and $\{p_n\}, \{q_n\}$ are sequences of nonnegative real numbers and k and l are positive integers. Our conditions are more relaxed when the equation is of first order.

Let $\rho = \max\{k, l\}$. By a solution of (1.1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq -\rho$ and satisfies equation (1.1) for $n \geq 0$. A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative.

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of (1.1).

In the sequel, we shall need the following conditions:

(C_1) $\liminf_{n\to\infty} q_n > 0$;

(C_2) $0 \leq p_n < 1$;

(C_3) $-1 < -P \leq p_n \leq 0$, where $P > 0$ is a constant;

We need the following result proved in [13] for our subsequent discussion.

1.1. Lemma. *Assume that for large n,*

$$(p_n, p_{n+1}, ..., p_{n+k-1}) \neq 0.$$

Then

$$\Delta x_n + p_n x_{n-k}^{\alpha} = 0 \quad , n = 0, 1, 2, ...$$

has an eventually positive solution if and only if the corresponding inequality

$$\Delta x_n + p_n x_{n-k}^{\alpha} \leq 0 \quad , n = 0, 1, 2, ...$$

has an eventually positive solution.

Furthermore, we need following lemmas proved in [1].

1.2. Lemma. *(Discrete Kneser's Theorem) Let z_n be defined for $n \geq a$, and $z_n > 0$ with $\Delta^m z_n$ of constant sign for $n \geq a$ and not identically zero. Then, there exists an integer j, $0 \leq j \leq m$ with $(m+j)$ odd for $\Delta^m z_n \leq 0$, and $(m+j)$ even for $\Delta^m z_n \geq 0$, such that*

$$j \leq m-1 \text{ implies } (-1)^{j+i} \Delta^i z_n > 0, \quad \text{for all } n \geq a, \ \ j \leq i \leq m-1,$$

and

$$j \geq 1 \text{ implies } \Delta^i z_n > 0, \quad \text{for all large } n \geq a, 1 \leq i \leq j-1.$$

1.3. Lemma. *Let z_n be defined for $n \geq a$, and $z_n > 0$ with $\Delta^m z_n \leq 0$ for $n \geq a$ and not identically zero. Then, there exists a large $n_1 \geq a$ such that*

$$z_n \geq \frac{(n-n_1)^{m-1}}{(m-1)!} \Delta^{m-1} z_{2^{m-j-1}n} \quad , n \geq n_1,$$

where j is defined in Lemma 1.2. Further, if z_n is increasing, then

$$z_n \geq \frac{1}{(m-1)!} \left(\frac{n}{2^{m-1}}\right)^{m-1} \Delta^{m-1} z_n \quad , n \geq 2^{m-1} n_1.$$

2. Sufficient Conditions For Oscillations Of Equation (1.1)

2.1. Theorem. *Assume $1 \leq \alpha < \infty$, (C_1) and (C_2) hold.*
(a) Let m be even. If the difference equation

$$\Delta w_n + q_n \left(\frac{1-p_{n-l}}{(m-1)!}\right)^{\beta} \left(\frac{n-l}{2^{m-1}}\right)^{(m-1)\beta} w_{n-l}^{\beta} = 0 \tag{2.1}$$

is oscillatory, then all solutions of (1.1) are oscillatory.
(b) Let m be odd. Then, every solution of (1.1) either oscillates or tends to zero as $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1.1), with $x_{n-\rho} > 0$ for all $n \geq n_0$. Setting $z_n = x_n + p_n x_{n-k}^{\alpha}$, we get $z_n \geq x_n > 0$ and

$$\Delta^m z_n = -q_n x_{n-l}^{\beta} < 0 \quad , n \geq n_0. \tag{2.2}$$

It is clear from *Lemma 1.2* that $\Delta^i z_n$ is eventually strictly monotonic for $i = 1, 2, 3, ..., m-1$ and it is also of constant sign. By *Lemma 1.2* that for $m \geq 2$

$$\Delta^{m-1} z_n > 0 \quad , n \geq n_0 \tag{2.3}$$

Now, we claim that $\lim_{n\to\infty} x_n = 0$. Summing (2.2) from $n_1 \geq n_0 + l$ to ∞

$$\sum_{s=n_4}^{\infty} \Delta^m z_s = -\sum_{s=n_4}^{\infty} q_s x_{s-l}^{\beta}$$

or

$$0 < \Delta^{m-1} z_{n_4} - L = \sum_{s=n_4}^{\infty} q_s x_{s-l}^{\beta}$$

where $0 \leq L := \lim_{s\to\infty} \Delta^{m-1} z_s < \infty$. Since $\sum_{s=n_4}^{\infty} q_s x_{s-l}^{\beta} < \infty$, we have $\lim_{s\to\infty} q_s x_{s-l}^{\beta} = 0$, in the view of (C_1), we see that $\lim_{s\to\infty} x_s = 0$ holds. Therefore, there is a $n \geq n_1$ such that

$$0 \leq x_n^{\alpha} \leq x_n \quad , n \geq n_1$$

or

$$0 \leq x_n^{\alpha-1} \leq 1 \quad , n \geq n_1. \tag{2.4}$$

Now, we claim that $\Delta z_n \leq 0$ eventually. This is obvious from equation (1.1) in the case $m = 1$. For $m \geq 2$, we suppose on the contrary, that $\Delta z_n > 0$ for $n \geq n_1$. Then, considering (2.4),

$$\begin{aligned} (1-p_n)z_n &\leq z_n - p_n z_{n-k} \\ &= x_n + p_n x_{n-k}^{\alpha} - p_n x_{n-k} - p_n p_{n-k} x_{n-2k}^{\alpha} \\ &= x_n + p_n x_{n-k}\left(x_{n-k}^{\alpha-1} - 1\right) - p_n p_{n-k} x_{n-2k}^{\alpha} \\ &\leq x_n. \end{aligned} \tag{2.5}$$

for $n \geq n_2 \geq n_1 + 2k$. Since z_n is positive and increasing, it follows from *Lemma 1.3* and (2.5)

$$x_n \geq (1-p_n)z_n \geq \frac{(1-p_n)}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{(m-1)} \Delta^{m-1} z_n \quad , n \geq 2^{m-1} n_2 \tag{2.6}$$

Using (2.6), we find

$$q_n \left(x_{n-l}\right)^{\beta} \geq q_n \left(\frac{(1-p_{n-l})}{(m-1)!}\left(\frac{n-l}{2^{m-1}}\right)^{(m-1)} \Delta^{m-1} z_{n-l}\right)^{\beta} \quad , n \geq n_3 \geq 2^{m-1} n_2 + l$$

and so from (2.2)

$$\Delta^m z_n \leq -q_n \left(\frac{1-p_{n-l}}{(m-1)!}\right)^{\beta} \left(\frac{n-l}{2^{m-1}}\right)^{(m-1)\beta} \left(\Delta^{m-1} z_{n-l}\right)^{\beta} \quad , n \geq n_3$$

or

$$\Delta^m z_n + q_n \left(\frac{1-p_{n-l}}{(m-1)!}\right)^{\beta} \left(\frac{n-l}{2^{m-1}}\right)^{(m-1)\beta} \left(\Delta^{m-1} z_{n-l}\right)^{\beta} \leq 0.$$

Thus, we see that $\left\{\Delta^{m-1} z_n\right\}$ is an eventually positive (see (2.3)) solution of

$$\Delta w_n + q_n \left(\frac{1-p_{n-l}}{(m-1)!}\right)^{\beta} \left(\frac{n-l}{2^{m-1}}\right)^{(m-1)\beta} w_{n-l}^{\beta} \leq 0. \tag{2.7}$$

Therefore by *Lemma 1.1*, (2.1) has eventually positive solution. This is a contradiction with the oscillatory of equation (2.1) under the assumption of Theorem 2.1. Hence, $\Delta z_n \leq 0$ eventually. Since $\Delta z_n \leq 0$ eventually, in *Lemma 1.2*, we must have $j = 0$ and

$$(-1)^i \Delta^i z_n > 0 \quad , 0 \leq i \leq m-1, n \geq n_1. \tag{2.8}$$

If m is even, (2.8) yields a contradiction to (2.3). This proves part (a) of the theorem. Now, let m be odd. Assume further that x_n is a non-oscillating solution which does not tend to zero as $n \to \infty$. As in the preceding case, (2.3) holds. Therefore, summing (2.2) from $n_1 \geq n_0 + l$ to ∞, we get

$$0 < \Delta^{m-1} z_{n_4} = \sum_{k=n_4}^{\infty} q_n x_{n-l}^{\beta} < \infty$$

which implies $\lim_{n\to\infty} x_n = 0$ in the view of (C_1). This contradiction completes the proof of part (b). □

2.2. Theorem. *Assume (C_1) and (C_3) hold. Then every solution of (1.1) either oscillates or tends to zero as $n \to \infty$.*

Proof. Let $\{x_n\}$ be a non-oscillatory solution of (1.1) which is not limiting to zero, with $x_{n-\rho} > 0$, for all $n \geq n_0$. Setting $z_n = x_n + p_n x_{n-k}^{\alpha}$, we get $z_n \leq x_n$, and also inequality (2.2) for $n \geq n_1$ where $n_1 \geq n_0$. Then, by *Lemma 1.2*, we have (2.3) for $n \geq n_1$. Summing (2.2), from $n_2 \geq n_1 + l$ to ∞, we get

$$0 < \Delta^{m-1} z_{n_4} - L = \sum_{s=n_4}^{\infty} q_s x_{s-l}^{\beta}$$

where $0 \leq L := \lim_{s\to\infty} \Delta^{m-1} z_s < \infty$. Since $\sum_{s=n_4}^{\infty} q_s x_{s-l}^{\beta} < \infty$, we have $\lim_{s\to\infty} q_s x_{s-l}^{\beta} = 0$, in the view of (C_1), we see that $\lim_{s\to\infty} x_s = 0$ holds. This contradiction completes the proof. □

3. Applications on (1.1)

In this section, we give general examples on the equation (1.1).

3.1. Example. Let m be an even positive integer, l be an odd positive integer, $\alpha \in [1, \infty)$ be a quotient of odd positive integer and $\beta \in (0, 1)$ be a quotient of odd positive integer, furthermore $0 < p < 1$ and $q > 0$. Consider

$$\Delta^m \left[x_n + p x_{n-k}^{\alpha}\right] + q x_{n-l}^{\beta} = 0. \tag{3.1}$$

Since every condition of *Theorem 2.1(a)* is satisfied. So, we associate (3.1) with

$$\Delta w_n + q \left(\frac{1-p}{(m-1)!} \right)^{\beta} \left(\frac{n-l}{2^{m-1}} \right)^{\beta(m-1)} w_{n-l}^{\beta} = 0. \tag{3.2}$$

Therefore,

$$\sum_{n=0}^{\infty} q \left(\frac{1-p}{(m-1)!} \right)^{\beta} \left(\frac{n-l}{2^{m-1}} \right)^{\beta(m-1)} = \infty,$$

every solution of (3.2) is oscillatory from *Theorem 1.2* in [6] and so is (3.1). Also,

$$x_n := \left(\frac{2^m}{q} \right)^{\frac{1}{\beta-1}} \cos(n\pi)$$

is such kind of a solution of (3.1).

3.2. Example. Let m be an odd positive integer and l be an even positive integer, $\alpha \in [1, \infty)$ be a quotient of odd positive integer and β be a quotient of odd positive integer, furthermore $p \in (-1, 1) - \{0\}$ and $q, k > 0$. Consider

$$\Delta^m \left[x_n + p x_{n-k}^{\alpha}\right] + q x_{n-l}^{\beta} = 0. \tag{3.3}$$

Since every condition of *Theorem 2.1(b)* is satisfied when $0 < p < 1$ and the case where $-1 < p < 0$ hold conditions of *Theorem 2.2*, every solution of (3.3) is oscillatory or tending to zero. Direct substitution of

$$x_n := \left(\frac{2^m}{q} \right)^{\frac{1}{\beta-1}} \sin\left(n\pi + \frac{\pi}{2}\right)$$

into (3.3) shows that x_n is an oscillatory solution of (3.3).

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