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Applications of soft union sets to h-semisimple and h-$quasi$-hemiregular hemirings

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Abstract

By means of soft intersection-union sum and product, we make a new approach to the hemiring theory via soft set theory with the concepts of soft union (h-ideals, h-*bi*-ideals, h-*quasi*-ideals and h-interior ideals). Also, we investigate some characteristics of h-semisimple and h-*quasi*-hemiregular hemirings using these kinds of soft union h-ideals.

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1. Introduction

The traditional classical models often fail to overcome the complexities arising in the modeling of uncertain data in many fields like economics, engineering, environmental science, sociology, medical science etc. Molodtsov [24] proposed the concept of soft set theory which is a completely new mathematical approach for modeling vagueness and uncertainty. At present works on the soft set theory are progressing very rapidly. Maji [22] presented some definitions on soft sets. Further, Ali and Sezgin et al. [4–6, 26] introduced some new operations on soft sets and obtained some important properties. Simultaneously, this theory is very much useful in some different research areas such as information sciences with intelligent systems, approximate reasoning, expert and decision support systems and decision making etc., for examples, see [8–10, 12, 23, 29, 34].

Recently, the algebraic structures of soft sets dealing with uncertainties have been studied by many authors. Feng [11] introduced the concepts of soft semirings and idealistic soft semirings, and investigated some characteristics of them. Jun [14, 15] applied soft set theory to BCK/BCI-algebras. Aktas [2] discussed some important properties

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of soft groups. Based on soft sets many algebraic structures such as soft rings [1], soft ordered semigroups [16], soft BCH-algebras [18] and soft int-groups [7] etc. have been introduced.

We note that the ideals of semirings play a crucial role in the structure theory, ideals of semirings do not in general coincide with the ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. By a hemiring, we mean a special semiring with a zero and a commutative addition. The properties of h-ideals of hemirings were thoroughly investigated by Torre [28]. Torre established some analogous ring theorems for hemirings using h-ideals. In particular, Jun [17] discussed some properties of hemirings. Some characteristics of h-hemiregular hemirings have been investigated by Zhan in [33]. Further, some properties of h-semisimple and h-*intra*-hemiregular hemirings have been established by Yin [30,31]. It is pointed out some generalized fuzzy h-ideals of hemirings were investigated by Allen, Ma et al, for examples, see [3, 13, 19–21].

Recently, Sezgin and Çağman applied soft union set theory to near-rings and rings [25, 27]. By means of this kind of new idea, Zhan [32] applied soft union set theory to hemirings and investigated some properties of SU-hemirings and SU-h-ideals. As a continuation of this paper, we organize the present paper as follows. In Section 2, we first highlight some basic concepts and results of hemirings and soft sets. Then in Section 3, we introduce the concept of SU-h-interior ideals of hemirings and its properties. Finally, we discuss some features of h-*quasi*-hemiregular hemirings and h-semisimple hemirings by means of SU-h-ideals, SU-h-*bi*-ideals, SU-h-*quasi*-ideals and SU-h-interior ideals in the Sections 4 and 5 respectively.

2. Preliminaries

A semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a \cdot (b + c) = a \cdot b + b \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c$$

are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring $(S, +, \cdot)$ with zero is called a hemiring if $(S, +)$ is commutative.

A subhemiring of a hemiring S is a subset A of S closed under addition and multiplication. A subset A of S is called a left(right) ideal of S if A is closed under addition and $SA \subseteq A(AS \subseteq A)$. A subset A is called an ideal if it is both a left ideal and a right ideal. A subset B of S is called a *bi*-ideal of S if B is closed under addition and multiplication such that $BSB \subseteq B$. A subset Q of S is called a *Quasi*-ideal of S if Q is closed under addition and $SQ \cap QS \subseteq Q$. A subset A of S is called an interior ideal of S if A is closed under addition and multiplication such that $SAS \subseteq A$.

A subhemiring (left ideal, right ideal, ideal, *bi*-ideal, interior ideal) of S is called an h-subhemiring (left h-ideal, right h-ideal, h-ideal, h-*bi*-ideal, h-interior ideal) of S, respectively, if for any $x, z \in S$, and $a, b \in A$, $x + a + z = b + z$ it follows $x \in A$.

The h-closure $\overline{A}$ of a subset A of S is defined as

$$\overline{A} = \{x \in S | x + a + z = b + z \text{ for some } a, b \in A,\ z \in S\}.$$

A *quasi*-ideal Q of S is called an h-*quasi*-ideal of S if $\overline{SQ} \cap \overline{QS} \subseteq Q$ and for any $x, z \in S$ and $a, b \in Q$ from $x + a + z = b + z$, it follow $x \in Q$.

From now we denote S as a hemiring, U as an initial universe, E as a set of parameters, $P(U)$ as the power set of U and $A, B, C \subseteq E$.

2.1. Definition. *[24] A soft set f_A over U is defined as $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}$.*

It is clear to see that a soft set is a parameterized family of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

2.2. Definition. *[8] (i) Let $f_A, f_B \in S(U)$. Then, f_A is called a soft subset of f_B, denoted by $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

(ii) Let $f_A, f_B \in S(U)$. Union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A \tilde{\cup} B}$, where $f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$

(iii) Let $f_A, f_B \in S(U)$. Intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A \tilde{\cap} B}$, where $f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

(iv) Let $f_A \in S(U)$ and $\alpha \subseteq U$. Then, lower α-inclusion of f_A, denoted by $L(f_A; \alpha)$, is defined as $L(f_A; \alpha) = \{x \in A | f_A(x) \subseteq \alpha\}$.

2.3. Definition. *[25] Let $A \subseteq S$. The soft characteristic function of the complement of A denoted by $\mathcal{S}_{A^C}$ and is defined as*

$$\mathcal{S}_{A^C}(x) = \begin{cases} \emptyset & \text{if } x \in A, \\ U & \text{if } x \in S \backslash A. \end{cases}$$

2.4. Definition. *[32] Let $f_S, g_S \in S(U)$. Then*
(1) Soft intersection-union sum $f_S \oplus g_S$ is defined by

$$(f_S \oplus g_S)(x) = \bigcap_{x+a_1+b_1+z=a_2+b_2+z} (f_S(a_1) \cup f_S(a_2) \cup g_S(b_1) \cup g_S(b_2))$$

and $(f_S \oplus g_S)(x) = U$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$.
(2) Soft intersection-union product $f_S \diamondsuit g_S$ is defined by

$$(f_S \diamondsuit g_S)(x) = \bigcap_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cup f_S(a'_j) \cup g_S(b_i) \cup g_S(b'_j))$$

for all $i = 1, 2, \cdots, n$,
and $(f_S \diamondsuit g_S)(x) = U$ if x cannot be expressed as $x + \sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z$.

It is easy to see that if $f_S(x) = \emptyset$ for all $x \in S$, then f_S is an SU-hemiring(SU-$left(right)$ h-ideal, SU-h-ideal, SU-h-bi-ideal, SU-h-$quasi$-ideal) of S over U. We denote such a kind of SU-hemiring(SU-$left(right)$ h-ideal) by $\tilde{\theta}$.

2.5. Definition. *[32] (i) A soft set f_S over U is called a soft union hemiring(briefly, SU-hemiring) of S over U if it satisfies: (SU_1) $f_S(x+y) \subseteq f_S(x+y)$ for all $x, y \in S$;*
(SU_2) $f_S(xy) \subseteq f_S(x) \cup f_S(y)$ for all $x, y \in S$;
(SU_3) $f_S(x) \subseteq f_S(a) \cup f_S(b)$ with $x + a + z = b + z$ for all $x, a, b, z \in S$.
(ii) A soft set f_S over U is called a soft union $left(right)$ h-ideal(briefly, SI-$left(right)$ hideal) of S over U if it satisfies (SU_1), (SU_3) and
(SU_4) $f_S(xy) \subseteq f_S(y)$ $(f_S(xy) \subseteq f_S(x))$ for all $x, y \in S$.
(iii) A soft set f_S over U is called a soft union bi-ideal(briefly, SU-h-bi-ideal) of S over U if it satisfies (SU_1), (SU_2), (SU_3) and
(SU_5) $f_S(xyz) \subseteq f_S(x) \cup f_S(z)$ for all $x, y \in S$.
(iv) A soft set f_S over U is called a soft union h-quasi-ideal(briefly, SU-h-quasi-ideal) of S over U if it satisfies (SU_1), (SU_3) and
(SU_6) $(f_S \diamondsuit \tilde{\theta}) \tilde{\cup} (\tilde{\theta} \diamondsuit f_S) \tilde{\supseteq} f_S$.

2.6. Definition. *[32] Let $A \subseteq S$. Then A is an h-subhemiring(left h-ideal, right h-ideal, h-ideal, h-bi-ideal, h-quasi-ideal) of $\mathcal{S}$ if and only if $\mathcal{S}_{A^C}$ is an SU-hemiring(SU-left h-ideal, SU-right h-ideal, SU-h-ideal, SU-h-bi-ideal, SU-h-quasi-ideal) of S over U.*

2.7. Theorem. [32] Let $f_S \in S(U)$. Then, we have
(i) f_S is an SU-hemiring of S over U if and only if it satisfies (SU_3) and
(SU_7) $f_S \oplus f_S \tilde{\supseteq} f_S$.
(SU_8) $f_S \diamondsuit f_S \tilde{\supseteq} f_S$.
(ii) f_S is an SU-$left(right)$ h-ideal of S over U if and only if it satisfies (SU_3), (SU_7) and
(SU_9) $\tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$ $(f_S \diamondsuit \tilde{\theta} \tilde{\supseteq} f_S)$.
(iii) f_S is an SU-h-bi-ideal of S over U if and only if it satisfies (SU_3), (SU_7), (SU_8) and
(SU_{10}) $f_S \diamondsuit \tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$.
(iv) f_S is an SU-h-$quasi$-ideal of S over U if and only if it satisfies (SU_3), (SU_7) and (SU_6).

3. SU-h-interior ideals

In this section, we introduce the concept of soft union h-interior ideals of hemirings and investigate some related properties.

3.1. Definition. *Let $f_S \in S(U)$. Then f_S is called a soft union h-interior ideal(briefly, SU-h-interior ideal) of S over U if it satisfies (SU_1), (SU_2), (SU_3) and*
(SU_{11}) *$f_S(xyz) \subseteq f_S(y)$ for all $x, y, z \in S$.*

3.2. Example. *Assume that $U = D_2 = \{(x, y)|x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$, Dihedral group, as the universal set. Let $S = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be the hemiring of non-negative integers module 6 as the set of parameters. Define a soft set f_S over U by $f_S(0) = \{x\}, f_S(1) = f_S(5) = \{e, x, y\}$, $f_S(2) = f_S(4) = \{e, y\}, f_S(3) = \{e, x, yx\}$.*
Then one can easily check that f_S is an SU-h-interior ideal of S over U.

3.3. Example. *Assume that $U = S_3$ is the symmetric group. Let $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the hemiring of non-negative integers module 4 as the set of parameters.*
Define a soft set f_S over U by $f_S(0) = \{(1), (12), (13)\}, f_S(1) = f_S(3) = \{(1)\}$ and $f_S(2) = \{(1), (12)\}$.
Then f_S is not an SU-h-interior ideal of S over U.

3.4. Theorem. A soft set f_S over U is an SU-h-interior ideal of S over U if and only if it satisfies (SU_3), (SU_7), (SU_8) and
(SU_{12}) $\tilde{\theta} \diamondsuit f_S \diamondsuit \tilde{\theta} \tilde{\supseteq} f_S$.

Proof. By Theorem 2.7, we only show that (SU_{11}) is equivalent to (SU_{12}).

Let $x \in S$. If $(\tilde{\theta} \diamondsuit f_S \diamondsuit \tilde{\theta})(x) = U$, then it is clear that $(\tilde{\theta} \diamondsuit f_S \diamondsuit \tilde{\theta})(x) \supseteq f_S(x)$, that is, $\tilde{\theta} \diamondsuit f_S \diamondsuit \tilde{\theta} \tilde{\supseteq} f_S$.
Otherwise, we have

$$
\begin{aligned}
&(\tilde{\theta}\diamondsuit f_S\diamondsuit\tilde{\theta})(x)\\
=&\bigcap_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z}((\tilde{\theta}\diamondsuit f_S)(a_i)\cup(\tilde{\theta}\diamondsuit f_S)(a'_j)\cup f_S(b_i)\cup f_S(b'_j))\\
=&\bigcap_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z}(\bigcap_{a_i+\sum_{k=1}^{m_i}a_{ik}b_{ik}+z_1=\sum_{l=1}^{j}a'_{jl}b'_{jl}+z_1}(\tilde{\theta}(a_{ik})\cup\tilde{\theta}(a'_{jl})\cup f_S(b_{ik})\cup f_S(b'_{jl}))\\
\cup&\bigcap_{a'_j+\sum_{k=1}^{m_i}a_{ik}b_{ik}+z_2=\sum_{l=1}^{n_j}a'_{jl}b'_{jl}+z_2}(\tilde{\theta}(a_{ik})\cup\tilde{\theta}(a'_{jl})\cup f_S(b_{ik})\cup f_S(b'_{jl}))\cup f_S(b_i)\cup f_S(b'_j))\\
=&\bigcap_{x+\sum_{i=1}^{m'}a_ib_ic_i+z'=\sum_{j=1}^{n'}a'_jb'_jc'_j+z'}(f_S(\sum_{i=1}^{m'}a_ib_ic_i)\cup f_S(\sum_{j=1}^{n'}a'_jb'_jc'_j))\\
\subseteq& f_S(x),
\end{aligned}
$$

which implies, $\tilde{\theta}\diamondsuit f_S\diamondsuit\tilde{\theta}\tilde{\subseteq}f_S$.

Conversely, assume that the condition (SU_{12}) holds. For any $x,y,z\in S$, we have

$$
\begin{aligned}
&f_S(xyz)\subseteq(\tilde{\theta}\diamondsuit f_S\diamondsuit\tilde{\theta})(xyz)\\
=&\bigcap_{xyz+\sum_{i=1}^{m}a_ib_i+z'=\sum_{j=1}^{n}a'_jb'_j+z}((\tilde{\theta}\diamondsuit f_S)(a_i)\cup(\tilde{\theta}\diamondsuit f_S)(a'_j)\cup\tilde{\theta}(b_i)\cup\tilde{\theta}(b'_j))\\
\subseteq&(\tilde{\theta}\diamondsuit f_S)(0)\cup(\tilde{\theta}\diamondsuit f_S)(xy)\cup\tilde{\theta}(0)\cup\tilde{\theta}(z)\\
\subseteq&(\tilde{\theta}\diamondsuit f_S)(xy)\\
=&\bigcap_{xy+\sum_{j=1}^{m}a_ib_i+z'=\sum_{j=1}^{n}a'_jb'_j+z'}(\tilde{\theta}(a_i)\cup\tilde{\theta}(a'_j)\cup f_S(b_j)\cup f_S(b'_j))\\
\subseteq&\tilde{\theta}(0)\cup\tilde{\theta}(x)\cup f_S(0)\cup f_S(y)\\
\subseteq&f_S(y).
\end{aligned}
$$

This implies that (SU_{11}) holds. □

3.5. Lemma. Let $f_S\in S(U)$. If f_S is an SU-h-ideal of S over U, then f_S is an SU-h-interior ideal of S over U.

Proof. We only need to prove the conditions (SU_2) and (SU_{11}) hold.
For any $x,y\in S$, we have $f_S(xy)\subseteq f_S(y)$ and $f_S(xy)\subseteq f_S(x)$ science S is an SU-h-ideal of S, and so $f_S(xy)\subseteq f_S(x)\cup f_S(y)$. Thus, (SU_2) holds.

For any $x,y,z\in S$, we have $f_S(xyz)=f_S((xy)z)\subseteq f_S(xy)\subseteq f_S(y)$. Thus, (SU_{11}) holds.

Hence, f_S is an SU-h-interior ideal of S over U. □

The following proposition is obvious.

3.6. Proposition. A non-empty subset A of S is an h-interior ideal of S if and only if the soft set f_S defined by

$$
f_S(x)=\begin{cases}\alpha & \text{if } x\in S\backslash A,\\ \beta & \text{if } x\in A,\end{cases}
$$

is an SU-h-interior ideal of S, where $\alpha,\beta\in U$ such that $\alpha\supseteq\beta$.

3.7. Corollary. Let $A\subseteq S$. Then A is an h-interior ideal of S if and only if $\mathcal{S}_{A^C}$ is an SU-h-interior ideal of S over U.

3.8. Theorem. (i) Let f_S be a soft set over U and $\alpha\subseteq U$ such that $\alpha\in I_m(f_S)$. If f_S is an SU-h-interior ideal of S over U, then $L(f_S;\alpha)$ is an an h-interior ideal of S.
(ii) Let f_S be a soft set over U, $L(f_S;\alpha)$ a lower h-interior ideal of f_S for each $\alpha\subseteq U$ and $I_m(f_S)$ an ordered set by inclusion. Then f_S is an SU-h-interior ideal of S over U.

Proof. (i) Since $f_S(x) = \alpha$ for some $x \in S$, then $\emptyset \neq L(f_S; \alpha) \subseteq S$. Let $x, y \in L(f_S; \alpha)$, then $f_S(x) \subseteq \alpha$ and $f_S(y) \subseteq \alpha$. Then $f_S(x+y) \subseteq f_S(x) \cup f_S(y) \subseteq \alpha \cup \alpha = \alpha$, $f_S(xy) \subseteq f_S(x) \cup f_S(y) \subseteq \alpha \cup \alpha = \alpha$, which implies, $x+y, xy \in L(f_S; \alpha)$.

Similarly, we can show that $xyz \in L(f_S; \alpha)$ for all $y \in L(f_S; \alpha)$.

Now, let $x, z \in S$ and $a, b \in L(f_S; \alpha)$ with $x + a + z = b + z$. Then $f_S(a) \subseteq \alpha$ and $f_S(b) \subseteq \alpha$. Thus $f_S(x) \subseteq f_S(a) \cup f_S(b) = \alpha \cup \alpha = \alpha$, which implies, $x \in L(f_S; \alpha)$. Therefore, $L(f_S; \alpha)$ is an h-subhemiring of S.

(ii) Let $x, y \in S$ be such that $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$. Then $x \in L(f_S; \alpha_1)$ and $y \in L(f_S; \alpha_2)$, and so $x \in L(f_S; \alpha_2)$. Since $L(f_S; \alpha)$ is an h-subhemiring of S for all $\alpha \subseteq U$, then $x + y \in L(f_S; \alpha_2)$ and $xy \in L(f_S; \alpha_2)$. Hence

$f_S(x+y) \subseteq \alpha_2 = \alpha_1 \cup \alpha_2 = f_S(x) \cup f_S(y)$ and $f_S(xy) \subseteq \alpha_2 = \alpha_1 \cup \alpha_2 = f_S(x) \cup f_S(y)$.

Similarly, we can show that $f_S(xyz) \subseteq f_S(y)$ for all $x, y, z \in S$.

Now, let $x, z, a, b \in S$ with $x + a + z = b + z$ be such that $f_S(a) = \alpha_1$ and $f_S(b) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$, then $a \in L(f_S; \alpha_1)$ and $b \in L(f_S; \alpha_2)$, and so $a \in L(f_S; \alpha_2)$. Since $L(f_S; \alpha)$ is an h-subhemiring of S for each $\alpha \subseteq U$, then $x \in L(f_S; \alpha_2)$. Thus $f_S(x) \subseteq \alpha_2 = \alpha_1 \cup \alpha_2 = f_S(a) \cup f_S(b)$. Therefore f_S is an SU-hemiring of S over U.

□

4. h-*quasi*-hemiregular hemirings

In this section, we investigate some characterizations of h-*quasi*-hemiregular hemirings by some kinds of SU-h-ideals.

4.1. Definition. *[19] A subset A of S is called idempotent if $A = \overline{A^2}$. A hemiring S is called left(right) h-quasi-hemiregular if every left(right) h-quasi-ideal is idempotent and is called h-quasi-hemiregular if every left h-ideal. Every right h-ideal are idempotent.*

4.2. Lemma. [19] A hemiring S is left h-*quasi*-hemiregular if and only if one of the following holds:

(1) There exist $c_i, d_i, c'_j, d'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} c_i x d_i x + z = \sum_{j=1}^{n} c'_j x d'_j x + z \text{ for all } x \in S;$$

(2) $x \in \overline{SxSx}$ for all $x \in S$;

(3) $A \subseteq \overline{SASA}$ for all $A \subseteq S$;

(4) $I \cap L = \overline{IL}$ for every h-ideal I and every *left* h-ideal of S.

4.3. Theorem. A hemiring S is *left*(*right*) h-*quasi*-hemiregular if and only if every SU-*left*(*right*) h-ideal of S over U is idempotent.

Proof. Let S be a *left* h-*quasi*-hemiregular hemiring and f_S any SU-*left* h-ideal of S over U. For any $x \in S$, then there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x d_i x + \sum_{j=1}^{n} c'_j x d'_j x + z$ since S is left h-*quasi*-hemiregular.

Thus,

$$\begin{aligned}
(f_S \diamondsuit f_S)(x) &= \bigcap_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cup f_S(a'_j) \cup f_S(b_i) \cup f_S(b'_j)) \\
&\subseteq f_S(c_i x) \cup f_S(c'_j x) \cup f_S(d_i x) \cup f_S(d'_j x) \\
&\subseteq f_S(x) \cup f_S(x) \cup f_S(x) \cup f_S(x) \\
&= f_S(x),
\end{aligned}$$

which implies, $(f_S\diamondsuit f_S)(x)\subseteq f_S(x)$, that is, $f_S\diamondsuit f_S\tilde{\subseteq} f_S$. Since f_S is an SU-$left$ h-ideal of S over U, then $f_S\diamondsuit f_S\tilde{\supseteq} f_S$ always holds. Then $f_S\diamondsuit f_S=f_S$.

Conversely, let L be any $left$ h-ideal of S. Then by Proposition 2.6, we have $\mathcal{S}_{L^C}$ is an SU-$left$ h-ideal of S over U. If there exists $x\in L$ and $x\notin\overline{L^2}$, then there do not exist $a_1,a_2,b_1,b_2,z\in S$ such that $x+a_1a_2+z=b_1b_2+z$. Then $\mathcal{S}_{L^C}=\emptyset$ and $\mathcal{S}_{L^C}=(\mathcal{S}_{L^C}\diamondsuit\mathcal{S}_{L^C})(x)=U$, contradiction. This implies that $L\subseteq\overline{L^2}$. On the other hand, $\overline{L^2}\subseteq L$ always holds. Thus $L=\overline{L^2}$. It follows from Lemma 4.2 that S is left h-$quasi$-hemiregular.

Similarly, we can show that the case for right h-$quasi$-hemiregular hemirings.

□

4.4. Theorem. Let S be a hemiring. Then the following are equivalent:
(1) S is $left$ h-$quasi$-hemiregular;
(2) $f_S\tilde{\cup} g_S=f_S\diamondsuit g_S$ for every SU-h-ideal f_S and every SU-$left$ h-ideal g_S of S over U;
(3) $f_S\tilde{\cup} g_S\tilde{\supseteq} f_S\diamondsuit g_S$ for every SU-h-ideal f_S and every SU-h-bi-ideal g_S of S over U;
(4) $f_S\tilde{\cup} g_S\tilde{\supseteq} f_S\diamondsuit g_S$ for every SU-h-ideal f_S and every SU-h-$quasi$-ideal g_S of S over U.

Proof. (1) $\Rightarrow$ (3) Let f_S and g_S be any SU-h-ideal and any SU-h-bi-ideal of S over U, respectively.
For any $x\in S$. By Lemma 4.2, we have $x\in\overline{SxSx}\subseteq\overline{S\overline{SxSx}Sx}\subseteq\overline{SxSxSx}$, and so, there exist $c_i,c'_j,d_i,d'_j,e_i,e'_j,z\in S$ such that $x+\sum_{i=1}^{m'}c_ixd_ixe_ix+z=\sum_{j=1}^{n'}c'_jxd'_jxe'_jx+z$.
Thus, we have

$$\begin{array}{rcl}(f_S\diamondsuit g_S)(x) & = & \displaystyle\bigcap_{x+\sum_{i=1}^{m}a_ib_i+z'=\sum_{j=1}^{n}a'_jb'_j+z'}(f_S(a_i)\cup f_S(a'_j)\cup g_S(b_i)\cup g_S(b'_j))\\ & \subseteq & f_S(c_ixd_i)\cup f_S(c'_jxd'_j)\cup g_S(xe_ix)\cup g_S(xe'_jx)\\ & \subseteq & f_S(x)\cup g_S(x)\\ & = & (f_S\tilde{\cup} g_S)(x),\end{array}$$

which implies, $f_S\diamondsuit g_S\tilde{\subseteq} f_S\tilde{\cup} g_S$.
It is clear that (3) $\Rightarrow$ (4) $\Rightarrow$ (2).
(2) $\Rightarrow$ (1) Let I and L be any h-ideal and any left h-ideal of S, respectively, Then $\mathcal{S}_{I^C}$ and $\mathcal{S}_{L^C}$ are SU-h-ideal and SU-$left$ h-ideals of S over U respectively. If there exists $x\in I\cap L$ such that $x\notin\overline{IL}$, then there do not exist $a_1,a_2\in I$, $b_1,b_2\in L$, $z\in S$ such that $x+a_1b_1+z=a_2b_2+z$. Then $(\mathcal{S}_{I^C}\diamondsuit\mathcal{S}_{L^C}(x)=U$. Since $x\in I\cap L$, then $x\in I$ and $x\in L$, and so $\mathcal{S}_{I^C}(x)=\mathcal{S}_{L^C}=\emptyset$. By the assumption, we have $(\mathcal{S}_{I^C}\diamondsuit\mathcal{S}_{L^C})(x)=(\mathcal{S}_{i^C}\tilde{\cup}\mathcal{S}_{L^C})(x)=\mathcal{S}_{I^C}(x)\cup\mathcal{S}_{L^C}(x)=\emptyset\cup\emptyset=\emptyset$, a contradiction. This implies that $I\cap L\subseteq\overline{IL}$. On the other hand, $\overline{IL}\subseteq I\cap L$ always holds. Thus, $I\cap L=\overline{IL}$. It follows from Lemma 4.2 that S is $left$ h-$quasi$-hemiregular. □

Similarly, we have the following result.

4.5. Theorem. Let S be a hemiring. Then the following are equivalent:
(1) S is $left$ h-$quasi$-hemiregular;
(2) $f_S\tilde{\cup} g_S\tilde{\cup} h_S\tilde{\supseteq} f_S\diamondsuit g_S\diamondsuit h_S$ for every SU-h-ideal f_S and every SU-$right$ h-ideal g_S and every SU-h-bi-ideal h_S of S over U;
(3) $f_S\tilde{\cup} g_S\tilde{\cup} h_S\tilde{\supseteq} f_S\diamondsuit g_S\diamondsuit h_S$ for every SU-h-ideal f_S and every SU-$right$ h-ideal g_S and every SU-h-$quasi$-ideal h_S of S over U.

Now we give an important property of h-$quasi$-hemiregular hemirings.

4.6. Theorem. A hemiring S is h-$quasi$-hemiregular if and only if $f_S=(\tilde{\theta}\diamondsuit f_S)^2\tilde{\cup}(f_S\diamondsuit\tilde{\theta})^2$ for every SU-h-$quasi$-ideal of S over U.

Proof. Let S be an h-*quasi*-ideal of S over U. We can check that $\tilde{\theta}\diamondsuit f_S$ and $f_S\diamondsuit\tilde{\theta}$ are SU-*left* h-ideal and SU-*right* h-ideal of S over U, respectively. Then by Theorem 4.3, $\tilde{\theta}\diamondsuit f_S$ and $f_S\diamondsuit\tilde{\theta}$ are idempotent. Hence, we have

$$(\tilde{\theta}\diamondsuit f_S)^2\tilde{\cup}(f_S\diamondsuit\tilde{\theta})^2 = (\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(f_S\diamondsuit\tilde{\theta})\tilde{\supseteq}f_S.$$

For any $x \in S$, then there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that $x + \sum_{i=1}^{m'} c_i x d_i x + z = \sum_{j=1}^{n'} c'_j x d'_j x + z$ as S is h-*quasi*-hemiregular. Thus, we have

$$\begin{aligned}(\tilde{\theta}\diamondsuit f_S)^2(x) &= \bigcap_{x+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} ((\tilde{\theta}\diamondsuit f_S)(a_i) \cup (\tilde{\theta}\diamondsuit f_S)(a'_j) \cup (\tilde{\theta}\diamondsuit f_S)(b_i) \cup (\tilde{\theta}\diamondsuit f_S)(b'_j)) \\ &\subseteq (\tilde{\theta}\diamondsuit f_S)(c_i x) \cup (\tilde{\theta}\diamondsuit f_S)(c'_j x) \cup (\tilde{\theta}\diamondsuit f_S)(d_i x) \cup (\tilde{\theta}\diamondsuit f_S)(d'_j x) \\ &\subseteq f_S(x),\end{aligned}$$

which implies, $(\tilde{\theta}\diamondsuit f_S)^2\tilde{\subseteq}f_S$. Similarly, we can prove $(f_S\diamondsuit\tilde{\theta})^2\tilde{\subseteq}f_S$, and so $(\tilde{\theta}\diamondsuit f_S)^2\tilde{\cup}(f_S\diamondsuit\tilde{\theta})^2 \subseteq f_S$. Hence $(\tilde{\theta}\diamondsuit f_S)^2\tilde{\cup}(f_S\diamondsuit\tilde{\theta})^2 = f_S$.

Conversely, assume that the given conditions hold. Let f_S be any SU-*left* h-ideal of S over U, then f_S is an SU-h-*quasi*-ideal of S over U. Thus,

$$f_S = (\tilde{\theta}\diamondsuit f_S)^2\tilde{\cup}(f_S\diamondsuit\tilde{\theta})^2\tilde{\supseteq}(\tilde{\theta}\diamondsuit f_S)^2\tilde{\supseteq}f_S\diamondsuit f_S\tilde{\supseteq}\tilde{\theta}\diamondsuit f_S\tilde{\supseteq}f_S.$$

Thus, $f_S = f_S\diamondsuit f_S$. By Theorem 4.3, S is *left* h-*quasi*-hemiregular. Similarly, we can show that S is *right* h-*quasi*-hemiregular. Hence S is h-*quasi*-hemiregular. □

Similar to Lemma 4.2, we have the following:

4.7. Lemma. A hemiring S is both *left* h-*quasi*-hemiregular and h-intra-hemiregular if and only if for any $x \in S$, there exist $c_i, d_i, c'_i, d'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x^2 d_i x + z = \sum_{j=1}^{n} c'_j x d'_j x + z$.

By Theorem 4.3 and Lemma 4.7, we can get the following result:

4.8. Theorem. Let S be a hemiring. Then the following are equivalent:
(1) S is both *left* h-*quasi*-hemiregular and h-intra-hemiregular;
(2) $f_S\tilde{\cup}g_S\tilde{\supseteq}f_S\diamondsuit g_S$ for every SU-*left* h-ideal f_S and every SU-h-*bi*-ideal g_S of S over U;
(3) $f_S\tilde{\cup}g_S\tilde{\supseteq}f_S\diamondsuit g_S$ for every SU-*left* h-ideal f_S and every SU-h-*quasi*-ideal g_S of S over U.

5. h-semisimple hemirings

In this section, we present some properties of h-semisimple hemirings by means of SU-h-ideals and SU-h-interior ideals.

5.1. Definition. *[30] A subset A of S is called idempotent if $A = \overline{A^2}$. A hemiring S is called idempotent if every h-ideal is idempotent.*

5.2. Lemma. [30] A hemiring S is h-semisimple if and only if one of the following holds:
(1) There exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that $x+\sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z$ for all $x \in S$;
(2) $x \in \overline{SxSxS}$ for all $x \in S$;
(3) $A \subseteq \overline{SASAS}$ for all $A \subseteq S$.

5.3. Theorem. Let S be an h-semisimple hemiring. Then a soft set f_S over U is an SU-h-ideal of S over U if and only if it is an SU-h-interior ideal of S over U.

Proof. If f_S is an SU-h-ideal of S over U, then by Lemma 3.5, we know that f_S is an SU-h-interior ideal of S over U.

Conversely, assume that f_S is an SU-h-interior ideal of S over U. For any $x, y \in S$, then there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z$ since S is h-semisimple, and so, $xy + \sum_{i=1}^{m} c_i x d_i e_i f_i y + zy = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j y + zy$.

Thus,

$$f_S(xy) \subseteq f_S(\sum_{i=1}^{m} c_i x d_i e_i x f_i y) \cup f_S(\sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j y) \subseteq f_S(x),$$

which implies, f_S is an SU-*right* h-ideal of S over U. Similarly, we can prove that f_S is an SU-*left* h-ideal of S over U. Hence, f_S is an SU-h-ideal of S over U. □

5.4. Theorem. A hemiring S is h-semisimple if and only if for any SU-h-interior ideals f_S and g_S, we have $f_S \tilde{\cup} g_S = f_S \diamondsuit g_S$.

Proof. Let S be an h-semisimple hemiring, f_S and g_S two SU-h-interior ideals of S over U. Then by Theorem 5.3, f_S and g_S are two SU-h-ideal of S over U. Thus, we have

$$f_S \diamondsuit g_S \tilde{\supseteq} f_S \diamondsuit \tilde{\theta} \tilde{\supseteq} f_S \text{ and } f_S \diamondsuit g_S \tilde{\supseteq} \tilde{\theta} \diamondsuit g_S \tilde{\supseteq} g_S.$$

This proves that $f_S \diamondsuit g_S \tilde{\supseteq} f_S \tilde{\cup} g_S$.

For any $x \in S$, then there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z$ since S is h-semisimple.

Thus,

$$\begin{aligned} (f_S \diamondsuit g_S)(x) &= \bigcap_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cup f_S(a'_j) \cup g_S(b_i) \cup g_S(b'_j)) \\ &\subseteq f_S(c_i x d_i) \cup f_S(c'_j x d'_j) \cup g_S(e_i x f_i) \cup g_S(e'_j x f'_j) \\ &\subseteq f_S(x) \cup g_S(x) \\ &\subseteq (f_S \tilde{\cup} g_S)(x), \end{aligned}$$

which implies, $(f_S \diamondsuit g_S)(x) \subseteq (f_S \tilde{\cup} g_S)(x)$, that is, $f_S \diamondsuit g_S \tilde{\subseteq} f_S \diamondsuit g_S$. Thus, $f_S \diamondsuit g_S = g_S \tilde{\cup} g_S$.

Conversely, let A be any h-ideal of S, then it is an h-interior ideal of S. Then by Corollary 3.7, $\mathcal{S}_{A^C}$, is an SU-h-interior ideal of S over U. If there exists $x \in A$ such that $x \notin \overline{A^2}$, then there do not exist $a_1, a_2, b_1, b_2 \in A$, and $z \in S$ such that $x + a_1 b_1 + z = a_2 b_2 + z$. Thus, $(\mathcal{S}_{A^C} \diamondsuit \mathcal{S}_{A^C})(x) = U$. Since $x \in A$, then $\mathcal{S}_{A^C}(x) = \emptyset$. By the assumption, we have $\mathcal{S}_{A^C}(x) = (\mathcal{S}_{A^C} \diamondsuit \mathcal{S}_{A^C})(x) = U$, contradiction. This means that $A \subseteq \overline{A^2}$. On the other hand, $\overline{A^2} \subseteq A$ always, hold. Thus, $A = \overline{A^2}$. Hence S is h-semisimple. □

6. Conclusions

The aim of this paper is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. By means of soft intersection-union sum and soft intersection-union product, we apply soft set theory to h-semisimple and h-*quasi*-hemiregular hemirings. In our future work, we apply this theory to some applied fields, such as decision making, information sciences and intelligent systems.

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