Hacettepe Journal of Mathematics and Statistics
Volume 43 (5) (2014), 747 – 753

Some results on primeness in the near-ring of Lipschitz functions on a normed vector space

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Received 01 : 02 : 2013 : Accepted 10 : 06 : 2013

Abstract

In this note, we consider the equiprime and strongly prime radicals in the near-ring $\mathcal{L}_X$ of Lipschitz functions over a normed vector space X. We prove that the equiprime radical of $\mathcal{L}_X$ is trivial, and we also obtain upper and lower bounds on its strongly prime radical in terms of the ideal of bounded Lipschitz functions on X.

2000 AMS Classification: 16Y30.

Keywords: near-ring, Lipschitz, equiprime, strongly prime

1. Introduction

Recall that, given two metric spaces X and Y with metrics d_X and d_Y, respectively, a function $f : X \to Y$ is called *Lipschitz* if there exists some constant $M > 0$ such that, for every $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq M \cdot d_X(x_1, x_2)$. In this case we define the *Lipschitz number* of f as $L(f) = \sup\{\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \mid x_1 \neq x_2\}$. Lipschitz functions, which are easily seen to be uniformly continuous, have wide-ranging uses from existence theorems for ordinary differential equations to applications in image processing.

Spaces of Lipschitz functions have been of interest for over fifty years. In many of the papers on the topic, extra conditions are imposed on one or more of the underlying metric spaces, giving additional structure to certain subsets of the set of Lipschitz functions from X to Y. For example, if Y is a Banach space, the set of all bounded functions from X to Y having bounded Lipschitz number is a Banach space with norm $\|\cdot\| = \max(L(\cdot), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the usual supremum-norm (see, for example, [12]). Related studies can be found, *inter alia*, in [7], [15], and [16]; the book of Weaver [18] gives a unified treatment of many of the results in the area.

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Although most of these Banach space and Banach algebra investigations considered the product of bounded Lipschitz functions using standard multiplication, we focus instead on the composition operation for Lipschitz functions over a normed vector space. As a result, the governing algebraic structures are near-rings and near-algebras, the more general counterparts to rings and algebras, and our results may be considered as a generalization of the (bounded) linear operators often encountered in analysis. The dissertation of Irish [11] treats the important special case of near-algebras of Lipschitz functions on a Banach space. More recently in [8], the author and B. van der Merwe initiated a more general study of the near-ring of Lipschitz functions on a metric space. The present article furthers this study by considering primeness in the near-ring of Lipschitz functions on a normed vector space. We begin with a "Preliminaries" section to give the requisite background for the main results of the paper.

2. Preliminaries

2.1. Definition. A *right near-ring* is a triple $(N, +, *)$ satisfying:

(1) $(N, +)$ is a (not necessarily abelian) group,
(2) $(N, *)$ is a semigroup, and
(3) for all $a, b, c \in N$, $(a + b) * c = a * c + b * c$.

Left-near-rings may be defined analogously by replacing the third condition with: for all $a, b, c \in N$, $c * (a + b) = c * a + c * b$. All near-rings throughout this paper will be right near-rings. A near-ring $(N, +, *)$ is called *zero-symmetric* if, for all $n \in N$, $n * 0_N = 0_N$, where 0_N is the neutral element of $(N, +)$.

If $(G, +)$ is any group (written additively, though not necessarily Abelian), then $\mathcal{M}(G)$, the set of all self-maps of G, is a near-ring under pointwise addition and function composition. The set of all zero-preserving self-maps of G, $\mathcal{M}_0(G)$, is a zero-symmetric subnear-ring of $\mathcal{M}(G)$ that provides an appropriate structure in which to study $\mathrm{End}(G)$, the set of endomorphisms of G, in case $\mathrm{End}(G)$ is not a ring. Further examples and applications of near-rings, along with many of the basic results of the theory of near-rings, may be found in the books of Clay [6], Meldrum [13], and Pilz [14].

2.2. Notation. Hereafter $\mathbb{F}$ will denote the field of scalars $\mathbb{R}$ or $\mathbb{C}$ with the standard metric given by $|\cdot|$. For a pointed metric space $(X, \rho; e_0)$, $\mathcal{L}_X$ will denote the set of basepoint preserving Lipschitz functions, which is a near-ring under certain conditions (see [8]). In particular, if (X, ρ, e_0) is a normed vector space over $\mathbb{F}$ with norm $\|\cdot\|$, $\rho(x_1, x_2) = \|x_1 - x_2\|$, and $e_0 = 0$, then $\mathcal{L}_X$ is a near-ring (in fact, a *normed near-algebra* as shown in [8]). We denote by $\mathcal{SL}_X$ the set of scalar-valued, zero-preserving Lipschitz functions on the metric space X. If X has a multiplication $\cdot : \mathbb{F} \times X \to X$ defined on it, we define the left multiplication of an element $\alpha \in X$ by an element $f \in \mathcal{SL}_X$ as a map $f_\alpha : X \to X$ given by $f_\alpha(x) = f(x) \cdot \alpha$ (or simply $f(x)\alpha$) for any $x \in X$.

While homomorphisms of near-rings are defined as those for rings, requiring two-sided ideals to be the kernels of homomorphisms gives rise to a somewhat more complicated elementwise definition.

2.3. Definition. A subset I of a near-ring $(N, +, *)$ is *two-sided ideal* of N if:

(1) $(I, +)$ is a normal subgroup of $(N, +)$,
(2) $IN := \{i * n \mid i \in I \text{ and } n \in N\} \subseteq I$, and
(3) for all $n, m \in N$ and $i \in I$, $n(m + i) - nm \in I$.

As in the case of noncommutative rings, there are multiple notions of "prime" for ideals in near-rings. The reader is referred to [9] for a discussion of different, nonequivalent near-ring extensions of the standard definition: an ideal I of a near-ring N is *prime* (or *0-prime*) if for all ideals A, B of N, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. In the sequel, we shall make use of the following kinds of primeness.

2.4. Definition. An ideal I of a near-ring N is called *equiprime* if $a \in N \setminus I$ and $x, y \in N$ with, for all $n \in N$, $anx - any \in I$, implies $x - y \in I$. A near-ring N is called equiprime if the zero ideal is an equiprime ideal of N.

Equiprime ideals are prime ideals for near-rings; the converse, however, is not true. In [4], though, it is shown that the equiprime condition is equivalent to the standard prime condition in case N is a ring. Of significance from the radical-theoretic point of view, defining the equiprime radical of a near-ring N by the intersection of all equiprime ideals of N leads to a Kurosh-Amitsur prime radical (i.e., a Hoehnke radical map that is complete and idempotent) for near-rings [1].

2.5. Definition. An ideal I of a near-ring N is called *strongly prime* if $a \in N \setminus I$ implies that there exists a finite subset F of N such that $aFx \subset I$ implies $x \in I$. A near-ring N is called strongly prime if the zero ideal is a strongly prime ideal of N.

In the next section, we show that the near-ring $\mathcal{L}_X$ of Lipschitz functions over a normed vector space X is equiprime but not strongly prime. Moreover, we find upper and lower bounds on the strongly prime radical of $\mathcal{L}_X$ in terms of the ideal of bounded Lipschitz functions on X.

3. Primeness Results

It is known (cf. [17]) that, for any group G, $\mathcal{M}_0(G)$ is equiprime. Furthermore, there is a growing number of papers ([10], [2], [5], [3]) that treat questions of primeness in $\mathcal{N}_0(G)$, the near-ring of continuous, zero-preserving maps on G, for certain classes of groups G. The reader may find it of interest to compare/contrast our results and proof methods with those found in these articles.

We begin with the following result, which is well-known (see, e.g., [18]).

3.1. Lemma. *For any metric space (X, ρ) and any closed subset $C \subseteq X$, the map $\phi : X \to \mathbb{R}$ given by $\phi(x) := \rho(x, C)$ is Lipschitz.*

3.2. Proposition. *Let $(X, \rho; 0)$ be a pointed metric space with a multiplication $\cdot : \mathbb{F} \times X \to X$ such that: 1) $\mathbb{R}X \subseteq X$ with $1x = x$ for all $x \in X$ and 2) left multiplication of any element in X by any element in $\mathcal{SL}_X$ yields an element in $\mathcal{L}_X$. If $\mathcal{L}_X$ is a near-ring with zero at the basepoint of X, then $\mathcal{L}_X$ is equiprime.*

Proof. We must establish that, given $a, f, g \in \mathcal{L}_X$, $anf = ang$ for all $n \in \mathcal{L}_X$ implies $a = 0$ or $f = g$. Suppose, then, that $anf = ang$ for all $n \in \mathcal{L}_X$ and that $a \neq 0$. Thus there exists some $\alpha \in X$ such that $a(\alpha) \neq 0$. If $f \neq g$ as well, then there exists some $x_0 \in X$, $x_0 \neq 0$, for which $f(x_0) \neq g(x_0)$. Without loss of generality, suppose $f(x_0) \neq 0$. Define $n : X \to X$ via $n(x) = \frac{\rho(x, \{g(x_0), 0\})}{\rho(f(x_0), \{g(x_0), 0\})}\alpha$. Then $n \in \mathcal{L}_X$ by part 2) of the hypotheses and Lemma 3.1 since (X, ρ) is T_1. However, $a(n(f(x_0))) = a(\alpha) \neq 0 = a(n(g(x_0)))$, a contradiction. □

Throughout this paper, for a normed vector space $(X, \|\cdot\|)$ over $\mathbb{F}$, we use the usual metric induced by the norm to regard X as a metric space with basepoint 0. We then immediately have the following.

3.3. Corollary. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$. Then the near-algebra $\mathcal{L}_X$ is equiprime.*

The next result is used to show directly that $\mathcal{L}_X$ is not strongly prime in case $(X, \|\cdot\|)$ is a normed vector space over $\mathbb{F}$, as well as to establish the nontriviality of the lower bound we will determine for the strongly prime radical of $\mathcal{L}_X$.

3.4. Lemma. *Suppose $(X, \|\cdot\|)$ is a normed vector space over $\mathbb{F}$. Let r_1, r_2 be real numbers with $r_2 > r_1 > 0$. Then for any fixed $x_0 \in X$ the map $a_{x_0,r_1,r_2} : X \to X$ defined via*

$$a_{x_0,r_1,r_2}(x) = \begin{cases} (\|x\| - r_1)x_0 & \text{if } r_1 \leq \|x\| \leq \frac{r_1+r_2}{2} \\ (r_2 - \|x\|)x_0 & \text{if } \frac{r_1+r_2}{2} < \|x\| \leq r_2 \\ 0 & \text{otherwise.} \end{cases}$$

is an element of $\mathcal{L}_X$ with Lipschitz number at most $2\|x_0\|$.

Proof. Let $x_0 \in X$, $r_1, r_2 \in \mathbb{R}$, and $a_{x_0,r_1,r_2} : X \to X$ be as in the stated hypotheses. For simplicity, denote a_{x_0,r_1,r_2} by a. We consider five cases; the remaining cases are trivial or analogous.

Case 1: For $0 \leq \|x_1\| \leq r_1$ and $r_1 \leq \|x_2\| \leq \frac{r_1+r_2}{2}$, we find that

$$\begin{aligned} \|a(x_1) - a(x_2)\| &= \|0 - (\|x_2\| - r_1)x_0\| \\ &= (\|x_2\| - r_1)\|x_0\| \\ &\leq (\|x_2\| - \|x_1\|)\|x_0\| \\ &\leq (\|x_2 - x_1\|)\|x_0\|. \end{aligned}$$

Case 2: For $0 \leq \|x_1\| \leq r_1$ and $\frac{r_1+r_2}{2} \leq \|x_2\| \leq r_2$, we find that

$$\begin{aligned} \|a(x_1) - a(x_2)\| &= \|0 - (r_2 - \|x_2\|)x_0\| \\ &= (r_2 - \|x_2\|)\|x_0\| \\ &\leq (\|x_2\| - \|x_1\|)\|x_0\| \\ &\leq (\|x_2 - x_1\|)\|x_0\|. \end{aligned}$$

Case 3: For $r_1 \leq \|x_1\| \leq \frac{r_1+r_2}{2}$ and $r_1 \leq \|x_2\| \leq \frac{r_1+r_2}{2}$, we find that

$$\begin{aligned} \|a(x_1) - a(x_2)\| &= \|(\|x_1\| - r_1)x_0 - (\|x_2\| - r_1)x_0\| \\ &= |(\|x_1\| - \|x_2\|)| \cdot \|x_0\| \\ &\leq (\|x_2 - x_1\|)\|x_0\|. \end{aligned}$$

Case 4: For $r_1 \leq \|x_1\| \leq \frac{r_1+r_2}{2}$ and $\frac{r_1+r_2}{2} \leq \|x_2\| \leq r_2$, we find that

$$\begin{aligned} \|a(x_1) - a(x_2)\| &= \|(\|x_1\| - r_1)x_0 - (r_2 - \|x_2\|)x_0\| \\ &= |(\|x_1\| - \tfrac{r_1+r_2}{2} + \|x_2\| - \tfrac{r_1+r_2}{2})| \cdot \|x_0\| \\ &\leq (|\|x_1\| - \tfrac{r_1+r_2}{2}| + |\|x_2\| - \tfrac{r_1+r_2}{2}|) \cdot \|x_0\| \\ &\leq 2(\|x_2\| - \|x_1\|) \cdot \|x_0\| \\ &\leq 2(\|x_2 - x_1\|)\|x_0\|. \end{aligned}$$

Case 5: For $\frac{r_1+r_2}{2} \leq \|x_1\| \leq r_2$ and $\frac{r_1+r_2}{2} \leq \|x_2\| \leq r_2$, we find that

$$\begin{aligned} \|a(x_1) - a(x_2)\| &= \|(r_2 - \|x_1\|)x_0 - (r_2 - \|x_2\|)x_0\| \\ &= |(\|x_2\| - \|x_1\|)| \cdot \|x_0\| \\ &\leq (\|x_2 - x_1\|)\|x_0\|. \end{aligned}$$

□

3.5. Proposition. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$. Then $\mathcal{L}_X$ is not strongly prime.*

Proof. Let $A := \{x \in X \mid 1 \leq \|x\| \leq 2\}$. Then $X \setminus A$ is open in X. Fix $x_0 \in X \setminus \{0\}$ and define $a : X \to X$ via $a = a_{x_0,1,2} \in \mathcal{L}_X$ as in Lemma 3.4. Now suppose that $F := \{f_1, f_2, \ldots, f_n\} \subseteq \mathcal{L}_X$ is any finite subset. For $i = 1, 2, \ldots, n$, f_i is continuous, $V_i := f_i^{-1}(X \setminus A)$ is open in X, and $0 \in V_i$. Hence $V := \bigcap_{i=1}^{n} V_i$ is open and contains 0, so we may choose $v \in V$, $v \neq 0$ such that, for any $i = 1, 2, \ldots, n$ and any real number $t \in [0, 1]$, $f_i(tv) \notin A$. Define $g : X \to X$ via $g = a_{v,1,2} \in \mathcal{L}_X$ as in Lemma 3.4. Since for any $i = 1, 2, \ldots, n$ and any $x \in X$, we have $af_ig(x) = 0$ with $a \neq 0$ and $g \neq 0$, the result follows.

□

We now proceed to strengthen this result.

3.6. Lemma. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$ and let $\{x_i\} \subset X$ be a sequence such that $x_0 = 0$ and for all integers $i \geq 1$, $\|x_i - x_{i-1}\| < \frac{1}{2}$. Then $f : X \to X$ defined via*

$$f(x) = (\|x\| - (i-1))(x_i - x_{i-1}) + x_{i-1}, \text{ whenever } i - 1 \leq \|x\| \leq i$$

is an element of $\mathcal{L}_X$.

Proof. Suppose that $y_1, y_2 \in X$.

Case 1: If $\|y_1\|, \|y_2\| \in [n, n+1]$ for some integer $n \geq 0$, then

$$\begin{aligned} \|f(y_2) - f(y_1)\| &= \|(\|y_2\| - n)(x_{n+1} - x_n) + x_n \\ &\qquad - ((\|y_1\| - n)(x_{n+1} - x_n) + x_n)\| \\ &= |(\|y_2\| - \|y_1\|)| \cdot \|x_{n+1} - x_n\| \\ &\leq (\|y_2 - y_1\|) \cdot \tfrac{1}{2}. \end{aligned}$$

Case 2: If $\|y_1\| \in [n, n+1]$, $\|y_2\| \in [m, m+1]$ for some integers nonnegative integers $n \neq m$, then without loss of generality we may suppose that $0 \leq n < m$. In this case, we have

$$\begin{aligned} \|f(y_2) - f(y_1)\| &= \|(\|y_2\| - m)(x_{m+1} - x_m) + x_m \\ &\qquad - ((\|y_1\| - n)(x_{n+1} - x_n) + x_n)\| \\ &= \|(\|y_2\| - m)(x_{m+1} - x_m + x_n - x_{n+1}) + x_m \\ &\qquad + (\|y_2\| - \|y_1\|)(x_{n+1} - x_n) \\ &\qquad\quad - (m - n - 1)(x_{n+1} - x_n) - x_{n+1}\| \\ &\leq |(\|y_2\| - m)|\|x_{m+1} - x_m\| \\ &\qquad + |(\|y_2\| - m)|\|x_n - x_{n+1}\| \\ &\qquad\quad + |(\|y_2\| - \|y_1\|)|\|x_{n+1} - x_n\| \\ &\qquad\qquad + (m - n - 1)\|x_{n+1} - x_n\| + \|x_m - x_{n+1}\|, \end{aligned}$$

and since $\|y_2\| - m \leq \|y_2\| - \|y_1\|$, $m - n - 1 \leq \|y_2\| - \|y_1\|$, and $\|x_m - x_{n+1}\| < \frac{1}{2}(m - n - 1)$, it follows that

$$\|f(y_2) - f(y_1)\| < \frac{5}{2}(\|y_2\| - \|y_1\|) \leq \frac{5}{2}\|y_2 - y_1\|.$$

□

Let $\mathcal{BL}_X$ denote the set of bounded functions in $\mathcal{L}_X$.

3.7. Proposition. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$. Then $\mathcal{BL}_X$ is a strongly prime ideal of $\mathcal{L}_X$.*

Proof. It was shown in [8] that $\mathcal{BL}_X \lhd \mathcal{L}_X$. Given $a \in \mathcal{L}_X \setminus \mathcal{BL}_X$, there exists a sequence $\{y_i\} \subseteq X$ such that $\|a(y_i)\| \to \infty$ as $i \to \infty$. Since a is continuous, $\|y_i\| \to \infty$ as $i \to \infty$. If necessary, relabel and refine the sequence $\{y_i\}$ to find a supersequence, $\{x_i\} \supseteq \{y_i\}$ such that $x_0 = 0$ and for all integers $i \geq 1, \|x_i - x_{i-1}\| < \frac{1}{2}$. Then $\|a(x_i)\| \to \infty$ as $i \to \infty$. Now define a function f on X via:

$$f(x) = (\|x\| - (i-1))(x_i - x_{i-1}) + x_{i-1}, \text{whenever } i - 1 \leq \|x\| \leq i.$$

By Lemma 3.6, $f \in \mathcal{L}_X$. Now if $b \in \mathcal{L}_X \setminus \mathcal{BL}_X$, then there exists a sequence $\{t_i\} \subseteq X$ such that $\|b(t_i)\| \to \infty$ as $i \to \infty$. Furthermore, since b is continuous, given any positive integer N, there exists some $s_N \in X$ such that $\|b(s_N)\| = N$. Since $afb(s_N) = a(x_N)$, we conclude that $afb \notin \mathcal{BL}_X$, so the result follows. □

Let $\mathcal{S}(\mathcal{L}_X)$ denote the strongly prime radical of $\mathcal{L}_X$, i.e., the intersection of all strongly prime ideals of $\mathcal{L}_X$.

3.8. Theorem. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$. Define $P := \{f \in \mathcal{L}_X \mid$ there exists an open set U of X with $0 \in U \subseteq f^{-1}(0)\}$. Then $\{0\} \neq \mathcal{BL}_X \cap P \subseteq \mathcal{S}(\mathcal{L}_X) \subseteq \mathcal{BL}_X$.*

Proof. Suppose I is an ideal of $\mathcal{L}_X$ and that $\mathcal{BL}_X \cap P \nsubseteq I$. We show that I cannot be strongly prime. To this end, let $a \in (\mathcal{BL}_X \cap P) \setminus I$. Then a is bounded and there is some open set U in X such that $0 \in U$ and $a(U) = \{0\}$. Let $F = \{f_1, f_2, \ldots, f_n\}$ be any finite subset of $\mathcal{L}_X$ and define $V := \bigcap_{i=1}^{n} f_i^{-1}(U)$. Since U is open and each f_i is continuous, V is open in X with $0 \in V$. And since a is bounded and V is open, there exists some $\epsilon > 0$ such that, for all $x \in X, \epsilon \cdot a(x) \in V$, so that $\epsilon \cdot a$ is in $(\mathcal{BL}_X \cap P) \setminus I$, too. Since for any $x \in X$ and any $i \in \{1, 2, \ldots, n\}$ we have $a(f_i(\epsilon \cdot a(x))) \in a(f_i(V)) \subseteq a(U) = \{0\}$, we have shown that $\mathcal{BL}_X \cap P \nsubseteq I$ implies that I is not strongly prime. Thus, any strongly prime ideal of $\mathcal{L}_X$ must contain $\mathcal{BL}_X \cap P$, so that $\mathcal{BL}_X \cap P \subseteq \mathcal{S}(\mathcal{L}_X)$. By Proposition 3.7, we have that $\mathcal{S}(\mathcal{L}_X) \subseteq \mathcal{BL}_X$. Choosing some nonzero $x_0 \in X$ and considering $f = a_{x_0,1,2}$ as in Lemma 3.4, we see that for any $x \in X$, $\|f(x)\| \leq \frac{1}{2}\|x_0\|$ and, for $D = \{x \in X \mid 0 \leq \|x\| < 1\}$, $f(D) = \{0\}$. So $0 \neq f \in \mathcal{BL}_X \cap P$ follows, completing the proof.

□

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