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Characterization of Weakly Regular S-acts

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Abstract

We generalize the concept of weakly regularity in semigroups to S-acts, where S is a monoid. We prove among other results that if a monoid is von-Neumann regular then weakly regularity and von-Neumann regularity, in the context of S-acts, coincide. We also define locally projective S-acts, which is the generalization of projective S-acts. We consider many relationships between weakly regular S-acts and locally projective S-acts.

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1. Preliminaries

A right S-act is a triple (M, S, δ), where M is a nonempty set, S is a semigroup and $\delta : M \times S \to M$ is a mapping such that $\delta(m, st) = \delta(\delta(m, s), t)$ for all $m \in M$ and $s, t \in S$. For simplicity, we set $\delta(m, s) = ms$. We denote right S-act by M_S. Analogously, we can define a *left* S-act which we denote as $_SM$. An S_1-S_2-biact is a 5-tuple $(M, S_1, S_2, \delta_1, \delta_2)$, where (M, S_1, δ_1) is left S_1-act and (M, S_2, δ_2) is right S_2-act. That is, $s_1(ms_2) = (s_1m)s_2$ for all $s_1 \in S_1$, $s_2 \in S_2$ and $m \in M$. We denote S_1-S_2-biact by $_{S_1}M_{S_2}$. A right S-act is said to be *unitary* if S is a semigoup with identity 1 then $m1 = m$ for all $m \in M$. A nonempty subset N of a right S-act M_S is said to be S-subact of M_S if $NS \subseteq N$. Let M_S and A_S be right S-acts. A mapping $f : M_S \to A_S$ is called *S-homomorphism* if $f(ms) = f(m)s$ for all $m \in M$ and $s \in S$. The *S-monomorphism*, *S-epimorphism*, *S-isomorphism* and *S-endomorphism* are defined as usual. Simply, we use the abbreviation hom for homomorphism, mon for monomorphism, epi for epimorphism,

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iso for isomorphism and end for endomorphism. Let A_S and B_S be right S-acts. We denote the set containing all homs from A_S to B_S as

$$\mathcal{H}(A,B) = \{f \mid f \text{ is an } S\text{-homomorphism from } A_S \textit{ to } B_S\}.$$

Clearly, the set $\mathcal{H}(A,A)$ is a monoid with respect to the composition of mappings. Every right S-act A_S is a left $\mathcal{H}(A,A)$-act under the action $\psi a = \psi(a)$, where $\psi \in \mathcal{H}(A,A)$ and $a \in A$.

A nonempty subset U of a right S-actA_S is called a *generating set* of A_S if every element $a \in A$ can be represented as $a = us$ for some $u \in U, s \in S$. We say that A_S is *finitely generated* if $|U| < \infty$. We call A_S *cyclic* if generating set U of A_S is a singleton set. A generating set U of A_S is called a *basis* of A_S if every element $a \in A_S$ can be uniquely represented in the form $a = us$ for some $u \in U, s \in S$. That is, if $a = u_1s_1 = u_2s_2$ then $u_1 = u_2$ and $s_1 = s_2$. A right S-act is called *free* if it has a basis. A right S-act P_S is called *projective* if for every S-epi $g : M_S \to N_S$ and every S-hom $h : P_S \to N_S$ there exists an S-hom $k : P_S \to M_S$ such that $gk = h$, where M_S and N_S are any S-acts. Dual to projective S-acts, there is the notion of injective S-acts. A right S-act A_S is called *injective* if for any S-mon $\alpha : C_S \to B_S$ and S-hom $\beta : C_S \to A_S$, there is an S-hom $\mu : B_S \to A_S$ such that $\mu\alpha = \beta$, where B_S and C_S are any S-acts. For our later convenience we recall the following two results.

1.1. Proposition ([4]). *Let J be a nonempty set. Let $\dot{\bigcup} X_j$ be the disjoint union of right -acts X_j and take injections $\gamma_j : X_j \to \dot{\bigcup} X_j$ defined by $\gamma_j = I_{\dot{\bigcup} X_j|X_j}$, where $I_{\dot{\bigcup} X_j}$ denotes identity mapping. Then $\dot{\bigcup} X_j$ is an S-act and the injections γ_j are S-homs, for all $j \in J$. Moreover, for every right S-act K_S and for every family $\{k_i \in \mathcal{H}(X_j, K_S), j \in J\}$, the mapping $k : \dot{\bigcup} X_j \to K_S$ with $k(x) = k_j(x)$ for $x \in X_j$ is the unique S-hom such that $k\gamma_j = k_j$ for all $j \in J$.*

1.2. Proposition ([4]). *A right S-act M_S is projective if and only if $M_S = \dot{\bigcup} P_j$, where $P_j \cong e_jS$, for all $j \in J$.*

In the rest of the paper, by an S-act we always mean unitary right S-act. For the sake of clarity, we sometimes suppress S in the notation of S-acts.

2. Weakly Reagular S-acts

Following [3], we call a monoid S *right weakly regular*, if for all $t \in S$, t is in $(tS)^2$. Needless to say, S is called *left weakly regular* if for all $t \in S$, t is in $(St)^2$. In this section, we introduce the notion of weakly regular S-acts. The following is the formal definition of weakly regular S-acts.

2.1. Definition. An S-act M_S is called *weakly regular* if for all $m \in M$, there exist S-homs $\psi \in \mathcal{H}(M,M)$ and $\xi \in \mathcal{H}(M,S)$ such that $m = \psi(m)\xi(m)$.

2.2. Theorem. *A monoid S is left weakly regular if and only if the right S-act S_S is weakly regular S-act.*

Proof. $\Rightarrow$ Let S be a weakly regular monoid. For all $x \in S$, x is in $(Sx)^2$. That is, $x = yxzx$ for some $y, z \in S$. We define the mapping $\alpha : S_S \to S_S$ by $\alpha(a) = ya$, for all $a \in S$, where y is fixed. We also define the mapping $\beta : S_S \to S_S$ by $\beta(b) = zb$, for all $b \in S$, where z is fixed. Clearly, these two mappings are S-homs. The element x can be represented as $x = \alpha(x)\beta(x)$. Thus, S_S is weakly regular.

$\Leftarrow$ Suppose S_S is weakly regular. For all $t \in S_S$, there exist S-homs $\psi, \xi \in \mathcal{H}(S,S)$ such that $t = \psi(t)\xi(t) = \psi(1)t\ \xi(1)t = sts't$, where $s = \psi(1)$ and $s' = \xi(1)$. Thus, S is weakly regular. □

2.3. Corollary. *A monoid S is right weakly regular if and only if the left S-act ${}_SS$ is weakly regular S-act.*

Proof. The proof is similar to the above theorem. $\square$

2.4. Proposition. *A bisubact ${}_{\mathcal{H}(M,M)}N_S$ of a weakly regular biact ${}_{\mathcal{H}(M,M)}M_S$ is weakly regular.*

Proof. For all $n \in N$, there exist S-homs $\psi \in \mathcal{H}(M, M)$ and $\xi \in \mathcal{H}(M, S)$ such that $n = \psi(n)\xi(n)$. We know that ${}_{\mathcal{H}(M,M)}N_S$ is a left $\mathcal{H}(M, M)$-subact of ${}_{\mathcal{H}(M,M)}M$. Therefore, $\psi(n)$ is in N. But ${}_{\mathcal{H}(M,M)}N_S$ is a right S-subact of M_S too. Therefore, $\psi(n)\xi(n)$ is in N. Let $\hat{\psi}$ and $\hat{\xi}$ be restrictions of ψ and ξ respectively to N. We can rewrite the above equation as $n = \hat{\psi}(n)\hat{\xi}(n)$. Hence, ${}_{\mathcal{H}(M,M)}N_S$ is weakly regular. $\square$

2.5. Lemma. *Let M_S be an S-act. For any $m \in M$ and any S-hom $\xi \in \mathcal{H}(M, S)$, the mapping $m\xi : M_S \to M_S$ defined by $(m\xi)(x) = m \cdot \xi(x)$, for all $x \in M$, is S-end.*

Proof. Obvious. $\square$

We define the following notation as we are going to use it in our next result. To define we proceed as follows. Let A_S and B_S be two S-acts and X be a nonempty set. We define

$$\mathcal{H}(A, B)(X) = \{f(x) \mid f \in \mathcal{H}(A, B) \wedge x \in X\}.$$

2.6. Theorem. *An S-act M_S is weakly regular if and only if $N = N\mathcal{H}(M, S)(N)$ for all left $\mathcal{H}(M, M)$-subacts ${}_{\mathcal{H}(M,M)}N$ of the left $\mathcal{H}(M, M)$-act ${}_{\mathcal{H}(M,M)}M$.*

Proof. $\Rightarrow$ For all $n \in N$, there exist S-homs $\psi \in \mathcal{H}(M, M)$ and $\xi \in \mathcal{H}(M, S)$ such that $n = \psi(n)\xi(n)$. As ${}_{\mathcal{H}(M,M)}N$ is subact of ${}_{\mathcal{H}(M,M)}M$, it follows that $\psi(n)$ is in N. Therefore, n is in $N\mathcal{H}(M, S)(N)$. Hence, $N \subseteq N\mathcal{H}(M, S)(N)$. To prove that $N\mathcal{H}(M, S)(N) \subseteq N$, we proceed as follows. Let $n\xi(n')$ be in $N\mathcal{H}(M, S)(N)$, where $n, n' \in N$ and $\xi \in \mathcal{H}(M, S)$. We can write $n\xi(n') = (n\xi)(n)$. By Lemma 2.5, $n\xi$ is an S-hom from M_S to M_S. It follows that $(n\xi)(n')$ is in N because ${}_{\mathcal{H}(M,M)}N$ is subact of M_S. Thus, we conclude $N = N\mathcal{H}(M, S)(N)$.

$\Leftarrow$ For all $m \in M$, we know that ${}_{\mathcal{H}(M,M)}\mathcal{H}(M, M))(m)$ is subact of ${}_{\mathcal{H}(M,M)}M$. By assumption, we have $\mathcal{H}(M, M)(m) = \mathcal{H}(M, M)(m)\mathcal{H}(M, S)(\mathcal{H}(M, M)(m))$. It follows that,

$$m = I(m) = \psi(m)\xi(\gamma(m)) = \psi(m)(\xi\ \gamma)(m),$$

where $I \in \mathcal{H}(M, M)$ is an identity mapping, $\psi, \gamma \in \mathcal{H}(M, M)$ and $\xi,\ \xi\ \gamma \in \mathcal{H}(M, S)$. Thus, M_S is weakly regular. $\square$

Before we mention our next result, we recall the following definition.

2.7. Definition. An S-act A_S is a *retract* of an S-act B_S if there exist S-homs $\alpha : A_S \to B_S$ and $\beta : B_S \to A_S$ such that $\beta\ \alpha = I_A$, where I_A is the identity mapping from A_S to A_S.

2.8. Lemma. *Every retract of a weakly regular S-act is weakly regular.*

Proof. Let B_S be a retract of a weakly regular S-act M_S. This implies that there exist S-homs $\alpha : B_S \to M_S$ and $\beta : M_S \to B_S$ such that $\beta\ \alpha = I_B$. Let b be in B. We have

(2.1) $\quad b = \beta(\alpha(b)).$

As M_S is weakly regular, there exist S-homs $\psi \in \mathcal{H}(M, M)$ and $\xi \in \mathcal{H}(M, S)$ such that

(2.2) $\quad \alpha(b) = \psi(\alpha(b))\xi(\alpha(b)).$

From equations 2.1 and 2.2, we get

$$b = \beta(\psi(\alpha(b))\xi(b)) = (\beta\ \psi\ \alpha)(b)(\xi\ \alpha)(b),$$

where $\beta\ \psi\alpha \in \mathcal{H}(B, B)$ and $\xi\ \alpha \in \mathcal{H}(B, S)$. Thus, B_S is weakly regular. □

2.9. Definition. Let M_S be an S-act. An element θ in M_S is called a *fixed element* if $\theta t = \theta$ for all t in S.

2.10. Theorem. *Let $\{M_j \mid j \in J\}$ be a family of S-acts, where each M_j has a fixed element. Their disjoint union $\dot{\bigcup}_{j\in J} M_j$ is weakly regular if and only if each M_j is weakly regular.*

Proof. $\Rightarrow$ We show that, for all $j \in J$, M_j is a retract of $\dot{\bigcup} M_j$. To show this we proceed as follows. We define the mapping $\alpha_j : \dot{\bigcup} M_j \to M_j$ by

$$\alpha_j(x) = \begin{cases} x & \text{if } x \in M_j \\ \theta & \text{otherwise} \end{cases}$$

where θ is a fixed element in M_j. It is not hard to show that α_j is an S-hom . Let $\gamma_j : M_j \to \dot{\bigcup} M_j$ be injection, for all $j \in J$. It implies that $\alpha\ \gamma_j = I_{M_j}$. By Lemma 2.8, M_j is weakly regular.

$\Leftarrow$ Let m be in $\dot{\bigcup}_{j\in J} M_j$. This implies that, there exists an i in J such that m is in M_i. As M_i is weakly regular, there exist S-homs $\psi_i \in \mathcal{H}(M_i, M_i)$ and $\xi_i \in \mathcal{H}(M_i, S)$ such that

$$m = \psi_i(m)\xi_i(m). \tag{2.3}$$

We define S-hom $\psi : \dot{\bigcup} M_j \to \dot{\bigcup} M_j$ by $\psi(x) = \psi_j(x)$ (where $\psi_j \in \mathcal{H}(M_j, M_j)$) whenever $x \in M_j$, for all $j \in J$. Let, for all $j \in J$, $\gamma_j : M_j \to \dot{\bigcup} M_j$ be injections. Consider a family $\{\xi_j \in \mathcal{H}(M_j, S), j \in J\}$. By Proposition 1.1, there exists a unique S-hom $\xi : \dot{\bigcup} M_j \to S$ with $\xi(y) = \xi_j(y)$, for $y \in M_j$, such that $\xi\ \gamma_j = \xi_j$, for all $j \in J$. Thus, the above Equation 2.3 can be rewritten as

$$m = \psi(m)\xi\ \gamma_i(m) = \psi(m)\xi(m).$$

Thus, $\dot{\bigcup}_{j\in J} M_j$ is weakly regular. □

2.11. Proposition. *Let S_S be a weakly regular S-act. If e is an idempotent element in S, then eS_S is weakly regular.*

Proof. It is enough to show that eS_S is a retract of S_S. To show this we begin by defining the mapping $\alpha : S_S \to eS_S$ by $\alpha(t) = et$, for all $t \in S$. Clearly, α is S-hom. Suppose that $\beta : eS_S \to S_S$ is an inclusion mapping. Let et be an element of eS. We have $\alpha\ \beta(et) = \alpha(et) = e^2t = et$. This implies that $\alpha\ \beta = I_{eS}$. Thus, eS_S is a retract of S_S. □

2.12. Theorem. *A monoid S is a weakly regular S-act if and only if every projective S-act is weakly regular.*

Proof. $\Rightarrow$ Let A_S be a projective S-act. By Proposition 1.2, we have $A_S = \dot{\bigcup} P_j$, where P_j is isomorphic to e_jS, e_j is idempotent in S, for all $j \in J$. By Proposition 2.11, each e_jS is a weakly regular S-act. From Theorem 2.10, it follows that $\dot{\bigcup} e_jS$ is a weakly regular S-act. Hence, A_S is weakly regular.

$\Leftarrow$ Since every monoid S (considered as a right S-act) is projective. So by our assumption, S is weakly regular S-act. □

Let us recall that an S-act is called free if it has a basis. We borrow the following proposition from [4].

2.13. Proposition. *Every free S-act is projective.*

Now, we are ready to prove our next result.

2.14. Proposition. *As S-act S_S is weakly regular if and only if every free S-act is weakly regular.*

Proof. $\Rightarrow$ The proof is evident from Theorem 2.12 and Proposition 2.13.
$\Leftarrow$ As every monoid S is free with basis $\{1\}$, where 1 is the identity element in S. Therefore, S_S is free with basis $\{1\}$. We conclude that S_S is weakly regular. $\square$

2.15. Proposition. *Let M_S be a free S-act with basis $\{u_j\}$, $j \in J$. Then for all $j \in J$, S-act u_jS_S is a retract of S_S.*

Proof. For all $j \in J$: we define the mapping $\alpha_j : u_jS \to S_S$ by $\alpha_j(u_jx) = x$, for all $x \in S$. We also define the mapping $\beta_j : S_S \to u_jS$ by $\beta_j(y) = u_jy$, where y is in S. It is not hard to show that these mappings are S-homs . Clearly, $\beta\ \alpha = I_{u_jS}$. Hence, u_jS_S is a retract of S_S. $\square$

A semigroup X is called *von-Neumann regular* if for any $x \in X$, there exists an element y in X such that $x = xyx$. We extend the von-Neumann regularity of semigroups to S-acts through the following definition.

2.16. Definition. An S-act M_S is called *von-Neumann regular* if for all $m \in M$ there exists an S-act $\xi \in \mathcal{H}(M, S)$ such that $m = m\xi(m)$.

The immediate consequence of this definition is the following result, whose proof is straightforward.

2.17. Lemma. *Every von-Neumann regular S-act is weakly regular.* $\square$

2.18. Lemma. *If S is von-Neumann regular monoid then every weakly regular S-act is von-Neumann regular.*

Proof. Suppose M_S is a weakly regular. For all $m \in M$, there exist S-homs $\psi \in \mathcal{H}(M, M)$ and $\xi \in \mathcal{H}(M, S)$ such that

(2.4) $\quad m = \psi(m)\xi(m).$

As S is von-Neumann regular monoid, there exists an element $x \in S$ such that

(2.5) $\quad \xi(m) = \xi(m)x\xi(m).$

Putting Equation 2.5 in Equation 2.4, we get

(2.6) $\quad m = \psi(m)\xi(m)x\xi(m) = mx\xi(m).$

We define the mapping $\phi : M_S \to S_S$ by $\phi(m) = x\xi(m)$. Clearly, ϕ is S-hom . We rewrite Equation 2.6 as $m = m\phi(m)$. Hence, M_S is von-Neumann regular. $\square$

From the above two lemmas it follows that if a monoid S is von-Neumann regular then the concept of von-Neumann regularity and weak regularity coincides over S-acts. We formalize this observation in the following theorem.

2.19. Theorem. *If a monoid S is von-Neumann regular, then for an S-act M_S the following are equivalent:*

(1) M_S is weakly regular,
(2) M_S is von-Neumann regular. $\square$

3. Locally Projective S-acts

We recall that an S-act P_S is *projective* if for every S-epi $g : M_S \to N_S$ (where M_S and N_S are any two S-acts) and every S-hom $h : P_S \to N_S$, there exists an S-hom $k : P_S \to M_S$ such that $gk = h$. We generalize the concept of projective in the following definition.

3.1. Definition. An S-act M_S is called *locally projective* if for all $m \in M$ there exists an element $m' \in M$ and an S-hom $\xi \in \mathcal{H}(M, S)$ such that $m = m'\xi(m)$.

It follows immediately that a weakly regular S-act is locally projective. We formalize this in the following lemma.

3.2. Lemma. *Every weakly regular S-act is locally projective.* □

Our next lemma follows from the lemma above and Lemma 2.17.

3.3. Lemma. *For an S-act we have the following implications:*

von-Neumann regular ⇒ Weakly regular ⇒ Locally projective. □

3.4. Theorem. *Every projective S-act is locally projective.*

Proof. Let M_S be a projective S-act. By Proposition 1.2, we can write $M_S = \dot{\bigcup} P_j$, where $P_j \cong e_jS$, e_j is an idempotent element of S, for all $j \in J$. We represent the isomorphism between P_j and e_jS by α_j for all $j \in J$. Let m be in M. This implies that there exists an i in J for which m is in P_i. There exists an element s in S such that

$$\begin{aligned} m &= \alpha_i(e_i s) \\ &= \alpha_i(e_i(e_i s)) \\ &= \alpha_i(e_i)e_i s \\ &= m'e_i s, \end{aligned}$$

where $m' = \alpha_i(e_i) \in P_i$. As α_i is S-iso, we can define its invrese. Assume that $\alpha_i^{-1} : P_i \to e_iS$ is the inverse of α_i. Thus, $\alpha_i^{-1} = e_i s$. By Proposition 1.1, there exists the unique S-hom $\alpha : \dot{\bigcup} P_j \to e_iS$ with $\alpha(x) = \alpha_i^{-1}(x)$ for $x \in P_i$ such that $\alpha\beta_i = \alpha_i^{-1}$, where β_i is injection from P_i to $\dot{\bigcup} P_j$. It follows that

$$\begin{aligned} m &= m'e_i s \\ &= m'\alpha_i^{-1}(m) \\ &= m'(\alpha\beta_i)(m) \\ &= m'\alpha(m) \end{aligned}$$

Thus, $\dot{\bigcup} P_j$ is locally projective, so is M_S. □

3.5. Lemma. *A retract of a locally projective S-act is locally projective.*

Proof. Let an S-act M_S be locally projective. Suppose an S-act A_S is a retract of M_S. There exist S-homs $\alpha : A_S \to M_S$ and $\beta : M_S \to A_S$ such that $\beta\ \alpha = I_A$. To show A_S is locally projective, we proceed as follows. Let a be an element in A. We write

$$a = \beta\ \alpha(a). \tag{3.1}$$

We set $\alpha(a) = m$. As M_S is locally projective, for m there exists $m' \in M$ and $\xi \in \mathcal{H}(M, S)$ such that $m = m'\xi(m)$. Putting the value of m in Equation 3.1, we get

$$\begin{aligned} a &= \beta(m'\xi(m)) \\ &= \beta(m')\xi(m) \\ &= \beta(m')\xi(\alpha(a)) \\ &= a'(\xi\ \alpha)(a), \end{aligned}$$

where $a' = \beta(m')$. Thus, A_S is locally projective. □

3.6. Theorem. *Let $\{M_j \mid j \in J\}$ be a family of S-acts, where each M_j has a fixed element. Their disjoint union $\dot{\bigcup}_{j\in J} M_j$ is locally projective if and only if each M_j is locally projective.*

Proof. $\Rightarrow$ For all $j \in J$. Let $\gamma_j : M_j \to \dot{\bigcup} M_j$ be injection. We define the mapping $\alpha_j : \dot{\bigcup} M_j \to M_j$ by

$$\alpha_j(x) = \begin{cases} x & \text{if } x \in M_j \\ \theta & \text{otherwise} \end{cases}$$

where θ is a fixed element in M_j. It is not hard to show that α_j is S-hom . Clearly, $\alpha_j\ \gamma_j = I_{M_j}$. Each M_j is a retract of $\dot{\bigcup} M_j$. By Lemma 3.5, each M_j is locally projective.

$\Leftarrow$ Let m be an element in $\dot{\bigcup} M_j$. This implies that there exists an $i \in J$ for which $m \in M_i$. As M_i is locally projective, there exists an element $m' \in M_i$ and an S-hom $\xi_i \in \mathcal{H}(M_i, S)$ such that

(3.2) $\qquad m = m'\xi_i(m).$

We assume that $\beta_j : M_j \to \dot{\bigcup}_{j\in J} M_j$ are injections and a family $\{\xi_j \in \mathcal{H}(M_j, S), j \in J\}$. By Proposition 1.1, there exists the unique S-hom $\bar{\xi} : \dot{\bigcup} M_j \to S_S$ with $\bar{\xi}(x) = \xi_j(x)$ (where $x \in M_j$) such that $\bar{\xi}\beta_j = \xi_j$ for all $j \in J$. Thus, Equation 3.2 can be written as:

$$\begin{aligned} m &= m'(\bar{\xi}\beta i)(m) \\ &= m'\bar{\xi}(\beta_i(m)) \\ &= m'\bar{\xi}(m). \end{aligned}$$

Hence, $\dot{\bigcup}_{j\in J} M_j$ is locally projective. □

3.7. Definition. An S-subact N_S of an S-act M_S is called *ideal pure* if

$$N_S\mathcal{I} = M_S\mathcal{I} \cap N_S,$$

for all left ideal $\mathcal{I}$ of S.

3.8. Proposition. *A subact of a locally projective S-act is locally projective if the subact is ideal pure.*

Proof. Let n be an element in N. As M_S is locally projective, there exists an $m \in M$ and an S-hom $\xi \in \mathcal{H}(M, S)$ such that $n = m\xi(n)$. Let $\hat{\xi}$ be the restriction of ξ to N_S, that is, $\xi\mid_{N_S} = \hat{\xi}$. We can rewrite the above equation as $n = m\hat{\xi}(n)$. For simplicity, we set $\hat{\xi}(n) = x$. Consider the left ideal $\mathcal{I} = Sx$ generated by x. As N_S is ideal pure, this implies that $N_S\mathcal{I} = M_S\mathcal{I} \cap N_S$. We get,

$$\begin{aligned} n &= mx \in M_S\mathcal{I} \cap N_S = N_S\mathcal{I} \\ &= n'tx \text{ for some } n' \in N, tx \in \mathcal{I} \\ &= n''x, \text{ where } n'' = n't \in N \\ &= n''\hat{\xi}(n). \end{aligned}$$

Hence, N_S is locally projective. □

3.9. Theorem. *The following are equivalent.*

(1) An S-act M_S is weakly regular.

(2) M_S is locally projective and every $\mathcal{H}(M,M)$-S-bisubact of ${}_{\mathcal{H}(M,M)}M_S$ is ideal pure.

(3) M_S is locally projective and for all $m \in M$, $\mathcal{H}(M,M)mS$ is ideal pure.

Proof. (1) ⇒ (2) Let M_S be weakly regular S-act. By Lemma 3.2, M_S is locally projective. To show that $\mathcal{H}(M,M)$-S-bisubact ${}_{\mathcal{H}(M,M)}N_S$ of ${}_{\mathcal{H}(M,M)}M_S$ is ideal pure, we proceed as follows. Let x be an element in $M_S\mathcal{I} \cap N_S$, where $\mathcal{I}$ is an ideal of S. As by Proposition 2.4, ${}_{\mathcal{H}(M,M)}N_S$ is weakly regular, therefore, for the element x, there exist $\psi \in \mathcal{H}(N,N)$ and $\xi \in \mathcal{H}(N,S)$ such that $x = \psi(x)\xi(x)$. As x is in $M_S\mathcal{I}$ too, there exist elements $m \in M$ and $t \in \mathcal{I}$ such that $x = mt$. We can write

$$\xi(x) = \xi(mt) = \xi(m)t \in S\mathcal{I} \subseteq \mathcal{I}.$$

It follows that $x = \psi(x)\xi(x)$ is in $N_S\mathcal{I}$. So, $M_S\mathcal{I} \cap N_S$ is contained in $N_S\mathcal{I}$. Clearly, $N_S\mathcal{I}$ is contained in N_S. Hence, $M_S\mathcal{I} \cap N_S = N_S\mathcal{I}$.

(2) ⇒ (3) As for all $m \in M$, $\mathcal{H}(M,M)mS_S$ is $\mathcal{H}(M,M)$-S-bisubact of ${}_{\mathcal{H}(M,M)}M_S$, so by our assumption in (2), $\mathcal{H}(M,M)mS_S$ is ideal pure.

(3) ⇒ (1) As M_S is locally projective, for all $m \in M$, there exists $m' \in M$ and S-hom $\xi \in \mathcal{H}(M,M)$ such that $m = m'\xi(m)$. By Lemma 2.5, the mapping $m'\xi$ is in $\mathcal{H}(M,M)$ and by the fact that S contains the identity element 1, it follows that the m is in

$$M_S\xi(m) \cap \mathcal{H}(M,M)mS.$$

By our assumption in (3), $\mathcal{H}(M,M)mS$ is ideal pure. Consider the left ideal $S\xi(m)$ of S. We can write

$$\begin{aligned} \mathcal{H}(M,M)mSS\xi(m) = \mathcal{H}(M,M)mS\xi(m) &= M_SS\xi(m) \cap \mathcal{H}(M,M)mS \\ &= M_S\xi(m) \cap \mathcal{H}(M,M)mS. \end{aligned}$$

This implies that m is in $\mathcal{H}(M,M)mS\xi(m)$. This implies that there exists $\psi \in \mathcal{H}(M,M)$ and $u \in S$ such that

$$\begin{aligned} m &= \psi mt\xi(m) \\ &= (\psi(mt))\xi(m) \\ &= \psi(m)t\xi(m), \end{aligned}$$

where $t\xi \in \mathcal{H}(M,M)$. Thus, M_S is weakly regular. □

Before we begin our next result we define PM-injective S-acts stated in [1]. Let M_S ba a fixed S-act. We say an S-act A_S is PM-injective if each S-hom from a cyclic S-subact mS (for all $m \in M$) of M_S to A_S extends to an S-hom from M_S to A_S.

3.10. Theorem. *The following are equivalent:*

(1) An S-act M_S is von-Neumann regular.

(2) M_S is locally projective and every S-act is PM-injective.

(3) M_S is locally projective and for each $m \in M$, mS_S is PM-injective.

Proof. (1) $\Rightarrow$ (2) Let an S-act M_S be von-Neumann regular. By Lemma 3.3, M_S is locally projective. Let Q_S be an S-act. To show that Q_S is PM-injective, we proceed as follows. Assume that $\beta : mS_S \to Q_S$ is S-hom from a cyclic S-subact mS (where $m \in M$) of the M_S to Q_S. We define the mapping $\alpha : S_S \to mS_S$ by $\alpha(t) = mt$. Clearly, α is S-hom . Consider an S-hom ξ from M_S to S_S. Such S-hom exsits as M_S is locally projective. Now, the mapping $\beta\alpha\xi : M_S \to Q_S$ is the required S-hom that extends β. Hence, Q_S is PM-injective.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) As M_S is locally projective, for all $m \in M$, there exists $m' \in M$ and S-hom $\xi \in \mathcal{H}(M, S)$ such that

(3.3) $\quad m = m'\xi(m).$

Let $\hat{\xi} : mS_S \to S_S$ be restriction of ξ. Let $I : mS_S \to mS_S$ be identity mapping. As mS_S is PM-injective, there exists an extension, say $\rho : M_S \to mS$, of I. We know that $\hat{\xi}\rho(m) = \xi(m)$. We can write

(3.4) $\quad m'\hat{\xi}\rho(m) = m'\xi(m).$

Consider Equation 3.3:

$$\begin{aligned} m &= m'\hat{\xi}(m) \\ &= m'\hat{\xi}\rho(m); \quad \text{Using Equation 3.4} \\ &= m'\hat{\xi}\rho(m'\xi(m)); \quad \text{Using Equation 3.3} \\ &= m'\hat{\xi}\rho(m')\xi(m) \\ &= m'\hat{\xi}(mt)\xi(m); \quad \text{Aussuming that } \rho(m') = mt \text{ for some } t \in S \\ &= m'\hat{\xi}(m)t\xi(m) \\ &= m'\xi(m)(t\xi)(m) \\ &= m(t\xi)(m), \end{aligned}$$

where $t\xi \in \mathcal{H}(M, S)$. Thus, M_S is von-Neumann regular. □

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