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Test for single-index composite quantile regression

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Abstract

It is known that composite quantile regression estimator could be much more efficient and sometimes arbitrarily more efficient than the least squares estimator. In this paper, tests for the index parameter and index function in the single-index composite quantile regression are considered. The asymptotic behaviors of the proposed tests are established and their limiting null distributions are demonstrated to follow an asymptotically χ^2-distribution. The simulation studies and a real data application are conducted to illustrate the finite sample performance of the proposed methods.

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1. Introduction

We consider the single-index model

$$Y = g_0(X^T\gamma_0) + \varepsilon, \tag{1.1}$$

where Y is a response variable and X is a vector of p-dimensional covariates; $g_0(\cdot)$ is an unknown univariate measurable function; γ_0 is the unknown single-index vector coefficient and $\|\gamma_0\| = 1$ for model identifiability, where $\|\cdot\|$ denotes the Euclidean norm; the error ε is independent of X with $E(\varepsilon) = 0$.

Single-index models (SIMs) provide an efficient way of coping with high-dimensional nonparametric estimation problems and avoid the "curse of dimensionality" by assuming that the response is only related to a single linear combination of the covariates. Various methods are available for fitting the SIMs. Härdle and Stoker (1989) employed the average derivative method (ADE) to study the SIMs. Ichimura (1993) studied the properties

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of a semiparametric least-squares estimator in a general single-index model. Yu and Ruppert (2002) proposed the penalized spline estimation procedure, while Xia and Härdle (2006) applied the minimum average variance estimation method, which was originally introduced by Xia et al. (2002) for dimension reduction. Wu et al. (2010) proposed a back-fitting algorithm which was shown to be more efficient than the ADE, to serve the same purpose in single-index quantile regression. Cui et al. (2011) considered an estimating function approach to generalized single-index models with an efficient fixed point algorithm. Zhu et al. (2012) proposed a computationally efficient two-step estimation procedure to estimate the parameters involved in the quantile regression function. Other works about SIMs can see Xue and Zhu (2006); Huang and Zhang (2011); Huang et al. (2013) and so on.

Some authors also studied testing problems for the index parameter γ_0 and the index function $g(\cdot)$ based on least square method. For example, Zhang et al. (2010) employed the generalized likelihood ratio (GLR) test method to the testing problem for the parametric components in the SIMs. The GLR test was first proposed by Fan et al. (2001) for nonparametric models. Huang and Zhang (2010) employed the GLR test for varying-coefficient single-index model. Liang et al. (2010) used GLR test and Wald test to deal with the testing issue of the index parameters and proposed a goodness-of-fit test to the nonparametric component in the partially linear single-index models. Huang and Zhang (2012) extended GLR test method to test the parametric parts in a single-index varying-coefficient model. Wong et al. (2013) constructed a generalized F-type testing statistic to test the significance of index parameters in varying-coefficient single-index models. However, the usual least squares method is sensitive to outliers and does not perform well when the error distribution is heavily skewed. Since outliers or aberrant observations are usually observed in regression model as well as in many other fields, a reliable treatment of outliers is an important step in highlighting features of a dataset. The least absolute deviation method is an obvious alternative to the least squares. However, the relative efficiency of the least absolute deviation can be arbitrarily small when compared with the least squares. Therefore, we do not consider it as a safe alternative to the least squares.

The composite quantile regression was first proposed by Zou and Yuan (2008) for estimating the regression coefficients in the classical linear regression model. Zou and Yuan (2008) showed that the relative efficiency of the CQR estimator compared with the least squares estimator is greated than 70% regardless of the error distribution. Moreover, CQR could be much more efficient and sometimes arbitrarily more efficient than the least squares. Based on CQR, Kai, Li and Zou (2010) proposed the local polynomial CQR estimators for estimating the nonparametric regression function and its derivative. It is shown that the local CQR method can significantly improve the estimation efficiency of the local least squares estimator for commonly-used non-normal error distributions. Kai, Li and Zou (2011) studied semiparametric CQR estimates for semiparametric varying-coefficient partially linear model. Jiang et al. (2012a) considered CQR method for random censored data. Recently, Jiang et al. (2012b) extended CQR method to single-index model. Furthermore, Jiang et al. (2013) proposed a computationally efficient two-step composite quantile regression for single-index model. These nice theoretical properties of CQR motivates us to consider testing problem for single index models based on CQR method so as to make test statistics more effective and robust.

To the best knowledge of the authors, tests for the hypothesis of the index parameter γ_0 and index function $g_0(\cdot)$ in single index composite quantile regression have not been considered so far in the literature, and the purpose of the present paper is to propose test method for this problem. The asymptotic behavior of the proposed test is demonstrate that its limiting null distribution follows a χ^2-distribution. It avoids density estimation by introducing a novel and intriguing resampling scheme, so that the distribution of the test statistic can be approximated.

The paper is organized as follows. In section 2, we introduce the test procedures for model (1.1), and the main theoretical restults are also given in this section. Both simulation examples and the application of real data are given in Section 3 to illustrate the proposed procedures. Final remarks are given in Section 4. All the conditions and technical proofs are deferred to the Appendix.

2. Methodology

2.1. Estimation. We first briefly recall the CQR method for SIMs. Suppose that $\{x_i, y_i\}_{i=1}^n$ is an independent identically distributed (i.i.d.) sample from (X, Y). For $x_i^T\gamma$ "close to" u, $g(x_i^T\gamma)$ can be approximated linearly by

$$g(x_i^T\gamma) \approx g(u) + g'(u)(x_i^T\gamma - u) = a + b(x_i^T\gamma - u),$$

where $a \triangleq g(u)$ and $b \triangleq g'(u)$. Let $\rho_{\tau_k}(r) = \tau_k r - rI(r < 0), k = 1, 2, \ldots, q$, be q check loss functions at q quantile positions: $\tau_k = k/(q+1)$. Let $K(\cdot)$ be the kernel weight function and h is the bandwidth.

The estimation procedures for estimating γ and $g(\cdot)$ are stated as follows:

Step 0 (Initialization step). Obtain initial $\hat{\gamma}^{(0)}$ from minimum average variance estimation (MAVE) of Xia and Härdle (2006). Standardize the initial estimate such that $\|\hat{\gamma}\| = 1$ and $\hat{\gamma}_1 > 0$.

Step 1. Given $\hat{\gamma}$, obtain $\{\hat{a}_{1j}, \ldots, \hat{a}_{qj}, \hat{b}_j\}_{j=1}^n$ by solving a series of the following

$$\min_{(a_{1j},\ldots,a_{qj},b_j)} \sum_{k=1}^{q}\sum_{i=1}^{n} \rho_{\tau_k}\{y_i - a_{kj} - b_j(x_i - x_j)^T\hat{\gamma}\}\omega_{ij},$$

where $\omega_{ij} = K\left(\frac{x_i^T\hat{\gamma} - x_j^T\hat{\gamma}}{h}\right) / \sum_{l=1}^n K\left(\frac{x_l^T\hat{\gamma} - x_j^T\hat{\gamma}}{h}\right)$ and with the bandwidth h chosen optimally.

Step 2. Given $\{\hat{a}_{1j}, \ldots, \hat{a}_{qj}, \hat{b}_j\}_{j=1}^n$, obtain $\hat{\gamma}$ by solving

$$\min_{\gamma} \sum_{j=1}^{n}\sum_{k=1}^{q}\sum_{i=1}^{n} \rho_{\tau_k}\{y_i - \hat{a}_{kj} - \hat{b}_j(x_i - x_j)^T\gamma\}\omega_{ij},$$

with ω_{ij} evaluated at γ and h from step 1.

Step 3. Repeat Steps 1 and 2 until convergence.

Step 4. Fix γ at its estimated value from Step 3. The final estimate of $g(\cdot)$ is $\hat{g}(u; h, \hat{\gamma}) = \frac{1}{q}\sum_{k=1}^q \hat{a}_k$, where

$$(\hat{a}_1, \ldots, \hat{a}_q, \hat{b}) = \arg\min_{(a_1,\ldots,a_q,b)} \sum_{k=1}^{q}\sum_{i=1}^{n} \rho_{\tau_k}\{y_i - a_k - b(x_i^T\hat{\gamma} - u)\}K\left(\frac{x_i^T\hat{\gamma} - u}{h}\right).$$

Remark 1. The choice of the bandwidths can be found in Remark 1 of Jiang et al. (2012b).

2.2. Testing index parameter. Consider the following testing problem

$$H_0 : \gamma_{01} = \cdots = \gamma_{0l} = 0 \longleftrightarrow H_1 : \textit{not all } \gamma_{0j} = 0,\ j \le l, l \le p. \tag{2.1}$$

Following the least absolute deviation analysis (Chen et al., 2008), we propose the composite quantile likelihood ratio test method under the null hypothesis. This method is an extension of the least absolute deviation techniques developed by Chen et al. (2008)

to the single index composite quantile regression. To test hypothesis (2.1), we define the composite quantile likelihood ratio test statistics as follows:

$$\lambda_n = \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_\tau\{y_i - \tilde{g}_k(x_{li}^T\tilde{\gamma}_l)\} - \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_\tau\{y_i - \hat{g}_k(x_i^T\hat{\gamma})\},$$

where $\tilde{\gamma}_l$ and $\hat{\gamma}$ are the estimators of γ_{0l} and γ_0 under H_0 and H_1, respectively. $\tilde{g}_k(x_{li}^T\tilde{\gamma}_l)$ and $\hat{g}_k(x_i^T\hat{\gamma})$ are the same definition $\hat{a}_k$ in Step 1 of Section 2.1 given $\tilde{\gamma}_l$ and $\hat{\gamma}$ under H_0 and H_1, respectively. These estimators can be obtained by using the same estimation method as that used in Section 2.1. $X_l = (X_{l+1}, \dots, X_p)$ and $\gamma_l = (\gamma_{l+1}, \dots, \gamma_p)$.

It is nature to see whether the asymptotic null distribution of the test statistic λ_n is a χ^2-distribution. The following theorem answers this question.

Theorem 1. Suppose that conditions (A1-A5) in the Appendix hold. Then, under H_0 (2.1), as $h \to 0$, and $nh \to \infty$,

$$\sigma_n^{-1}(\lambda_n - \mu_n + d_n) \xrightarrow{L} N(0,1),$$

where $\xrightarrow{L}$ stands for convergence in distribution. Furthermore, if $g(\cdot)$ is linear or $nh^{9/2} \to 0$, then, under H_0 (2.1), the test statistic $r_K\lambda_n$ approximately follows a χ^2-distribution with $r_K\mu_n$ degrees of freedom, namely

$$r_K\lambda_n \sim \chi^2(r_K\mu_n),$$

where

$$r_K = 2\mu_n/\sigma_n^2, \quad \mu_n = A(\mu_{pn} - \mu_{ln}), \quad \sigma_n^2 = A^2(\sigma_{pn}^2 + \sigma_{ln}^2), \quad A = \sum_{k=1}^{q}\frac{\tau_k(1-\tau_k)}{f(c_k)},$$

$$\mu_{pn} = \frac{|\Omega|}{h}\left(K(0) - \frac{1}{2}\int_\Omega K^2(t)dt\right), \quad \mu_{ln} = \frac{|\Omega_l|}{h}\left(K(0) - \frac{1}{2}\int_{\Omega_l} K^2(t)dt\right),$$

$$\sigma_{pn}^2 = 2\frac{|\Omega|}{h}\int_\Omega\left(K(t) - \frac{1}{2}K*K(t)\right)^2 dt, \quad \sigma_{ln}^2 = 2\frac{|\Omega_l|}{h}\int_{\Omega_l}\left(K(t) - \frac{1}{2}K*K(t)\right)^2 dt,$$

$$d_n = O_p(\sqrt{n}h^2) + O_p(nh^4),$$

where $K*K$ denotes the convolution of K, $|\Omega|$ is the length of the support of the density $f_1(\cdot)$ of $X^T\gamma$, and $|\Omega_l|$ is the length of the support of the density $f_2(\cdot)$ of $X_l^T\gamma_l$.

2.3. Testing index function. After obtaining nonparametric estimates of $g_0(\cdot)$, researchers frequently ask whether certain parametric models fit the nonparametric components. Without loss of generality, a simple linear model under the null hypothesis is considered. Accordingly, the null and alternative hypotheses are given as follows

$$H_0 : g_0(u) = \alpha + \beta u \longleftrightarrow H_1 : g_0(u) \neq \alpha + \beta u, \ for\ some\ u, \tag{2.2}$$

where α and β are unknown constant parameters. The hypothesis test (2.2) considers a semiparametric null hypothesis versus a semiparametric alternative hypothesis. We consider the following test statistic

$$\tilde{\lambda}_n = \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_{\tau_k}\{y_i - \hat{\alpha}_k - \hat{\beta}x_i^T\hat{\gamma}\} - \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_{\tau_k}\{y_i - \hat{g}(x_i^T\hat{\gamma})\},$$

where $\hat{\alpha}_k$ and $\hat{\beta}$ are the linear composite quantile regression estimators of $\alpha + \tau_k$ and β.

The following theorem shows that the asymptotic null distribution of the test statistic $\tilde{\lambda}_n$ is still following a χ^2-distribution.

Theorem 2. Suppose that conditions (A1-A5) in the Appendix hold. Then, under H_0 (2.2), as $h \to 0$, and $nh \to \infty$,

$$\tilde{\sigma}_n^{-1}(\tilde{\lambda}_n - \tilde{\mu}_n + d_n) \xrightarrow{L} N(0,1).$$

Furthermore, if $g(\cdot)$ is linear or $nh^{9/2} \to 0$, then, under H_0, the test statistic $\tilde{r}_K \tilde{\lambda}_n$ approximately follows a χ^2-distribution with $\tilde{r}_K \tilde{\mu}_n$ degrees of freedom, namely

$$\tilde{r}_K \tilde{\lambda}_n \sim \chi^2(\tilde{r}_K \tilde{\mu}_n),$$

where $\tilde{r}_K = 2\tilde{\mu}_n / \tilde{\sigma}_n^2$, $\tilde{\mu}_n = A\mu_{pn}$ and $\tilde{\sigma}_n^2 = A^2 \sigma_{pn}^2$.

2.4. Bootstrap test. To implement the proposed test, one usually needs to derive the null distribution of the test statistic λ_n (or $\tilde{\lambda}_n$). Although Theorem 1 (or Theorem 2) provides us an asymptotic distribution on λ_n (or $\tilde{\lambda}_n$). It is rather difficult and inaccurate to estimate the nuisance parameter $f(c_k)$. In this section, the bootstrap test is used to implement our proposed test procedure. Following Jiang et al. (2014), we rewrite the details as follows.

Step 1. Compute the test statistic λ_n (or $\tilde{\lambda}_n$) and the residuals $\hat{\varepsilon}_i$, $i = 1, \ldots, n$ from the unrestricted model (1.1) by using the estimation procedure in Section 2.

Step 2. Generate n i.i.d. random variables $\hat{\varepsilon}_1^*, \ldots, \hat{\varepsilon}_n^*$ from the centered empirical distribution of $\hat{\varepsilon}$, and compute $y_i^* = \tilde{g}(x_{li}^T \tilde{\gamma}_{0l}) + \hat{\varepsilon}_i^*$ (or $y_i^* = \sum_{k=1}^q \hat{\alpha}_k + \hat{\beta} x_i^T \hat{\gamma} + \hat{\varepsilon}_i$), $i = 1, \ldots, n$. This forms a bootstrap sample $\{x_i, y_i^*\}_{i=1}^n$.

Step 3. Using the bootstrap sample in step 2, obtain the test statistic λ_n^* in the same manner as λ_n (or $\tilde{\lambda}_n$). Further repeat this many times to generate a sample of statistic λ_n^*.

Step 4. Use the bootstrap sample in step 3 to determine the quantiles of the test statistic under H_0. Furthermore, obtain the p-value by calculating the percentage of observations from the bootstrap sample of λ_n^* whose value exceeds λ_n (or $\tilde{\lambda}_n$).

3. Numerical studies

In this section, we first use Monte Carlo simulation studies to assess the finite sample performance of the proposed procedures and then demonstrate the application of the proposed methods with the real data analysis. The performances of CQR method with different q are very similar (see the simulation part in Kai et al., 2010; Jiang et al., 2012b and Jiang et al., 2013a). Furthermore, Kai et al. (2010, 2011) suggest q=9 for CQR method. Therefore, we only consider q=9 for our proposed test method (CQR_9). Moreover, we compare our method with the method based on least squares (LS) of Zhang et al. (2010) for index parameter and Liang et al. (2010) for index function.

3.1. Simulation example for index parameter. We conduct a small simulation study with $n = 100$ and the data are generated from the following "sine-bump" model

$$Y = sin\{\pi(\gamma_0^T X - A)/(B - A)\} + 0.1\varepsilon,$$

where X is uniformly distributed on $[0,1]^3$, $A = \sqrt{3}/2 - 1.645/\sqrt{12}$ and $B = \sqrt{3}/2 + 1.645/\sqrt{12}$. In our simulation, we consider four error distributions for $\varepsilon : N(0,1)$; $t(3)$; $t(5)$ and $U(-1,1)$ distribution. All of the simulations are run for 1000 replicates and number of resample is $N = 1000$. The null hypothesis is taken as $H_0 : \gamma_{03} = 0$; here the true parameters γ_{01}, γ_{02} are taken as $1/\sqrt{5}$, $2/\sqrt{5}$, respectively.

For the power assessment, we evaluate the power in a sequence of alternatives with parameters γ_{03}, and according to each γ_{03}, the true parameters γ_{01}, γ_{02} are taken as some fixed values for the model identifiability. Tables 1 lists the power functions at significance level $\alpha = 0.05$ for four random error distributions. The results show that the empirical significance levels of two test method are close to the nominal significance levels when the null is true under normal errors. For other non-normal errors, our proposed method (CQR_9) is better than LS method.

Table 1 Empirical significant levels and powers for Example 3.1.

Method	γ_{03}	N(0,1)	t(3)	t(5)	U(-1,1)
LS	0.00	0.045	0.038	0.041	0.044
	0.01	0.098	0.130	0.142	0.163
	0.02	0.250	0.225	0.166	0.331
	0.03	0.558	0.266	0.382	0.510
	0.04	0.570	0.428	0.414	0.898
	0.05	0.744	0.485	0.700	0.979
	0.06	0.913	0.564	0.734	1.000
	0.07	0.940	0.772	0.958	1.000
	0.08	0.964	0.826	0.960	1.000
	0.09	1.000	0.857	1.000	1.000
	0.10	1.000	0.940	1.000	1.000
CQR_9	0.00	0.047	0.048	0.053	0.050
	0.01	0.073	0.176	0.176	0.181
	0.02	0.151	0.314	0.194	0.426
	0.03	0.439	0.375	0.441	0.669
	0.04	0.600	0.675	0.488	0.900
	0.05	0.760	0.720	0.723	0.984
	0.06	0.877	0.800	0.755	1.000
	0.07	0.918	0.868	0.959	1.000
	0.08	0.980	0.895	0.980	1.000
	0.09	1.000	0.959	1.000	1.000
	0.10	1.000	1.000	1.000	1.000

3.2. Simulation example for index function. To examine the performance of the test statistic in Section 2.3, we generated 1000 realizations from the model given below with simple size $n = 100$.

$$Y = g_0(X^T\gamma) + 0.5\varepsilon.$$

Considering the following hypotheses:

$$H_0 : g_0(\mathbb{X}) = \theta_0 + \theta_1\mathbb{X} \longleftrightarrow H_1 : g_0(\mathbb{X}) = \theta_0 + \theta_1\mathbb{X} + r\exp(\pi\mathbb{X}),$$

where r ranges from 0.0 to 1.0. $\mathbb{X} = X^T\gamma$, X is uniformly distributed on $[0, 1]^2$, and the true parameters $\gamma_1 = 1/\sqrt{5}$, $\gamma_2 = 2/\sqrt{5}$, $\theta_0 = 2$ and $\theta_1 = 1$. The random errors and others are generated from the same way as in Section 3.1.

The simulation results are reported in Table 2. The performance of test statistic for index function is the same as that of test statistic for index parameter. Therefore, the test statistic for testing index function is also available.

3.3. Walking behavior data. We also illustrate the methodology via an application to a walking behavior data set. The data set used here consists of weekly measurements of the number of walking times and individual attributes factors based on a travel survey of 126 individuals from Lujiazui Garden neighborhood in Shanghai between February 2012 and April 2012. Five individual attributes factors: age X_1, gender X_2, the highest level of education X_3, number of household X_4, income X_5 are considered. Our main interest is to study the relationship between individual attributes and the number of weekly walking times (Y) in Lujiazui Garden neighborhood in Shanghai.

Table 2 Empirical significant levels and powers for Example 5.2.

Method	r	N(0,1)	t(3)	t(5)	U(-1,1)
LS	0.0	0.044	0.036	0.043	0.040
	0.1	0.221	0.251	0.257	0.302
	0.2	0.329	0.280	0.299	0.554
	0.3	0.482	0.302	0.361	0.758
	0.4	0.593	0.380	0.444	0.924
	0.5	0.758	0.498	0.579	0.994
	0.6	0.888	0.557	0.687	0.996
	0.7	0.946	0.635	0.836	1.000
	0.8	0.970	0.730	0.874	1.000
	0.9	0.994	0.798	0.924	1.000
	1.0	1.000	0.892	0.972	1.000
CQR_9	0.0	0.048	0.050	0.052	0.044
	0.1	0.220	0.526	0.440	0.326
	0.2	0.242	0.683	0.504	0.402
	0.3	0.372	0.742	0.543	0.786
	0.4	0.680	0.823	0.580	0.943
	0.5	0.720	0.905	0.743	0.954
	0.6	0.788	0.960	0.907	0.998
	0.7	0.914	0.987	0.942	1.000
	0.8	0.960	1.000	0.980	1.000
	0.9	1.000	1.000	0.992	1.000
	1.0	1.000	1.000	1.000	1.000

The single-index model

$$Y = g_0(\gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3 + \gamma_4 X_4 + \gamma_5 X_5) + \varepsilon, \tag{3.1}$$

is fitted to the given data. We apply the proposed test method to see whether index function is statistically significant. The bootstrap times is 1000 and only $q = 9$ is considered in this example. The p-value for above test problem are all 0.001. Therefore, the single-index model (3.1) is suit to fit the given data under significance level $\alpha = 0.05$. Furthermore, a natural question is whether the coefficients of $X_1 - X_5$ are statistically significant. To answer this question, the proposed test method for index parameter is employed. The p-values for this problem are summarized in Table 3, which indicates that X_2 and X_5 are not significant under significance level $\alpha = 0.05$.

Finally, the single-index model

$$Y = g_0(\gamma_1 X_1 + \gamma_3 X_3 + \gamma_4 X_4) + \varepsilon,$$

is fitted to the given data. The estimated parametric coefficients are $\gamma_1 = 0.199$; $\gamma_3 = -0.601$ and $\gamma_4 = -0.774$.

Table 3 P-value of the GAVT_γ test for Walking behavior data.

	X_1	X_2	X_3	X_4	X_5
γ	0.143	0.024	-0.566	-0.798	-0.141
P-value	0.044	0.414	0.006	0.029	0.333

4. Conclusion

The aim of this paper is to provide test statistic for linear hypothesis testing problem in SIMs based on composite quantile regression. The proposed inference procedure via resampling avoids the difficulty of density estimation. All simulation studies confirm that the performance of the proposed method works well. The approach described here may be extended to varying coefficient models (Kai et al., 2011), which would be considered in future research.

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Appendix

To prove main results in this paper, the following technical conditions are imposed.

A1. The kernel $K(\cdot)$ is a symmetric density function with finite support.

A2. The density functions of $X^T\gamma$ and $X_l^T\gamma_l$ are compactly supported, bounded, Lipschitz continuous and bounded away from zero by a constant uniformly for γ and γ_l in a neighborhood of γ_0 and γ_{0l}, respectively, and $X^T\gamma$ has a bounded support Ω for γ in a neighborhood of γ_0, $X_l^T\gamma_l$ has a bounded support Ω_l for γ_l in a neighborhood of γ_{0l}. Further the density of $X^T\gamma_0$ and $X_l^T\gamma_l$ are bounded away from 0 and ∞ on its support.

A3. The function $g_0(\cdot)$ has a continuous and bounded second derivative.

A4. The error terms ε has a symmetric distribution with a positive density $f(\cdot)$.

A5. $E|\varepsilon|^4 < \infty$.

Remark 2. Conditions (A1)-(A5) are standard conditions, which are commonly used in single-index regression model, see Wu et al. (2010) and Jiang et al. (2012b).

Lemma. Suppose that the conditions (A1-A5) above hold. Then, as $n \to \infty$, $h \to 0$, and $nh \to \infty$, uniformly in $u \in \Omega$,

$$\hat{\gamma} - \gamma_0 = O_p(n^{-1/2}),$$
$$\hat{g}_k(u) - g_0(u) - c_k = (R(u) + e(u))(1 + o_p(1)),$$

and under H_0 (2.1), uniformly in $t \in \Omega_l$,

$$\tilde{\gamma}_l - \gamma_{0l} = O_p(n^{-1/2}),$$
$$\tilde{g}_k(t) - g_0(t) - c_k = (R_l(t) + e_l(t))(1 + o_p(1)),$$

where $R(u) = \frac{1}{2}\nu_2 g_0''(u)h^2$, $e(u) = \frac{1}{n}f_1^{-1}(u)f^{-1}(c_k)\sum_{i=1}^n \eta_{i,k}K_h(x_k^T\gamma_0 - u)$, $\nu_2 = \int_\Omega u^2K(u)du$, $R_l(t) = \frac{1}{2}\nu_{2l}g_0''(t)h^2$, $e_l(t) = \frac{1}{n}f_1^{-1}(t)f^{-1}(c_k)\sum_{i=1}^n \eta_{i,k}K_h(x_{lk}^T\gamma_{0l} - t)$, $\nu_{2l} = \int_{\Omega_l} t^2K(t)dt$. $\eta_{i,k} = \tau_k - I\{\varepsilon_i \le c_k\}$.

Proof of Lemma. By the proof of Theorem 1 in Jiang et al. (2012b), we can obtain

$$\hat{g}_k(u) - g_0(u) - c_k = \frac{1}{n}f_1^{-1}(u)f^{-1}(c_k)\sum_{i=1}^n \eta_{i,k}^*(u)K_h(x_k^T\gamma_0 - u)(1 + o_p(1)),$$

where $\eta_{i,k}^*(u) = \tau_k - I\{\varepsilon_i \le c_k - [g_0(u_i) - g_0(u) - g_0'(u)(u_i - u)]\}$. By Taylor expression

$$\eta_{i,k}^*(u) = \eta_{i,k} + \frac{1}{2}\eta_{i,k}'(x_k^T\gamma_0 - u)^2(1 + o(1)),$$

then, we can obtain

$$\hat{g}_k(u) - g_0(u) - c_k = \left[\frac{1}{n}f_1^{-1}(u)f^{-1}(c_k)\sum_{i=1}^n \eta_{i,k}K_h(x_k^T\gamma_0 - u) + \frac{1}{2}\nu_2 g_0''(u)h^2(1 + o(1))\right](1 + o_p(1)).$$

The proof of $\hat{\gamma} - \gamma_0 = O_p(n^{-1/2})$ can immediately be obtained from Theorem 3 of Jiang et al. (2012b). This completes the proof of Lemma.

Now we proceed to prove the theorems.

Proof of Theorem 1. We can rewrite λ_n as

$$\begin{aligned}
\lambda_n =& \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k - \left(\tilde{g}_k(x_{li}^T\tilde{\gamma}_l) - g_0(x_{li}^T\gamma_{0l}) - c_k\right)\right\} \\
&- \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k - \left(\hat{g}_k(x_i^T\hat{\gamma}) - g_0(x_{li}^T\gamma_{0l}) - c_k\right)\right\} \\
=& \sum_{k=1}^{q}\left[\sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k - \left(\tilde{g}_k(x_{li}^T\tilde{\gamma}_l) - g_0(x_{li}^T\gamma_{0l}) - c_k\right)\right\} - \sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k\right\}\right] \\
&- \sum_{k=1}^{q}\left[\sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k - \left(\hat{g}_k(x_i^T\hat{\gamma}) - g_0(x_{li}^T\gamma_{0l}) - c_k\right)\right\} - \sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k\right\}\right] \\
\triangleq& \sum_{k=1}^{q}\lambda_{1k} - \sum_{k=1}^{q}\lambda_{2k}.
\end{aligned}$$

By Lemma and the proof of Theorem 1 in Jiang et al. (2014), we can obtain

$$\begin{aligned}
\lambda_{1k} =& -\frac{\tau_k(1-\tau_k)|\Omega_l|}{hf(c_k)}\left(K(0) - \frac{1}{2}\int_{\Omega_l}K^2(t)dt\right) \\
&- \frac{1}{n}\sum_{i,s=1,i\neq s}^{n}\frac{\eta_{i,k}\eta_{s,k}}{f_2(x_{li}^T\gamma_{0l})f(c_k)}\left(K_h(x_{li}^T\gamma_{0l} - x_{ls}^T\gamma_{0l}) - \frac{1}{2}K_h * K_h(x_{li}^T\gamma_{0l} - x_{ls}^T\gamma_{0l})\right) \\
&+ O_p(\sqrt{n}h^2) + O_p(nh^4) \triangleq -A_k\mu_{ln} - W_{lk} + d_n, \\
\lambda_{2k} =& -\frac{\tau_k(1-\tau_k)|\Omega|}{hf(c_k)}\left(K(0) - \frac{1}{2}\int_{\Omega}K^2(t)dt\right) \\
&- \frac{1}{n}\sum_{i,s=1,i\neq s}^{n}\frac{\eta_{i,k}\eta_{s,k}}{f_2(x_i^T\gamma_0)f(c_k)}\left(K_h(x_i^T\gamma_0 - x_s^T\gamma_0) - \frac{1}{2}K_h * K_h(x_i^T\gamma_0 - x_s^T\gamma_0)\right) \\
&+ O_p(\sqrt{n}h^2) + O_p(nh^4) \triangleq -A_k\mu_{pn} - \overline{W}_k + d_n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_n &= A(\mu_{pn} - \mu_{ln}) + \sum_{k=1}^{q}(\overline{W}_k - W_{lk}) + \sum_{k=1}^{q}d_n, \\
&\triangleq \mu_n + W_n + d_n.
\end{aligned}$$

Next we discuss the asymptotic distribution of W_n. Note that,

$$E(W_n) = 0, \quad Var(W_n) = \sigma_n^2(1 + o(1)),$$

where $\sigma_n^2 = A^2(\sigma_{pn}^2 + \sigma_{ln}^2)$. By Proposition 3.2 in De Jong (1987), W_n is asymptotically normal,

$$\sigma_n^{-1}W_n \xrightarrow{L} N(0,1),$$

which implies that,

$$\sigma_n^{-1}(\lambda_n - \mu_n + d_n) \xrightarrow{L} N(0,1).$$

The proof is completed.

Proof of Theorem 2. $\tilde{\lambda}_n$ can be rewrite as

$$\begin{aligned}\tilde{\lambda}_n =& \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k + [c_k + \alpha - \hat{\alpha}_k] + (\beta\gamma_0^T - \hat{\beta}\hat{\gamma}^T)x_i\right\} \\ &- \sum_{k=1}^{q}\sum_{i=1}^{n}\rho_{\tau}\left\{\varepsilon_i - c_k + [c_k + g_0(x_i^T\gamma_0) - \hat{g}_k(x_i^T\hat{\gamma})]\right\} \\ =& \sum_{k=1}^{q}\left[\sum_{i=1}^{n}\rho_{\tau_k}\left\{\varepsilon_i - c_k + [c_k + \alpha - \hat{\alpha}_k] + (\beta\gamma_0^T - \hat{\beta}\hat{\gamma}^T)x_i\right\} - \sum_{i=1}^{n}\rho_{\tau}\left\{\varepsilon_i - c_k\right\}\right] \\ &+ \sum_{k=1}^{q}\left[\sum_{i=1}^{n}\rho_{\tau}\left\{\varepsilon_i - c_k + [c_k + g_0(x_i^T\gamma_0) - \hat{g}_k(x_i^T\hat{\gamma})]\right\} - \sum_{i=1}^{n}\rho_{\tau}\left\{\varepsilon_i - c_k\right\}\right] \\ \triangleq& \sum_{k=1}^{q}\tilde{\lambda}_{1k} - \sum_{k=1}^{q}\lambda_{2k}.\end{aligned}$$

By the Theorem 1 in Zou et al. (2008), we can obtain following results,

$$\hat{\alpha}_k - \alpha - c_k = O_p(n^{-1/2}), \quad \hat{\beta}\hat{\gamma}^T - \beta\gamma_0^T = O_p(n^{-1/2}),$$

then, we have $\sum_{k=1}^{q}\tilde{\lambda}_{1k} = O_p(1)$, and by the proof of Theorem 1, then we can obtain

$$\tilde{\lambda}_n = A\mu_{pn} + \sum_{k=1}^{q}\overline{W}_k + d_n = \tilde{\mu}_n + \sum_{k=1}^{q}\overline{W}_k + d_n.$$

Similar to the proof of Theorem 1, which implies that,

$$\tilde{\sigma}_n^{-1}(\tilde{\lambda}_n - \tilde{\mu}_n + d_n) \xrightarrow{L} N(0,1).$$

The proof is completed.

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