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Some new integral inequalities for functions whose nth derivatives in absolute value are (α, m)-convex functions

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Abstract: In this paper, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we established some new integral inequalities for functions whose nth derivatives in absolute value are (α, m)-convex functions.

Keywords: Convex function, (α, m)-convex function, Hölder integral inequality and power-mean integral inequality.

1 Introduction

In this work, we establish some new integral inequalities for functions whose nth derivatives in absolute value are (α, m)-convex functions. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For some inequalities, generalizations and applications concerning convexity see [5, 7-9, 12, 14, 16-18, 20]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [1, 3, 4, 6, 16, 19, 20], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of (α, m)-convex functions see for instance the recent papers [1, 2, 10, 11, 13, 15] and the references within these papers.

Definition 1. *A function $f : I \subseteq R \to R$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.

In [18], G. Toader defined m-convexity as the following.

Definition 2. *The function $f : [0,b] \to R,\ b > 0$, is said to be m-convex, where $m \in [0,1]$, if we have*

$$f(tx + m(1-t)y) \leq t f(x) + m(1-t) f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$. Denote by $K_m(b)$ the set of the m-convex functions on $[0,b]$ for which $f(0) \leq 0$.

In [17], V.G. Miheşan defined (α, m)-convexity as the following.

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Definition 3. *The function* $f:[0,b]\to R,\ b>0$*, is said to be* (α,m)*-convex, where* $(\alpha,m)\in[0,1]^2$ *, if we have*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

for all $x,y\in[0,b]$*and* $t\in[0,1]$*. Denote by* $K_m^{\alpha}(b)$ *the class of all* (α,m)*-convex functions on* $[0,b]$ *for which* $f(0)\le 0$*. It can be easily seen that for* $(\alpha,m)=(1,m)$*,* (α,m)*-convexity reduces to m-convexity;* $(\alpha,m)=(\alpha,1)$*,* (α,m)*-convexity reduces to* α*-convexity and for* $(\alpha,m)=(1,1)$*,* (α,m)*-convexity reduces to the concept of usual convexity defined on* $[0,b]$*,* $b>0$*. For recent results and generalizations concerning* (α,m)*-convex functions, see ([2] and [15]).*

Let $0<a<b$, throughout this paper we will use

$$A(a,b)=\frac{a+b}{2},\ L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ a\neq b,\ p\in R,\ p\neq -1,0,$$

for the arithmetic, geometric, logarithmic, generalized logarithmic mean for $a,b>0$ respectively.

2 Main Results

Especially we note that;

(i) In case of $m=1$ and $\alpha=s$, the results are obtained in this paper reduce to the results obtained for s-convex functions in the first sense in [14].

(ii) In case of $\alpha=m=1$, the results are obtained in this paper reduce to the results obtained for convex functions in [16].

We will use the following Lemma [16] in order to obtain the main results.

Lemma 1. *Let* $f:I\subseteq R\to R$ *be n-times differentiable mapping on* I° *for* $n\in N$ *and* $f^{(n)}\in L[a,b]$*, where* $a,b\in I^{\circ}$ *with* $a<b$*, we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)\,dx=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)\,dx.$$

where an empty sum is understood to be nil.

Theorem 1. *For all* $n\in N$*; let* $f:I\subseteq[0,\infty)\to R$*be n-times differentiable function on* I° *and* $a,b\in I^{\circ}$ *with* $a<b$*. If* $f^{(n)}\in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is* (α,m)*-convex on* $[a,b]$*, then the following inequality holds.*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}(b-a)L_{np}^n(a,b)\left[\frac{f(b)-mf\left(\frac{a}{m}\right)}{\alpha+1}+mf\left(\frac{a}{m}\right)\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$*.*

Proof. Since $\left|f^{(n)}\right|^q$ for $q>1$ is (α,m)-convex on $[a,b]$, using Lemma 1, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+m\frac{b-x}{b-a}\frac{a}{m}\right)\right|^q\le\left[\frac{x-a}{b-a}\right]^{\alpha}f(b)+m\left[1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right]f\left(\frac{a}{m}\right)$$

we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left(\left[\frac{x-a}{b-a}\right]^{\alpha} f(b)+m\left[1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right] f\left(\frac{a}{m}\right)\right)dx\right)^{\frac{1}{q}}$$
$$=\frac{1}{n!}\left(\frac{b^{np+1}}{np+1}-\frac{a^{np+1}}{np+1}\right)^{\frac{1}{p}}\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\left[\frac{f(b)}{(b-a)^{\alpha}}-m\frac{f\left(\frac{a}{m}\right)}{(b-a)^{\alpha}}\right]+m\frac{f\left(\frac{a}{m}\right)}{(b-a)^{\alpha}}(b-a)^{\alpha+1}\right)^{\frac{1}{q}}$$
$$=\frac{1}{n!}\left(\frac{b^{np+1}-a^{np+1}}{np+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{\alpha+1}\left[f(b)-mf\left(\frac{a}{m}\right)\right]+mf\left(\frac{a}{m}\right)(b-a)\right)^{\frac{1}{q}}$$
$$=\frac{1}{n!}(b-a)^{\frac{1}{p}}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left(\frac{b-a}{\alpha+1}\left[f(b)-mf\left(\frac{a}{m}\right)\right]+mf\left(\frac{a}{m}\right)(b-a)\right)^{\frac{1}{q}}$$
$$=\frac{1}{n!}(b-a)^{\frac{1}{p}+\frac{1}{q}}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left[\frac{f(b)-mf\left(\frac{a}{m}\right)}{\alpha+1}+mf\left(\frac{a}{m}\right)\right]^{\frac{1}{q}}$$
$$=\frac{1}{n!}(b-a)L_{np}^{n}(a,b)\left[\frac{f(b)-mf\left(\frac{a}{m}\right)}{\alpha+1}+mf\left(\frac{a}{m}\right)\right]^{\frac{1}{q}}$$

This completes the proof of theorem.

Corollary 1. *Under the conditions Theorem 1 for $n=1$we have the following inequality,*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq L_p(a,b)\left[\frac{f(b)-mf\left(\frac{a}{m}\right)}{\alpha+1}+mf\left(\frac{a}{m}\right)\right]^{\frac{1}{q}}.$$

Theorem 2. *For all $n\in N$; let $f:I\subseteq[0,\infty)\to R$ be n-times differentiable function and $0\leq a<b$. If $\left|f^{(n)}\right|^q\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q\geq 1$ is (α,m)-convex on $[a,b]$, then the following inequality holds.*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leq\frac{1}{n!}(b-a)^{1-\frac{1}{q}}L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[\frac{M}{(b-a)^{\alpha}}\left[f(b)-mf\left(\frac{a}{m}\right)\right]+m(b-a)f\left(\frac{a}{m}\right)L_n^n(a,b)\right]^{\frac{1}{q}}$$

where $M=M(a,b,\alpha,n)=\int_a^b x^n(x-a)^{\alpha}dx$.

Proof. From Lemma 1 and Power-mean integral inequality, we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$
$$\leq\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$\leq\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left(\left[\frac{x-a}{b-a}\right]^{\alpha}f(b)+m\left[1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right]f\left(\frac{a}{m}\right)\right)dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\frac{f(b)}{(b-a)^{\alpha}}\int_a^b x^n (x-a)^{\alpha}dx + \frac{mf\left(\frac{a}{m}\right)}{(b-a)^{\alpha}}\int_a^b x^n\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{f(b)}{(b-a)^{\alpha}}M + mf\left(\frac{a}{m}\right)\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right) - \frac{mf\left(\frac{a}{m}\right)}{(b-a)^{\alpha}}M\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)^{1-\frac{1}{q}}\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{1-\frac{1}{q}}$$

$$\times\left(\frac{M}{(b-a)^{\alpha}}\left[f(b)-mf\left(\frac{a}{m}\right)\right] + m(b-a)f\left(\frac{a}{m}\right)\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)^{1-\frac{1}{q}}L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[\frac{M}{(b-a)^{\alpha}}\left[f(b)-mf\left(\frac{a}{m}\right)\right] + m(b-a)f\left(\frac{a}{m}\right)L_n^n(a,b)\right]^{\frac{1}{q}}$$

This completes the proof of theorem.

Corollary 2. *Under the conditions Theorem 2.2 for $n=1$we have the following inequality.*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le A^{\frac{q-1}{q}}(a,b)\left[\frac{(\alpha+1)b+a}{(\alpha+1)(\alpha+2)}\left(f(b)-mf\left(\frac{a}{m}\right)\right) + mf\left(\frac{a}{m}\right)A(a,b)\right]^{\frac{1}{q}}.$$

Proposition 1. *Under the conditions Corollary 2 for $q=1$ we have the following inequality.*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(\alpha+1)b+a}{(\alpha+1)(\alpha+2)}\left(f(b)-mf\left(\frac{a}{m}\right)\right) + mf\left(\frac{a}{m}\right)A(a,b).$$

Corollary 3. *Under the conditions Theorem 2 for $q=1$ we have the following inequality.*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \le \frac{1}{n!}\left[\frac{M}{(b-a)^{\alpha}}\left(f(b)-mf\left(\frac{a}{m}\right)\right) + m(b-a)f\left(\frac{a}{m}\right)L_n^n(a,b)\right].$$

Theorem 3. *For all $n\in N$; let $f: I\subseteq[0,\infty)\to R$ be n-times differentiable function and $0\le a<b$. If $\left|f^{(n)}\right|^q \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is (α,m)-convex on $[a,b]$, then the following inequality holds.*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right|$$
$$\le \frac{1}{n!}(b-a)L_{p\left(n-\frac{1}{q}\right)}^{\frac{qn-1}{q}}(a,b)\left(\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left[\frac{(\alpha+1)b+a}{(\alpha+1)(\alpha+2)}\right] + mf\left(\frac{a}{m}\right)\frac{a+b}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $\left|f^{(n)}\right|^q$ for $q>1$ is (α,m)-convex on $[a,b]$, using Lemma 2.1 and the Hölder integral inequality, we have the following inequality.

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \le \frac{1}{n!}\int_a^b x^{n-\frac{1}{q}}.x^{\frac{1}{q}}\left|f^{(n)}(x)\right|dx$$
$$\le \frac{1}{n!}\left(\int_a^b\left(x^{n-\frac{1}{q}}\right)^p dx\right)^{\frac{1}{p}}\left(\int_a^b\left(x^{\frac{1}{q}}\right)^q\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{p\frac{qn-1}{q}}dx\right)^{\frac{1}{p}}\left(\int_a^b x\left(\left[\frac{x-a}{b-a}\right]^{\alpha} f(b)+m\left[1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right] f\left(\frac{a}{m}\right)\right)dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\int_a^b x^{p\frac{qn-1}{q}}dx\right)^{\frac{1}{p}}\left(\left[\frac{f(b)}{(b-a)^{\alpha}}-\frac{mf\left(\frac{a}{m}\right)}{(b-a)^{\alpha}}\right]\int_a^b x(x-a)^{\alpha}dx+mf\left(\frac{a}{m}\right)\int_a^b xdx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{\frac{1}{p}}\left[\frac{b^{p\frac{qn-1}{q}+1}-a^{p\frac{qn-1}{q}+1}}{\left(p\frac{qn-1}{q}+1\right)(b-a)}\right]^{\frac{1}{p}}$$

$$\times\left(\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left[\frac{(b-a)^2}{\alpha+2}+a\frac{b-a}{\alpha+1}\right]+mf\left(\frac{a}{m}\right)\left[\frac{(b-a)(b+a)}{2}\right]\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)L_{p\left(n-\frac{1}{q}\right)}^{\frac{qn-1}{q}}(a,b)\left(\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left[\frac{(\alpha+1)b+a}{(\alpha+1)(\alpha+2)}\right]+mf\left(\frac{a}{m}\right)\frac{a+b}{2}\right)^{\frac{1}{q}}$$

This completes the proof of theorem.

Corollary 4. *Under the conditions Theorem 2.3 for $n=1$we have the following inequality,*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq A^{\frac{1}{p}}(a,b)\left(\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left[\frac{(\alpha+1)b+a}{(\alpha+1)(\alpha+2)}\right]+mf\left(\frac{a}{m}\right)\frac{a+b}{2}\right)^{\frac{1}{q}}.$$

3 Conclusions

In this paper, by using an integral identity we obtained some new integral inequalities for functions whose nth derivatives in absolute value are (α,m)-convex functions.

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