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Semilinear problems involving nonlinear operators of monotone type

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Abstract

This is a survey article on semilinear problems with a non-symmetric linear part and a nonlinear part of monotone type in real Hilbert spaces. We study the solvability of semilinear inclusions in the nonresonance and resonance cases. Semilinear systems consisting of semilinear equations of different types are discussed.

Keywords: semilinear inclusion, semilinear system, operators of monotone type, degree theory.
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1. Introduction

Semilinear problems with nonlinear operators of monotone type have been studied in several ways of approach; see [1, 2, 5, 15]. Mawhin and Willem [15] employed the Leray-Schauder theory combined with monotone type operators in Galerkin arguments. For applications to nonlinear wave equations, we refer to [3, 6, 11, 14]. Berkovits and Fabry [1, 2] treated semilinear equations based on a degree theory as an extension of the Leray-Schauder degree utilizing compact embeddings.

Let H be a real separable Hilbert space. We first consider a semilinear equation of the form

$$Lu - Nu = h, \tag{1.1}$$

where L is a closed densely defined linear operator on H with a compact resolvent and N is a nonlinear operator. In the self-adjoint case, it is known that equation (1.1) has a solution provided that

$$\|Nu - \frac{\lambda_1}{2}u\| \le \mu\|u\| + \nu \qquad \text{for all } u \in H,$$

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where λ_1 is the first positive eigenvalue of L and $\mu \in [0, \lambda_1/2)$, $\nu \in [0, \infty)$ are constants. More generally, if $\operatorname{Ker} L = \operatorname{Ker} L^*$, L^* being the adjoint operator of L, then, as shown in [8], there exists a positive number ρ such that

$$\|Lu - \frac{\rho}{2}u\| \geq \frac{\rho}{2}\|u\| \qquad \text{for all } u \in D(L). \tag{1.2}$$

In this case, equation (1.1) admits a solution if there are constants $\mu \in [0, \rho/2)$ and $\nu \in [0, \infty)$ such that

$$\|Nu - \frac{\rho}{2}u\| \leq \mu\|u\| + \nu \qquad \text{for all } u \in H.$$

The existence proof was mainly based on the Leray-Schauder theory. When $\dim \operatorname{Ker} L = \infty$ and N is of class (S_+) or of class $(S_+)_P$, it was studied in [3, 12] based on suitable degree theory. Here $P : H \to \operatorname{Ker} L$ denotes the orthogonal projection.

However, if $\operatorname{Ker} L \not\subset \operatorname{Ker} L^*$, then the only number for which (1.2) holds is $\rho = 0$; see [8]. To overcome the difficulty, one needs a linear homeomorphism $J : H \to H$ such that $J(\operatorname{Ker} L) = \operatorname{Ker} L^*$, as in [9], in which case we can find a positive number ρ such that

$$\left\|Lu - \frac{\rho}{2}Ju\right\| \geq \frac{\rho}{2}\|Ju\| \qquad \text{for all } u \in D(L). \tag{1.3}$$

Berkovits and Fabry [1] showed the solvability of semilinear problem (1.1) under the condition

$$\left\|Nu - \frac{\rho}{2}Ju\right\| \leq \mu\|Ju\| + O(\|u\|^\alpha) \qquad \text{for all } u \in H,\ \|u\| \to \infty.$$

In fact, inequalities of type (1.3) are essential for deriving a priori estimates needed in the use of degree; see [1, 9].

Next, Berkovits and Fabry [1, 2] considered a system of semilinear equations in the form:

$$\begin{cases} L_1u_1 - N_1(u_1, u_2) = h_1, \\ L_2u_2 - N_2(u_1, u_2) = h_2, \end{cases}$$

where L_1, L_2 are closed densely defined linear operators with $\dim \operatorname{Ker} L_1 = \infty$ and $\dim \operatorname{Ker} L_2 < \infty$ and N_1, N_2 are nonlinear operators.

In this aspect, we are now interested in the case where the nonlinear operator N is set-valued. We consider a semilinear inclusion of the form

$$h \in Lu - Nu, \tag{1.4}$$

where L is a non-symmetric closed densely defined linear operator with a compact resolvent and N is a nonlinear set-valued operator of monotone type associated with linear homeomorphism J and orthogonal projection P. For non-symmetric densely defined linear operators, see *e.g.*, [1, 7, 9, 10]. Moreover, we observe the following semilinear system

$$\begin{cases} L_1u_1 - N_{1,1}(u_1) - N_{1,2}(u_2) \ni h_1, \\ L_2u_2 - N_2(u_1, u_2) = h_2, \end{cases} \tag{1.5}$$

where $N_{1,1}$ is a bounded upper semicontinuous operator of monotone type, and $N_{1,2}$, N_2 are bounded continuous operators. The system (1.5) can be written as $h \in Lu - Nu$, where $L = (L_1, L_2)$ and $N = (N_{1,1} + N_{1,2}, N_2)$ are as above.

In this note, we establish the existence of a solution of semilinear inclusion (1.4) in the nonresonance and resonance cases. The method is to use a topological degree theory for a class of these semilinear operators in real Hilbert spaces. Moreover, we are concerned with the solvability of the above semilinear system (1.5), where a key tool is the nonresonance theorem for semilinear inclusions. It is emphasized that our degree theoretic approach enables us to deal with semilinear systems having nonlinear terms of different types.

2. Degree theory

In this section, we introduce a topological degree theory for semilinear operators in real Hilbert spaces by means of the Leray-Schauder degree.

Definition 2.1. Let X, Y be two Hausdorff topological spaces. A set-valued operator $F : X \to 2^Y$ is said to be:

(1) *upper semicontinuous* if the set $F^{-1}(A) = \{u \in X \mid Fu \cap A \neq \emptyset\}$ is closed in X for every closed set A in Y;

(2) *upper demicontinuous* if $F^{-1}(A)$ is closed in X for every weakly closed set A in Y;

(3) *bounded* if it maps bounded sets into bounded sets;

(4) *compact* if it is upper semicontinuous and the image of any bounded set is relatively compact;

(5) *of Leray-Schauder type* if it is of the form $I - C$, where I denotes the identity operator and C is compact.

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space. Given a nonempty subset Ω of H, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in H, respectively. Let $B_r(u)$ denote the open ball in H of radius $r > 0$ centered at u. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Definition 2.2. Given a bounded linear operator $T : H \to H$, a set-valued operator $F : \Omega \subset H \to 2^H \setminus \emptyset$ is said to be:

(1) of class $(S_+)_T$, written $F \in (S_+)_T$, if for any sequence (u_n) in Ω, $u_n = v_n + z_n,\ v_n \in \operatorname{Ker} T,\ z_n \in (\operatorname{Ker} T)^\perp$ and for any sequence (w_n) in H with $w_n \in Fu_n$ such that $u_n \rightharpoonup u,\ v_n \to v$, and

$$\limsup_{n\to\infty} \langle w_n, T(u_n - u)\rangle \leq 0,$$

we have $u_n \to u$;

(2) *T-pseudomonotone*, written $F \in (PM)_T$, if for any sequence (u_n) in Ω, $u_n = v_n + z_n,\ v_n \in \operatorname{Ker} T,\ z_n \in (\operatorname{Ker} T)^\perp$ and for any sequence (w_n) in H with $w_n \in Fu_n$ such that $u_n \rightharpoonup u,\ v_n \to v$, and

$$\limsup_{n\to\infty} \langle w_n, T(u_n - u)\rangle \leq 0,$$

we have $\lim_{n\to\infty} \langle w_n, T(u_n - u)\rangle = 0$ and if $u \in \Omega$ and $w_j \rightharpoonup w$ for some subsequence (w_j) of (w_n) then $w \in Fu$;

(3) *T-quasimonotone*, written $F \in (QM)_T$, if for any sequence (u_n) in Ω, $u_n = v_n + z_n,\ v_n \in \operatorname{Ker} T,\ z_n \in (\operatorname{Ker} T)^\perp$ and for any sequence (w_n) in H with $w_n \in Fu_n$ such that $u_n \rightharpoonup u$ and $v_n \to v$, we have

$$\liminf_{n\to\infty} \langle w_n, T(u_n - u)\rangle \geq 0.$$

If all operators are assumed to be bounded and upper demicontinuous, it is easy to see that $(S_+)_T \subset (PM)_T \subset (QM)_T$ and the class $(S_+)_T$ is stable under $(QM)_T$-perturbations. If $J : H \to H$ is a linear homeomorphism and P is an orthogonal projection to a closed subspace of H, it is clear that $(S_+)_J \subset (S_+)_{JP}$ and, moreover, $(S_+)_J = (S_+)_{JP}$ if $\dim \operatorname{Ker} P < \infty$.

We give a typical example of an operator which is of class $(S_+)_{JP}$ but not of class $(S_+)_J$. See [2, Lemma 2.2] for the case $J = I$.

Example 2.3. Let E be a closed subspace of a real separable Hilbert space H. Let $P : H \to E$ and $\tilde{P} : H \to E^{\perp}$ be the orthogonal projections, respectively. If $J : H \to H$ is a linear homeomorphism, then $F := J(P - \tilde{P})$ is of class $(S_+)_{JP}$. However, it is not of class $(S_+)_J$ if $\dim E^{\perp} = \infty$.

As in [4], we adopt an elliptic super-regularization method to develop a degree theory for set-valued operators of class $(S_+)_{JP}$, which is proposed in [1] in the single-valued case.

Let H be a real separable Hilbert space. Suppose that $L : D(L) \subset H \to H$ is a closed densely defined linear operator with closed range $\operatorname{Im} L$. Then its adjoint $L^* : D(L^*) \subset H \to H$ is also a closed densely defined linear operator with closed range $\operatorname{Im} L^*$ and we have the orthogonal decompositions

$$H = \operatorname{Ker} L \oplus \operatorname{Im} L^* = \operatorname{Ker} L^* \oplus \operatorname{Im} L.$$

Let $P : H \to \operatorname{Ker} L$, $\tilde{P} : H \to \operatorname{Im} L^*$, $Q : H \to \operatorname{Ker} L^*$, and $\tilde{Q} : H \to \operatorname{Im} L$ denote the orthogonal projections, respectively. Let $J : H \to H$ be a linear homeomorphism such that

$$J(\operatorname{Ker} L) = \operatorname{Ker} L^*.$$

Let $\Psi : \operatorname{Ker} L \to \operatorname{Ker} L$ be a compact self-adjoint linear injection. Suppose that $N : \overline{G} \to \mathcal{K}(H)$ is a bounded upper semicontinuous operator of class $(S_+)_{JP}$, where G is any bounded open set in H. To each $F = \tilde{P} - [K\tilde{Q}(J^{-1})^* - P]J^*N$, we associate a family of operators

$$F_\lambda = I - [K\tilde{Q}(J^{-1})^* - \lambda\Psi^2 P]J^*N \qquad \text{for } \lambda > 0.$$

Then each $F_\lambda : \overline{G} \to \mathcal{K}(H)$ is a set-valued operator of Leray-Schauder type. Here $\mathcal{K}(H)$ denotes the collection of nonempty compact convex values. For the Leray-Schauder degree for these set-valued operators, see [13].

We introduce a topological degree for a semilinear class involving nonlinear set-valued operators of class $(S_+)_{JP}$.

Definition 2.4. Let L, K, N, J, Ψ be indicated as above and let G be a bounded open set in H. If $h \notin (L - N)(\partial G \cap D(L))$, then a degree is defined as an integer-valued function as follows:

$$\mathit{deg}\,(L - N, G, h) = \lim_{\lambda\to\infty} d_{LS}(F_\lambda, G, h_\lambda),$$

where $h_\lambda = K\tilde{Q}h - \lambda\Psi^2 PJ^*h$. Here d_{LS} denotes the Leray-Schauder degree.

We state some of basic properties of the above degree which follow from the corresponding properties of the Leray-Schauder degree.

Theorem 2.5. *Let L and N be as in Definition 2.4. Suppose that G is any bounded open set in H and $h \notin (L - N)(\partial G \cap D(L))$. Then the above degree has the following properties:*

(a) (Existence) *If $\mathit{deg}\,(L - N, G, h) \neq 0$, then the semilinear inclusion $h \in Lu - Nu$ has at least one solution in $G \cap D(L)$.*

(b) (Additivity) *If G_1 and G_2 are disjoint open subsets of G such that $h \notin (L - N)[\overline{G}\backslash(G_1 \cup G_2)]$, then we have*

$$\mathit{deg}\,(L - N, G, h) = \mathit{deg}\,(L - N, G_1, h) + \mathit{deg}\,(L - N, G_2, h).$$

(c) (Homotopy invariance) *Suppose that $N : [0, 1] \times \overline{G} \to \mathcal{K}(H)$ is a bounded upper semicontinuous homotopy of class $(S_+)_{JP}$. If $h : [0, 1] \to H$ is a continuous curve in H such that*

$$h(t) \notin Lu - N(t, u) \quad \text{for all } (t, u) \in [0, 1] \times (\partial G \cap D(L)),$$

then the value of $\mathit{deg}\,(L - N(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.

The basic idea for the following result is the Borsuk's antipodal theorem stated in [16, Theorem 16.B], for instance. It will be a key tool for proving main theorems on semilinear inclusions.

Lemma 2.6. *Let $B : H \to H$ be a bounded linear operator of class $(S_+)_{JP}$ such that $L - B$ is injective. If $h \in (L - B)(G \cap D(L))$, where G is any bounded open set in H, then we have*

$$deg\,(L - B, G, h) \neq 0.$$

In particular, if $J : H \to H$ is a linear homeomorphism such that $L - \alpha J$ is injective for some positive scalar α, then

$$deg\,(L - \alpha J, B_r(0), 0) \neq 0 \qquad \text{for any positive number } r.$$

3. Semilinear inclusions

In this section, we are concerned with the solvability of semilinear inclusions in the nonresonance and resonance cases, based on the degree theory in the previous section.

Let H be a real separable Hilbert space. Suppose that $L : D(L) \subset H \to H$ is a closed densely defined linear operator with closed range and that $K : \operatorname{Im} L \to \operatorname{Im} L^* \cap D(L)$, being the inverse of the restriction of L to $\operatorname{Im} L^* \cap D(L)$, is compact. Let $P : H \to \operatorname{Ker} L$, $\tilde{P} : H \to \operatorname{Im} L^*$, $Q : H \to \operatorname{Ker} L^*$, and $\tilde{Q} : H \to \operatorname{Im} L$ be the orthogonal projections, respectively. Let $J : H \to H$ be a linear homeomorphism. Set

$$\mathcal{A}_J := \{\, \rho \in \mathbb{R} \mid \|Lu\|^2 \geq \rho\langle Lu, Ju\rangle \quad \text{for all } u \in D(L)\}.$$

It is easily seen that

$$\mathcal{A}_J = \left\{ \rho \in \mathbb{R} \mid \left\|Lu - \frac{\rho}{2}Ju\right\| \geq \left\|\frac{\rho}{2}Ju\right\| \quad \text{for all } u \in D(L)\right\}. \tag{3.1}$$

It is known in [9] that the set $\mathcal{A}_J$ is a closed interval containing 0, and if $J(\operatorname{Ker} L) \subset \operatorname{Ker} L^*$ then 0 is an interior point of $\mathcal{A}_J$.

First, we prove the existence of a solution for semilinear inclusions in the nonresonance case. For related results, see [1, 9, 12].

Theorem 3.1. *Let L, K and P be as above. Suppose that $J : H \to H$ is a linear homeomorphism such that*

$$J(Ker\,L) = Ker\,L^*, \tag{3.2}$$

and that $N : H \to \mathcal{K}(H)$ is a bounded upper semicontinuous operator. Suppose that there are numbers $\rho \in (0, \sup \mathcal{A}_J]$, $\mu \in [0, \rho/2)$, and $\alpha \in [0, 1)$ such that

$$\left\|a - \frac{\rho}{2}Ju\right\| \leq \mu\|Ju\| + O(\|u\|^\alpha) \quad \text{for } u \in H, \|u\| \to \infty, \text{and } a \in Nu. \tag{3.3}$$

If the operator N is JP-pseudomonotone, then for every $h \in H$, the semilinear inclusion

$$h \in Lu - Nu$$

has a solution in $D(L)$.

Proof. Let h be any element of H. For $t \in [0, 1)$, we consider the equation

$$th \in Lu - (1 - t)\frac{\rho}{2}Ju - tNu. \tag{3.4}$$

Since $\rho \in \mathcal{A}_J$ implies that the linear operator $L - (\rho/2)J$ is injective, (3.4) has only the trivial solution when $t = 0$. We first show that the set of solutions of (3.4)

$$S = \left\{u \in D(L) \mid th \in Lu - (1 - t)\frac{\rho}{2}Ju - tNu \quad \text{for some } t \in [0, 1)\right\}$$

is bounded in H. Assume the contrary, then there are sequences (u_n) in $D(L)$ and (t_n) in $[0,1)$ with $\|u_n\| \to \infty$ such that

$$t_n h = Lu_n - (1-t_n)\frac{\rho}{2}Ju_n - t_n a_n \qquad \text{for all } n \in \mathbb{N},$$

where $a_n \in Nu_n$. For all $n \in \mathbb{N}$, we have by (3.1) and hypothesis (3.3)

$$\begin{aligned}\frac{\rho}{2}\|Ju_n\| &\le \left\|Lu_n - \frac{\rho}{2}Ju_n\right\| \\ &\le \left\|a_n - \frac{\rho}{2}Ju_n + h\right\| \\ &\le \mu\|Ju_n\| + \|h\| + O(\|u_n\|^\alpha)\end{aligned}$$

and hence

$$\left(\frac{\rho}{2} - \mu\right)\|u_n\| \le \|h\| + O(\|u_n\|^\alpha),$$

which is impossible, because of $\mu < \rho/2$. Now we can choose a positive number R such that

$$th \notin Lu - (1-t)\frac{\rho}{2}Ju - tNu \quad \text{for all } (t,u) \in [0,1) \times D(L) \text{ with } \|u\| \ge R. \tag{3.5}$$

For each fixed $t \in (0,1)$, we define $\Phi_t : [0,1] \times \overline{B_R(0)} \to \mathcal{K}(H)$ by

$$\Phi_t(\lambda, u) := (1-\lambda t)\frac{\rho}{2}Ju + \lambda t Nu \qquad \text{for } (\lambda,u) \in [0,1] \times \overline{B_R(0)}.$$

Then it is obvious that Φ_t is a bounded upper semicontinuous homotopy of class $(S_+)_{JP}$. Theorem 2.5 and Lemma 2.6 imply, in view of (3.5), that

$$deg\left(L - (1-t)\frac{\rho}{2}J - tN, B_R(0), th\right) = deg\left(L - \frac{\rho}{2}J, B_R(0), 0\right) \neq 0.$$

This implies that, for a sequence (t_n) in $(0,1)$ with $t_n \to 1$, there exists a corresponding sequence (u_n) in $B_R(0) \cap D(L)$ such that

$$t_n h = Lu_n - (1-t_n)\frac{\rho}{2}Ju_n - t_n a_n,$$

where $a_n \in Nu_n$. Without loss of generality, we may suppose that $u_n \rightharpoonup u$ and $a_n \rightharpoonup a$ for some $u, a \in H$. Since $J(\operatorname{Ker} L) = (\operatorname{Im} L)^\perp$ by (3.2) and $Lu_n - a_n \to h$, we have

$$\lim_{n\to\infty} \langle a_n, JP(u_n - u)\rangle = 0.$$

Since $\tilde{P}u_n = K\tilde{Q}Lu_n$ and K is compact, we see that $\tilde{P}u_n \to \tilde{P}u$. Hence it follows from $N \in (PM)_{JP}$ that $a \in Nu$. Since the graph of L is weakly closed and $Lu_n \rightharpoonup a + h$, we obtain that

$$u \in D(L) \qquad \text{and} \qquad h \in Lu - Nu.$$

This completes the proof. □

Next, we deal with the solvability of the semilinear inclusion under an h-dependent resonance type condition when $\mu = \rho/2$. For some results, we refer to [1, 9, 12].

Theorem 3.2. *Let L, K, J and P be the same as in Theorem 3.1. Suppose that $N : H \to \mathcal{K}(H)$ is a JP-pseudomonotone bounded upper semicontinuous operator and that there exist numbers $\rho \in (0, \sup \mathcal{A}_J)$ and $\alpha \in [0,1)$ such that*

$$\left\|a - \frac{\rho}{2}Ju\right\| \le \frac{\rho}{2}\|Ju\| + O(\|u\|^\alpha) \quad \text{for } u \in H, \|u\| \to \infty, \text{and } a \in Nu. \tag{3.6}$$

Let $h \in H$ be given. Suppose that for any sequence (u_n) in $D(L)$ such that $\|u_n\| \to \infty$ and $\|Lu_n\| = o(\|u_n\|)$ for $n \to \infty$, there is an $n_0 \in \mathbb{N}$ such that

$$\langle a_n + h, JPu_n\rangle \geq 0 \quad \text{for all } n \geq n_0 \text{ and all } a_n \in Nu_n. \tag{3.7}$$

Then the semilinear inclusion

$$h \in Lu - Nu$$

has a solution in $D(L)$.

Proof. It suffices to show that the solution set

$$\left\{u \in D(L) \mid th \in Lu - (1-t)\frac{\rho}{2}Ju - tNu \quad \text{for some } t \in (0,1)\right\}$$

is bounded in H. The rest of proof performs in an analogous way to that of Theorem 3.1. □

To deal with periodic problems, we present a more explicit form of Theorem 3.1.

Theorem 3.3. *Let L and K be as above. Let $N : H \to H$ be the Nemytskii operator induced by a real-valued function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, where Ω is a bounded domain in $\mathbb{R}^p$, such that*

(g1) g satisfies the Carathéodory condition, that is, $g(\cdot, s)$ is measurable on Ω for all $s \in \mathbb{R}$ and $g(x, \cdot)$ is continuous on $\mathbb{R}$ for almost all $x \in \Omega$;

(g2) g satisfies the growth condition, that is, there exist a nonnegative function $k \in H$ and a positive constant c such that

$$|g(x,s)| \leq k(x) + c|s| \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R};$$

(g3) $g(x, s)$ is nondecreasing in s, that is,

$$(g(x,s) - g(x,\eta))\,(s-\eta) \geq 0 \quad \text{for almost all } x \in \Omega \text{ and all } s, \eta \in \mathbb{R}.$$

Suppose that there are numbers $\rho \in (0, \sup \mathcal{A}_I]$, $\mu \in [0, \rho/2)$, and $\beta \in [0, \infty)$ such that

$$\left|g(x,s) - \frac{\rho}{2}s\right| \leq \mu|s| + \beta \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R}.$$

Then the semilinear equation

$$Lu - Nu = h$$

has a solution in $D(L)$ for every $h \in H$.

Proof. The Nemytskii operator $N : H \to H$ defined by

$$Nu(x) := g(x, u(x)) \qquad \text{for } \ u \in H \text{ and } x \in \Omega$$

is clearly bounded, continuous, and monotone on H; see *e.g.*, [17]. By hypothesis, we have

$$\left\|Nu - \frac{\rho}{2}u\right\| \leq \mu\|u\| + \xi \qquad \text{for all } u \in H,$$

where ξ is some positive constant. Applying Theorem 3.1 with $J = I$ and $\alpha = 0$, the equation $Lu - Nu = h$ has a solution for every $h \in H$. □

As an application, we consider the following periodic problem

$$c_1 \frac{\partial u}{\partial x_1} + c_2 \frac{\partial u}{\partial x_2} - g\,(x_1, x_2, u) = h(x_1, x_2), \tag{3.8}$$

where $c_1, c_2 \in \{1, -1\}$ are constants and $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is 2π-periodic in each of the variables x_1, x_2, and h is given.

We want to find solutions u which have the same type of periodicity in the variables x_1, x_2, that is,

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2) = u(x_1, x_2 + 2\pi).$$

To solve problem (3.8), we will consider the corresponding semilinear equation in the sense of Theorem 3.3. The differential operator $u \mapsto c_1 \partial u/\partial x_1 + c_2 \partial u/\partial x_2$ has an abstract realization which will be denoted by L. In the following, let L_c denote the complexification of the operator L and $\sigma(L_c)$ denote the spectrum of L_c, respectively.

Let $\Omega = (0, 2\pi) \times (0, 2\pi)$ and let $H = L^2(\Omega)$ be the real Hilbert space. Let $(\varphi_{mn})_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ be an orthonormal basis for the space H, where

$$\varphi_{mn}(x_1, x_2) = \frac{1}{2\pi} e^{i(mx_1 + nx_2)}.$$

Every element $u \in H$ can be represented in the form:

$$u = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} u_{mn}\,\varphi_{mn} \qquad \text{with } u_{mn} = \langle u, \varphi_{mn}\rangle.$$

Let $L : D(L) \subset H \to H$ be a linear operator defined by

$$Lu := \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} i(mc_1 + nc_2) u_{mn}\,\varphi_{mn},$$

where

$$D(L) = \{\, u \in H \mid \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} |(mc_1 + nc_2) u_{mn}|^2 < \infty \,\}.$$

Then it is known in [10] that L is a closed densely defined linear operator such that $L^* = -L$, $\dim \operatorname{Ker} L = \infty$, and $\sigma(L_c) \subset i\mathbb{R}$. If $\hat{L}$ denotes the restriction of L to $\operatorname{Im} L \cap D(L)$, then the inverse $\hat{L}^{-1} : \operatorname{Im} L \to \operatorname{Im} L \cap D(L)$ given by

$$\hat{L}^{-1} u := \sum_{(m,n)\in\Gamma} [i(mc_1 + nc_2)]^{-1} u_{mn}\,\varphi_{mn},$$

where $\Gamma = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \,|\, mc_1 + nc_2 \neq 0\}$, is compact, on observing that the spectrum of $(\hat{L}^{-1})_c$ has no limit point except 0 and $\operatorname{Ker}(\hat{L}^{-1} - \lambda I)$ is finite dimensional for any nonzero $\lambda \in \sigma((\hat{L}^{-1})_c)$. A similar argument about the compactness can be found in [11].

Suppose that $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is 2π-periodic in each of the first and second variables that satisfies all the assumptions on g in Theorem 3.3 with $\Omega = (0, 2\pi) \times (0, 2\pi)$. Let $N : H \to H$ be the Nemytskii operator induced by the function g. We say that a point $u \in H$ is a *weak solution* of problem (3.8) if the following relation holds for all $v \in C^1$:

$$\langle u, -c_1 v_{x_1} - c_2 v_{x_2}\rangle - \langle Nu, v\rangle = \langle h, v\rangle,$$

where C^1 denotes the space of continuously differentiable functions $v : \overline{\Omega} \to \mathbb{R}$ such that $v(x_1 + 2\pi, x_2) = v(x_1, x_2) = v(x_1, x_2 + 2\pi)$.

It can equivalently be written as $Lu - Nu = h$ with $u \in D(L)$. In view of Theorem 3.3, given periodic problem (3.8) has a weak solution.

4. Semilinear systems

In this section, we first observe under which conditions the operators are of class $(S_+)_{JP}$ or JP-pseudomonotone and then establish a nonresonance existence result for semilinear systems.

Let H_1, H_2 be real separable Hilbert spaces. Then $H = H_1 \times H_2$ is the Hilbert space with inner product defined by

$$\langle u, v\rangle = \langle u_1, v_1\rangle + \langle u_2, v_2\rangle \quad \text{for } u = (u_1, u_2), v = (v_1, v_2) \in H_1 \times H_2.$$

Suppose that for $k = 1, 2$, $L_k : D(L_k) \subset H_k \to H_k$ is a closed densely defined linear operator with $\operatorname{Im} L_k = (\operatorname{Ker} L_k^*)^\perp$ and $K_k : \operatorname{Im} L_k \to \operatorname{Im} L_k^*$, the inverse of the restriction of L_k to $\operatorname{Im} L_k^* \cap D(L_k)$, is compact. If $L : D(L) \subset H \to H$ is defined by

$$Lu = (L_1u_1, L_2u_2) \qquad \text{for } u = (u_1, u_2) \in D(L),$$

then $K : \operatorname{Im} L \to \operatorname{Im} L^*$, the inverse of the restriction of L to $\operatorname{Im} L^* \cap D(L)$, is compact, where

$$Ku = (K_1u_1, K_2u_2) \qquad \text{for } u = (u_1, u_2) \in \operatorname{Im} L.$$

For $k = 1, 2$, let $P_k : H_k \to \operatorname{Ker} L_k$ and $Q_k : H_k \to \operatorname{Ker} L_k^*$ be the orthogonal projections, and $\tilde{P}_k = I - P_k$, $\tilde{Q}_k = I - Q_k$. Then $P : H \to \operatorname{Ker} L$ and $Q : H \to \operatorname{Ker} L^*$ are the orthogonal projections and $\tilde{P} = I - P$, $\tilde{Q} = I - Q$, where

$$Pu = (P_1u_1, P_2u_2) \quad \text{and} \quad Qu = (Q_1u_1, Q_2u_2) \quad \text{for } u = (u_1, u_2) \in H.$$

For $k = 1, 2$, let $J_k : H_k \to H_k$ be a linear homeomorphism such that

$$J_1(\operatorname{Ker} L_1) = \operatorname{Ker} L_1^* \qquad \text{and} \qquad J_2(\operatorname{Ker} L_2) = \operatorname{Ker} L_2^*.$$

Then we have $J(\operatorname{Ker} L) = \operatorname{Ker} L^*$, where

$$Ju = (J_1u_1, J_2u_2) \qquad \text{for } u = (u_1, u_2) \in H.$$

Regarding semilinear systems, it is remarkable that any monotone type hypothesis on the second component N_2 is not required for $N = (N_1, N_2)$ to be of class $(S_+)_{JP}$ or JP-pseudomonotone.

Proposition 4.1. *Let $L = (L_1, L_2), J = (J_1, J_2)$, and $P = (P_1, P_2)$ be as above such that $\dim Ker\, L_1 = \infty$ and $\dim Ker\, L_2 < \infty$. Suppose that $N = (N_1, N_2) : H \to 2^H$ is bounded, where $N_1(v, z) = N_{1,1}(v) + N_{1,2}(z)$, such that the following conditions are satisfied:*

(a) $N_{1,1} : H_1 \to 2^{H_1}$ is of class $(S_+)_{J_1P_1}$;

(b) $N_{1,2} : H_2 \to H_1$ is continuous;

(c) $N_2 : H \to H_2$ is demicontinuous.

Then the operator N is of class $(S_+)_{JP}$.

Proof. Let (u_n) be any sequence in H and (a_n) any sequence in H with $a_n \in Nu_n$ such that

$$u_n \rightharpoonup u, \quad \tilde{P}u_n \to \tilde{P}u, \quad \text{and} \quad \limsup_{n\to\infty} \langle a_n, JP(u_n - u)\rangle \le 0. \tag{4.1}$$

Let $u_n = (v_n, z_n)$ and $u = (v, z)$. Since $\dim \operatorname{Ker} L_2 < \infty$ implies $P_2z_n \to P_2z$, we have

$$z_n = P_2z_n + \tilde{P}_2z_n \to P_2z + \tilde{P}_2z = z \quad \text{in } H_2.$$

Hence it follows from condition (b) and (4.1) that

$$\begin{aligned}\limsup_{n\to\infty} \langle b_n - N_{1,2}(z_n), J_1P_1(v_n - v)\rangle &= \limsup_{n\to\infty} \langle b_n, J_1P_1(v_n - v)\rangle \\ &= \limsup_{n\to\infty} \langle a_n, JP(u_n - u)\rangle \\ &\le 0,\end{aligned}$$

where $a_n = (b_n, c_n) \in Nu_n$, that is, $b_n \in N_{1,1}(v_n)+N_{1,2}(z_n)$ and $c_n = N_2(u_n)$. Since $N_{1,1}$ is of class $(S_+)_{J_1P_1}$, we get $v_n \to v$ in H_1 so that $u_n = (v_n, z_n) \to (v, z) = u$ in H. Therefore, N is of class $(S_+)_{JP}$. □

Remark 4.2. If $N_{1,1} : H_1 \to H_1$ is strongly monotone, that is, there is a positive constant c such that

$$\langle N_{1,1}(v) - N_{1,1}(v'), v - v'\rangle \ge c\,\|v - v'\|^2 \quad \text{for all } v, v' \in H_1,$$

then it is of class (S_+) and hence of class $(S_+)_{J_1P_1}$. More generally, if there is a positive constant c such that

$$\langle N_{1,1}(v) - N_{1,1}(v'), J_1P_1(v - v')\rangle \ge c\,\|P_1(v - v')\|^2 \quad \text{for all } v, v' \in H_1,$$

then it is of class $(S_+)_{J_1P_1}$.

Proposition 4.3. *Let $L = (L_1, L_2), J = (J_1, J_2)$, and $P = (P_1, P_2)$ be as above such that $\dim Ker L_1 = \infty$ and $\dim Ker L_2 < \infty$. Suppose that $N = (N_1, N_2) : H \to 2^H$ is bounded, where $N_1(v, z) = N_{1,1}(v) + N_{1,2}(z)$, such that the following conditions are satisfied:*

(a) $N_{1,1} : H_1 \to 2^{H_1}$ is J_1P_1-pseudomonotone;

(b) $N_{1,2} : H_2 \to H_1$ is continuous;

(c) $N_2 : H \to H_2$ is weakly continuous.

Then the operator N is JP-pseudomonotone.

We present a simple example of a JP-pseudomonotone operator which is not J-pseudomonotone.

Example 4.4. Let $H = H_1 \times H_2$, $L = (L_1, L_2), J = (J_1, J_2)$, and $P = (P_1, P_2)$ be as above such that $\dim \operatorname{Ker} L_1 = \infty$ and $\dim \operatorname{Ker} L_2 < \infty$. For $u = (u_1, u_2) \in H_1 \times H_2$, let

$$N_{1,1}(u_1) = J_1(P_1 - \tilde{P}_1)u_1, \quad N_{1,2}(u_2) = u_1^*, \quad \text{and} \quad N_2(u_1, u_2) = P_2u_2,$$

where u_1^* is a fixed element in H_1. Then $N = (N_{1,1} + N_{1,2}, N_2) : H \to H$ is JP-pseudomonotone. But it is not J-pseudomonotone if $\dim E_1^\perp = \dim E_2^\perp = \infty$, where $E_1 = \operatorname{Ker} L_1$ and $E_2 = \operatorname{Ker} L_2$.

Finally, we show the existence of a solution for semilinear systems in a more concrete situation by using the nonresonance theorem for semilinear inclusions. For related results, see [1, 12].

Theorem 4.5. *Let $L = (L_1, L_2), J = (J_1, J_2)$, and $P = (P_1, P_2)$ be as above such that*

$$\dim Ker L_1 = \infty \quad \text{and} \quad \dim Ker L_2 < \infty.$$

Suppose that $N = (N_1, N_2) : H \to \mathcal{K}(H)$ is a bounded upper semicontinuous operator which satisfies the conditions of Proposition 4.1 or Proposition 4.3. Moreover, suppose that there are numbers $\rho \in (0, \rho_*]$, $\mu_1, \mu_2 \in [0, \rho/2)$, and $\alpha \in [0, 1)$, where $\rho_* = \min\{\sup \mathcal{A}_{J_1}, \sup \mathcal{A}_{J_2}\}$, such that

$$\begin{aligned}\left\|a_1 - \frac{\rho}{2}J_1u_1\right\| &\le \mu_1\|J_1u_1\| + O(\|u\|^\alpha), \\ \left\|N_2(u_1, u_2) - \frac{\rho}{2}J_2u_2\right\| &\le \mu_2\|J_2u_2\| + O(\|u\|^\alpha),\end{aligned}$$

for $u = (u_1, u_2) \in H = H_1 \times H_2, \|u\| \to \infty$, *and* $a_1 \in N_1(u_1, u_2) = N_{1,1}(u_1) + N_{1,2}(u_2)$. *Then for every* $(h_1, h_2) \in H_1 \times H_2$, *the semilinear system*

$$\begin{cases} L_1 u_1 - N_{1,1}(u_1) - N_{1,2}(u_2) \ni h_1 \\ L_2 u_2 - N_2(u_1, u_2) = h_2 \end{cases}$$

has a solution in $D(L_1) \times D(L_2)$.

Proof. We may use an equivalent norm on the space $H = H_1 \times H_2$ given by

$$\|(u_1, u_2)\|_1 := \|u_1\| + \|u_2\| \qquad \text{for } (u_1, u_2) \in H_1 \times H_2.$$

Proposition 4.1 or Proposition 4.3 implies that N is JP-pseudomonotone. Apply Theorem 3.1 with $\mu = \max\{\mu_1, \mu_2\}$. □

We close this section by taking into account possible candidates for N_2 appearing in Proposition 4.1 or Proposition 4.3.

Let Ω be a bounded domain in $\mathbb{R}^p$ and let $H = L^2(\Omega)$ be the real Hilbert space. Let

$$g_2(x, s, p) := \alpha \sin s + \beta p,$$

where α, β are constants. Then g_2 satisfies the Carathéodory condition and the growth condition. So the Nemytskii operator $N_2 : H \times H \to H$ induced by g_2

$$N_2(u, v)(x) := \alpha \sin u(x) + \beta v(x) \quad \text{for } (u, v) \in H \times H \text{ and } x \in \Omega,$$

is bounded and continuous on $H \times H$. In particular, the Nemytskii operator

$$N_2(u, v)(x) = \beta v(x) \quad \text{for } (u, v) \in H \times H \text{ and } x \in \Omega,$$

is linear and bounded and hence weakly continuous, as required in Proposition 4.3.

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