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ON SUBSEQUENTIALLY CONVERGENT SEQUENCES

SEFA ANIL SEZER AND İBRAHİM ÇANAK

Abstract. In this study we obtain some sufficient conditions under which subsequential convergence of a sequence of real numbers follows from its boundedness. Eventually, we obtain crucial information about the subsequential behavior of sequences.

1. Introduction

It is well known that convergence of a sequence $\{s_n\}$ of real numbers implies its boundedness, yet the converse is not necessarily true is clear from the example of $\{\sin(n\pi/2)\}$. Since boundedness is a necessary condition for convergence of $\{s_n\}$, we put the following question: Under which conditions we get information on the convergence behavior of bounded sequences. In the case where $\{s_n\}$ is monotonic and bounded, we have its convergence. On the other hand, Bolzano-Weierstrass theorem states that every bounded sequence has at least one accumulation point. However, there are some bounded sequences such as $\{\sin(\log n)\}$ whose accumulation points lie on a finite interval and all points in this interval are accumulation points of the sequence. In this case we just have convergence of some subsequences of $\{s_n\}$. Motivated by this idea, Stanojević [10] defined a new kind of convergence as follows.

Definition 1. *A sequence $\{s_n\}$ is said to be subsequentially convergent if there exists a finite interval I such that all accumulation points of the sequence $\{s_n\}$ are in I and every point of I is an accumulation point of $\{s_n\}$.*

Throughout this paper, we adopt the following familiar conventions:

(i) $a_n = o(b_n)$ means $a_n/b_n \to 0$ as $n \to \infty$,
(ii) $a_n = O(b_n)$ means $|a_n| \leq Hb_n$ for sufficiently large n, where H is a positive constant,

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(iii) $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Note that every convergent sequence is subsequentially convergent. Further, it is obvious that subsequential convergence implies boundedness. But the converse is not always valid, provided by the example $\{(-1)^n\}$. The first theorem which reveals that the converse is valid under certain conditions was obtained by Dik [3] as stated below.

Theorem 2. *If $\{s_n\}$ is a bounded sequence such that $\Delta s_n = o(1)$ as $n \to \infty$, then $\{s_n\}$ is subsequentially convergent.*

Using Theorem 2 we can easily show that $s_n = \{\sin(\log n)\}$ is subsequentially convergent. Indeed, since $\{s_n\}$ is bounded and

$$\begin{aligned} |\Delta s_n| = |\Delta \sin(\log n)| &= |\sin(\log n) - \sin(\log(n-1))| \\ &\leq |\log n - \log(n-1)| = o(1),\ n \to \infty, \end{aligned}$$

$\{s_n\}$ is subsequentially convergent by Theorem 2.

Subsequential convergence was studied in a number of papers such as Çanak and Totur [1, 2], Dik [3], Dik et al. [4] and Sezer and Çanak [8]. In this paper we investigate conditions under which subsequential convergence of $\{s_n\}$ follows from its boundedness.

2. Preliminaries

In this section, we present some fundamental definitions, identities and lemmas which will be needed in the sequel.

The logarithmic mean of $\{s_n\}$ is defined by

$$t_n^{(1)}(s) = \frac{1}{\ell_n}\sum_{k=0}^{n}\frac{s_k}{k+1}, \quad where \quad \ell_n = \sum_{k=0}^{n}\frac{1}{k+1} \sim \log n, \quad n = 0, 1, 2, \dots . \tag{1}$$

Definition 3. *A sequence $\{s_n\}$ is said to be summable to a finite number L by the logarithmic mean method $(\ell, 1)$ if $\lim_{n\to\infty} t_n^{(1)}(s) = L$. In this case, we write $s_n \to \xi(\ell, 1)$.*

The difference between a sequence s_n and its logarithmic mean $t_n^{(1)}(s)$, that is known as the logarithmic Kronecker identity (see [9]) is given by

$$s_n - t_n^{(1)}(s) = v_n^{(0)}(\Delta s) \tag{2}$$

where $v_n^{(0)}(\Delta s) = \dfrac{1}{\ell_n}\displaystyle\sum_{k=1}^{n}\ell_{k-1}\Delta s_k$ and $\Delta s_n = s_n - s_{n-1}$ with $s_{-1} = 0$.

Since identity (2) can be rewritten as

$$s_n = v_n^{(0)}(\Delta s) + \sum_{k=1}^{n}\frac{v_k^{(0)}(\Delta s)}{(k+1)\ell_{k-1}} + s_0, \tag{3}$$

$\{v_n^{(0)}(\Delta s)\}$ is said to be a logarithmic generator sequence of $\{s_n\}$.

For every nonnegative integer r, we introduce $t_n^{(r)}(s)$ and $v_n^{(r)}(\Delta s)$ by

$$t_n^{(r)}(s) = \begin{cases} \dfrac{1}{\ell_n} \displaystyle\sum_{k=0}^{n} \dfrac{t_k^{(r-1)}(s)}{k+1} & , r \geq 1 \\ s_n & , r = 0 \end{cases}$$

and

$$v_n^{(r)}(\Delta s) = \begin{cases} \dfrac{1}{\ell_n} \displaystyle\sum_{k=0}^{n} \dfrac{v_k^{(r-1)}(\Delta s)}{k+1} & , r \geq 1 \\ v_n^{(0)}(\Delta s) & , r = 0 \end{cases}$$

respectively.

The classical logarithmic control modulo of the oscillatory behavior of $\{s_n\}$ is given by

$$\omega_n^{(0)}(s) = \alpha_n \Delta s_n \sim n \log n \Delta s_n, \tag{4}$$

where $\alpha_n = (n+1)\ell_{n-1}$. The general logarithmic control modulo of the oscillatory behavior of $\{s_n\}$ of integer order $r \geq 1$ is recursively defined by

$$\omega_n^{(r)}(s) = \omega_n^{(r-1)}(s) - t_n^{(1)}(\omega^{(r-1)}(s)). \tag{5}$$

For every nonnegative integer r, we have

$$\big(\alpha_n \Delta\big)_r s_n = \big(\alpha_n \Delta\big)_{r-1}\big(\alpha_n \Delta s_n\big) = \alpha_n \Delta\big(\big(\alpha_n \Delta\big)_{r-1} s_n\big),$$

where $\big(\alpha_n \Delta\big)_0 s_n = s_n$ and $\big(\alpha_n \Delta\big)_1 s_n = \alpha_n \Delta s_n$.

The next lemma provides a different representation of $\{\omega_n^{(r)}(s)\}$.

Lemma 4. (Sezer and Çanak, [9]) *For every integer $r \geq 1$, the assertion*

$$\omega_n^{(r)}(s) = \big(\alpha_n \Delta\big)_r v_n^{(r-1)}(\Delta s)$$

is valid.

Definition 5. *A sequence $\{s_n\}$ is called slowly oscillating with respect to summability $(\ell, 1)$ if*

$$\lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n < k \leq [n^\lambda]} |s_k - s_n| = 0 \tag{6}$$

or equivalently

$$\lim_{\lambda \to 1^-} \limsup_{n \to \infty} \max_{[n^\lambda] \leq k < n} |s_n - s_k| = 0, \tag{7}$$

where $[n^\lambda]$ denotes the integer part of n^λ.

Note that if the two-sided condition $n \log n \Delta s_n = O(1)$ is satisfied, then (6) holds.

There are subsequentially convergent sequences which are not slowly oscillating with respect to summability $(\ell, 1)$, and vice versa. For instance, $\{\log(\log n\})$ is subsequentially convergent but not slowly oscillating with respect to summability

$(\ell,1)$, conversely, the sequence $\left\{\sin\left(\sum_{k=1}^{n}\frac{\log k}{k}\right)\right\}$ is slowly oscillating with respect to summability $(\ell,1)$ but not subsequentially convergent.

The following lemma indicates that slow oscillation of $\{s_n\}$ is a Tauberian condition for $(\ell,1)$ summability.

Lemma 6. *If $\{s_n\}$ is $(\ell,1)$ summable to L and slowly oscillating with respect to summability $(\ell,1)$, then it converges to the same value.*

3. Main Results

In this section we present our main theorems.

Theorem 7. *If $\{s_n\}$ is bounded and $\{\Delta s_n\}$ is slowly oscillating with respect to summability $(\ell,1)$, then $\{s_n\}$ is subsequentially convergent.*

Proof. Considering identity (2), we have

$$\Delta s_n = \frac{v_n^{(0)}(\Delta s)}{\alpha_n} + \Delta v_n^{(0)}(\Delta s). \tag{8}$$

Since $\{s_n\}$ be bounded, then so is $\{v_n^{(0)}(\Delta s)\}$. By identity (8) and slow oscillation of $\{\Delta s_n\}$, $\{\Delta v_n^{(0)}(\Delta s)\}$ is slowly oscillating with respect to summability $(\ell,1)$. Also, since

$$\frac{1}{\ell_n}\sum_{k=0}^{n}\frac{\Delta v_k^{(0)}(\Delta s)}{k+1} = \frac{1}{\ell_n}\sum_{k=0}^{n}\frac{\frac{v_k^{(0)}(\Delta s)}{k+2}}{k+1} + \frac{1}{\ell_n}\frac{v_n^{(0)}(\Delta s)}{n+2} \to 0 \text{ as } n\to\infty,$$

$\{\Delta v_n^{(0)}(\Delta s)\}$ is $(\ell,1)$ summable to 0. Hence, we obtain $\Delta v_n^{(0)}(\Delta s) = o(1)$ by using Lemma 6. Also, by (8), $\Delta s_n = o(1)$. Therefore, proof of Theorem 7 follows from Theorem 2. □

Remark 8. *Notice that the following conditions are some of the classical Tauberian conditions for the $(\ell,1)$ summability which imply slow oscillation of $\{\Delta s_n\}$:*

(i) *$\{s_n\}$ is slowly oscillating with respect to summability $(\ell,1)$,* (Kwee, [6])
(ii) $\omega_n^{(0)}(s) = O(1)$, (Kwee, [6])
(iii) $\omega_n^{(0)}(s) = o(1)$, (Ishiguro, [5])
(iv) *$\{v_n^{(0)}(\Delta s)\}$ is slowly oscillating with respect to summability $(\ell,1)$,* (Sezer and Çanak, [9])
(v) $v_n^{(0)}(\Delta s) = o(1)$ (Kwee, [7])

In the next theorems, we propose new conditions imposed on the general logarithmic control modulo of the oscillatory behavior of $\{s_n\}$.

Theorem 9. *If $\{s_n\}$ is bounded and $\{\Delta(t_n^{(1)}(\omega^{(r)}(s)))\}$ is slowly oscillating with respect to summability $(\ell,1)$ for some nonnegative integer r, then $\{s_n\}$ is subsequentially convergent.*

Proof. Suppose $s_n = O(1)$. We see by using (2) that $v_n^{(0)}(\Delta s) = t_n^{(1)}(\omega^{(0)}(s)) = O(1)$. From the identity

$$t_n^{(1)}(\omega^{(0)}(s)) - t_n^{(2)}(\omega^{(0)}(s)) = t_n^{(1)}(\omega^{(1)}(s)),$$

we get $t_n^{(1)}(\omega^{(1)}(s)) = O(1)$. Continuing in the same fashion, we obtain

$$t_n^{(1)}(\omega^{(r)}(s)) = O(1)$$

for all integer $r \geq 0$, which is equivalent to

$$(\alpha_n \Delta)_r v_n^{(r)}(\Delta s) = O(1). \tag{9}$$

Hence, we observe

$$\begin{aligned} t_n^{(1)}(\Delta(t^{(1)}(\omega^{(r)}(s)))) &= t_n^{(1)}(\Delta((\alpha_n \Delta)_r v_n^{(r)}(\Delta s))) \\ &= \frac{1}{\ell_n} \sum_{k=0}^{n} \frac{(\alpha_k \Delta)_r v_k^{(r)}(\Delta s) - (\alpha_{k-1} \Delta)_r v_{k-1}^{(r)}(\Delta s)}{k+1} \\ &= \frac{1}{\ell_n} \sum_{k=0}^{n} \frac{\frac{(\alpha_k \Delta)_r v_k^{(r)}(\Delta s)}{k+2}}{k+1} + \frac{1}{\ell_n} \frac{(\alpha_n \Delta)_r v_n^{(r)}(\Delta s)}{n+2} \to 0 \end{aligned}$$

as $n \to \infty$. Combining the hypothesis of Theorem 9 and Lemma 6 yields

$$\Delta(t_n^{(1)}(\omega^{(r)}(s))) = \Delta((\alpha_n \Delta)_r v_n^{(r)}(\Delta s)) = o(1). \tag{10}$$

Considering identity

$$\omega_n^{(r)}(s) - t_n^{(1)}(\omega^{(r)}(s)) = \omega_n^{(r+1)}(s),$$

we have

$$\Delta((\alpha_n \Delta)_{r-1} v_n^{(r-1)}(\Delta s)) = \frac{(\alpha_n \Delta)_r v_n^{(r)}(\Delta s)}{\alpha_n} + \Delta((\alpha_n \Delta)_r v_n^{(r)}(\Delta s)).$$

Now, using (9) and (10), we have

$$\Delta((\alpha_n \Delta)_{r-1} v_n^{(r-1)}(\Delta s)) = o(1). \tag{11}$$

In the light of (10) and (11), if we continue in the same manner, then we get

$$\Delta v_n^{(0)}(\Delta s) = o(1).$$

Therefore, taking the identity

$$\Delta s_n = \frac{v_n^{(0)}(\Delta s)}{\alpha_n} + \Delta v_n^{(0)}(\Delta s)$$

into account together with the assumption $s_n = O(1)$, we conclude $\Delta s_n = o(1)$. This completes the proof. □

Remark 10. *The following results are noteworthy.*

(i) *If* $\{t_n^{(1)}(\omega^{(r)}(s))\}$ *is slowly oscillating with respect to summability* $(\ell,1)$*, then so is* $\{\Delta(t_n^{(1)}(\omega^{(r)}(s)))\}$ *of its backward difference.*

(ii) *Set* $r = 0$ *in* $\{t_n^{(1)}(\omega^{(r)}(s))\}$*. Then slow oscillation of* $\{v_n^{(0)}(\Delta s)\} = \{t_n^{(1)}(\omega^{(0)}(s))\}$ *is sufficient for subsequential convergence of a bounded sequence.*

(iii) *Two-sided condition* $n\log n \Delta v_n^{(0)}(\Delta s) = O(1)$ *implies slow oscillation of* $\{v_n^{(0)}(\Delta s)\}$*.*

Theorem 11. *Let* $\{s_n\}$ *be a bounded sequence and* $\{A_n\}$ *be a sequence satisfying*

$$\frac{1}{\ell_n}\sum_{k=0}^{n}\frac{|A_k|^p}{k+1} = O(1), p > 1. \tag{12}$$

If

$$\omega_n^{(r)}(s) = O(A_n) \tag{13}$$

for some nonnegative integer r*, then* $\{s_n\}$ *is subsequentially convergent.*

Proof. By (12), we see that $\left\{\sum_{j=0}^{n}\frac{A_j}{\alpha_j}\right\}$ is slowly oscillating with respect to summability $(\ell,1)$. Indeed,

$$\begin{aligned}
\max_{n<k\le[n^\lambda]}\left|\sum_{j=n+1}^{k}\frac{A_j}{\alpha_j}\right| &\le \max_{n<k\le[n^\lambda]}\sum_{j=n+1}^{k}\left|\frac{A_j}{\alpha_j}\right| \le \sum_{j=n+1}^{[n^\lambda]}\frac{|A_j|}{(j+1)\ell_{j-1}} \\
&\le \frac{1}{\ell_n}\sum_{j=n+1}^{[n^\lambda]}\frac{|A_j|}{(j+1)} \\
&\le \frac{1}{\ell_n}\left(\sum_{j=n+1}^{[n^\lambda]}\frac{1}{j+1}\right)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[n^\lambda]}\frac{1}{j+1}|A_j|^p\right)^{\frac{1}{p}} \\
&\le \frac{(\ell_{[n^\lambda]}-\ell_n)^{\frac{1}{q}}}{(\ell_n)^{\frac{1}{q}}}\frac{(\ell_{[n^\lambda]})^{\frac{1}{p}}}{(\ell_n)^{\frac{1}{p}}}\left(\frac{1}{\ell_{[n^\lambda]}}\sum_{j=0}^{[n^\lambda]}\frac{|A_j|^p}{j+1}\right)^{\frac{1}{p}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Taking the limit supremum as $n\to\infty$ of both sides of the last inequality

$$\begin{aligned}
&\limsup_{n\to\infty}\max_{n<k\le[n^\lambda]}\left|\sum_{j=n+1}^{k}\frac{A_j}{\alpha_j}\right| \\
&\quad\le \limsup_{n\to\infty}\left(\frac{(\ell_{[n^\lambda]}-\ell_n)^{\frac{1}{q}}}{(\ell_n)^{\frac{1}{q}}}\frac{(\ell_{[n^\lambda]})^{\frac{1}{p}}}{(\ell_n)^{\frac{1}{p}}}\right)\limsup_{n\to\infty}\left(\frac{1}{\ell_{[n^\lambda]}}\sum_{j=0}^{[n^\lambda]}\frac{|A_j|^p}{j+1}\right)^{\frac{1}{p}}
\end{aligned}$$

$$= \lim_{n\to\infty}\left(\frac{\ell_{[n^\lambda]}-\ell_n}{\ell_n}\right)^{\frac{1}{q}} \lim_{n\to\infty}\left(\frac{\ell_{[n^\lambda]}}{\ell_n}\right)^{\frac{1}{p}} \limsup_{n\to\infty}\left(\frac{1}{\ell_{[n^\lambda]}}\sum_{j=0}^{[n^\lambda]}\frac{|A_j|^p}{j+1}\right)^{\frac{1}{p}}.$$

Hence, from (12) we get

$$\limsup_{n\to\infty}\max_{n<k\le[n^\lambda]}\left|\sum_{j=n+1}^{k}\frac{A_j}{\alpha_j}\right| \le (\lambda-1)^{\frac{1}{q}}\lambda^{\frac{1}{p}}H \tag{14}$$

for $H>0$. Now, letting $\lambda\to 1^+$ in (14) gives

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n<k\le[n^\lambda]}\left|\sum_{j=n+1}^{k}\frac{A_j}{\alpha_j}\right| \le \lim_{\lambda\to 1^+}(\lambda-1)^{\frac{1}{q}}\lambda^{\frac{1}{p}}H = 0.$$

Since slow oscillation of $\left\{\sum_{j=0}^{n}\frac{A_j}{\alpha_j}\right\}$ implies $\frac{A_n}{\alpha_n}=o(1)$, it follows from

$$\omega_n^{(r)}(s) = \alpha_n\Delta((\alpha_n\Delta)_{r-1}v_n^{(r-1)}(\Delta s)) = O(A_n)$$

that

$$\Delta((\alpha_n\Delta)_{r-1}v_n^{(r-1)}(\Delta s)) = o(1). \tag{15}$$

By the boundedness of $\{s_n\}$, we also have

$$t_n^{(1)}(\omega^{(m)}(s)) = O(1) \text{ for each integer } m\ge 0. \tag{16}$$

Considering (16) for $m=r-1$, we have

$$t_n^{(1)}(\omega^{(r-1)}(s)) = (\alpha_n\Delta)_{r-1}v_n^{(r-1)}(\Delta s) = O(1). \tag{17}$$

Now, construct identity below using the definition of general logarithmic control modulo

$$\Delta((\alpha_n\Delta)_{r-2}v_n^{(r-2)}(\Delta s)) = \frac{(\alpha_n\Delta)_{r-1}v_n^{(r-1)}(\Delta s)}{\alpha_n} + \Delta((\alpha_n\Delta)_{r-1}v_n^{(r-1)}(\Delta s)).$$

Thus, by (15) and (17), we have

$$\Delta((\alpha_n\Delta)_{r-2}v_n^{(r-2)}(\Delta s)) = o(1). \tag{18}$$

Taking (15) and (18) into account and proceeding likewise, we accomplish

$$\Delta v_n^{(0)}(\Delta s) = o(1). \tag{19}$$

Then, since

$$\Delta s_n = \frac{v_n^{(0)}(\Delta s)}{\alpha_n} + \Delta v_n^{(0)}(\Delta s)$$

and $\{s_n\}$ is bounded, we find $\Delta s_n = o(1)$, which completes the proof. □

Remark 12. *Considering special cases of Theorem 11, we obtain the following corollaries.*

(i) *Take* $A_n = 1$ *for all integer* $n \geq 0$*. Then* (12) *and* (13) *reduce to* $\omega_n^{(r)}(s) = O(1)$*.*

(ii) *Take* $A_n = \alpha_n \Delta t_n^{(1)}(\omega^{(r)}(s))$*, then by the condition*

$$\frac{1}{\ell_n}\sum_{k=0}^{n}\frac{|\alpha_k \Delta t_k(\omega^{(r)}(s))|^p}{k+1} = O(1),\ p > 1, \tag{20}$$

we get subsequential convergence of a bounded sequence $\{s_n\}$ *using Remark 10, since* (20) *necessitate that* $\{t_n^{(1)}(\omega^{(r)}(s))\}$ *is slowly oscillating with respect to summability* $(\ell, 1)$*.*

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Current address: SEFA ANIL SEZER: İstanbul Medeniyet University, Department of Mathematics, Turkey.
E-mail address: sefaanil.sezer@medeniyet.edu.tr
ORCID Address: http://orcid.org/0000-0002-8053-9991
Current address: İBRAHİM ÇANAK: Ege University, Department of Mathematics, Turkey.
E-mail address: ibrahim.canak@ege.edu.tr
ORCID Address: http://orcid.org/0000-0002-1754-1685