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On the growth estimates of entire functions of double complex variables

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Abstract: Recently Datta et al. [6] introduced the idea of relative type and relative weak type of entire functions of two complex variables with respect to another entire function of two complex variables and prove some related growth properties of it. In this paper, further we study some growth properties of entire functions of two complex variables on the basis of their relative types and relative weak types as introduced by Datta et al [6].

Keywords: Entire function of two complex variables, order (lower order) of entire function of two complex variables, relative order (relative lower order) of entire functions of two complex variable, relative type (relative weak type) of two complex variables.

1 Introduction, definitions and notations

Suppose f be an entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\}$. Then in the light of maximum principal and Hartogs's theorem $\{$[7], p.2, p.51$\}$, $M_f(r_1, r_2)$ is an increasing function of r_1, r_2. For any two entire functions f and g of two complex variables, the ratio $\frac{M_f(r_1, r_2)}{M_g(r_1, r_2)}$ as $r_1, r_2 \to \infty$ is called the *growth* of f with respect to g. Taking this into account, the following definition is well known.

Definition 1. ([7], p.339, see also [1]) *The* order ${}_{v_2}\rho_f$ *and* lower order ${}_{v_2}\lambda_f$ *of an entire function* $f(z_1, z_2)$ *are defined as*

$${}_{v_2}\rho_f = \limsup_{r_1, r_2 \to \infty} \frac{\log\log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} = \limsup_{r_1, r_2 \to \infty} \frac{\log\log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)}$$

and

$${}_{v_2}\lambda_f = \liminf_{r_1, r_2 \to \infty} \frac{\log\log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} = \liminf_{r_1, r_2 \to \infty} \frac{\log\log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)}.$$

We see that the *order* ${}_{v_2}\rho_f$ and *lower order* ${}_{v_2}\lambda_f$ of an entire function $f(z_1, z_2)$ is defined in terms of the growth of $f(z_1, z_2)$ with respect to the exponential function $\exp(z_1 z_2)$.

The rate of *growth* of an entire function generally depends upon the *order (lower order)* of it. The entire function with higher *order* is of faster growth than that of lesser *order*. But if *orders* of two entire functions are the same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their *types* and thus one can define *type* of an entire function $f(z_1, z_2)$ denoted by ${}_{v_2}\sigma_f$ in the following way.

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Definition 2. [6] *The* type ${}_{v_2}\sigma_f$ *of an entire function* $f(z_1,z_2)$ *is defined as*

$$_{v_2}\sigma_f = \limsup_{r_1,r_2\to\infty} \frac{\log M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\rho_f}},\ 0 < {}_{v_2}\rho_f < \infty .$$

Similarly, the lower type ${}_{v_2}\overline{\sigma}_f$ *of an entire function* $f(z_1,z_2)$ *may be defined as*

$$_{v_2}\overline{\sigma}_f = \liminf_{r_1,r_2\to\infty} \frac{\log M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\rho_f}},\ 0 < {}_{v_2}\rho_f < \infty .$$

Similarly in order to determine the relative growth of two entire functions of two complex variables having same non zero finite *lower order* one may introduce the concept of *weak type* ${}_{v_2}\tau_f$ of $f(z_1,z_2)$ of finite positive *lower order* ${}_{v_2}\lambda_f$ which is as follows.

Definition 3. [6] *The* weak type ${}_{v_2}\tau_f$ *of an entire function* $f(z_1,z_2)$ *of finite positive* lower order ${}_{v_2}\lambda_f$ *is defined by*

$$_{v_2}\tau_f = \liminf_{r_1,r_2\to\infty} \frac{\log M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\lambda_f}},\ 0 < {}_{v_2}\lambda_f < \infty .$$

Likewise, one may define the growth indicator ${}_{v_2}\overline{\tau}_f$ *of an entire function* $f(z_1,z_2)$ *of finite positive* lower order ${}_{v_2}\lambda_f$ *in the following way.*

$$_{v_2}\overline{\tau}_f = \limsup_{r_1,r_2\to\infty} \frac{\log M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\lambda_f}},\ 0 < {}_{v_2}\lambda_f < \infty .$$

Bernal (see [2], [3]) introduced the definition of *relative order* between two entire functions of single variable. During the past decades, several authors (see [8],[9],[10],[11]) made closed investigations on the properties of *relative order* of entire functions of single variable. Using the idea of Bernal's *relative order* (see [2], [3]) of entire functions of single variable, Banerjee and Datta [4] introduced the definition of *relative order* of entire functions of two complex variables to avoid comparing growth just with $\exp(z_1z_2)$ which is as follows.

$$_{v_2}\rho_g(f) = \inf\left\{\mu > 0 : M_f(r_1,r_2) < M_g\left(r_1^{\mu},r_2^{\mu}\right); r_1 \geq R(\mu), r_2 \geq R(\mu)\right\} = \limsup_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log(r_1r_2)}$$

where g is also an entire function holomorphic in the closed polydisc

$$U = \{(z_1,z_2) : \ |z_i| \leq r_i,\ i = 1,2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with the classical one [4] if $g(z_1,z_2) = \exp(z_1z_2)$.

Like wise, one can define the *relative lower order* of f with respect to g denoted by ${}_{v_2}\lambda_g(f)$ as follows :

$$_{v_2}\lambda_g(f) = \liminf_{r_1,r_2\to\infty} \frac{\log M_g^{-1}M_f(r_1,r_2)}{\log(r_1r_2)} .$$

Now in the case of *relative order* of entire functions of two complex variables, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* respectively in order to compare the relative growth of two entire functions of two complex variables having same non zero finite *relative order* or *relative lower order* with respect to another entire function of two complex variables. Recently Datta et al [6] introduced such definitions which are as follows.

Definition 4. [6] *Let* $f(z_1,z_2)$ *and* $g(z_1,z_2)$ *be any two entire functions such that* $0 < {}_{v_2}\rho_g(f) < \infty$*. Then the* relative type ${}_{v_2}\sigma_g(f)$ *of* $f(z_1,z_2)$ *with respect to* $g(z_1,z_2)$ *is defined as.*

$$_{v_2}\sigma_g(f) = \inf\left\{k > 0 : M_f(r_1,r_2) < M_g(kr_1^{{}_{v_2}\rho_g(f)}, kr_2^{{}_{v_2}\rho_g(f)}) \quad \textit{for all sufficiently large values of } r_1 \textit{ and } r_2\right\}.$$

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Equivalent formula for ${}_{v_2}\sigma_g(f)$ *is*

$${}_{v_2}\sigma_g(f) = \limsup_{r_1,r_2\to\infty} \frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\rho_g(f)}}.$$

Likewise, one can define the relative lower type *of an entire function* $f(z_1,z_2)$ *with respect to an entire function* $g(z_1,z_2)$ *denoted by* ${}_{v_2}\overline{\sigma}_g(f)$ *as follows*

$${}_{v_2}\overline{\sigma}_g(f) = \liminf_{r_1,r_2\to\infty} \frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\rho_g(f)}},\ 0 < {}_{v_2}\rho_g(f) < \infty\,.$$

Definition 5. [6] *The* relative weak type ${}_{v_2}\tau_g(f)$ *of an entire function* $f(z_1,z_2)$ *with respect to another entire function* $g(z_1,z_2)$ *having finite positive* relative lower order ${}_{v_2}\lambda_g(f)$ *is defined as*

$${}_{v_2}\tau_g(f) = \liminf_{r_1,r_2\to\infty} \frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\lambda_g(f)}}.$$

Also one may define the growth indicator ${}_{v_2}\overline{\tau}_g(f)$ *of an entire function* f *with respect to an entire function* g *in the following way*

$${}_{v_2}\overline{\tau}_g(f) = \limsup_{r_1,r_2\to\infty} \frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{{}_{v_2}\lambda_g(f)}},\ 0 < {}_{v_2}\lambda_g(f) < \infty\,.$$

Considering $g(z_1,z_2) = \exp(z_1z_2)$ one may easily verify that Definition 4 and Definition 5 coincide with Definition 2 and Definition 3 respectively.

In the paper we investigate some relative growth properties of entire functions of two complex variables with respect to another entire function of two complex variables on the basis of *relative type* and *relative weak type* of two complex variables. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [7].

2 Lemma

In this section we present a lemma due to Datta et al. [5].

Lemma 1. [5] *Let* f *and* g *be any two entire functions of two complex variables with*

$$0 \le_{v_2} \lambda_f \le_{v_2} \rho_f < \infty \text{ and } 0 \le_{v_2} \lambda_g \le_{v_2} \rho_g < \infty.$$

Then

$$\frac{{}_{v_2}\lambda_f}{{}_{v_2}\rho_g} \le {}_{v_2}\lambda_g(f) \le \min\left\{\frac{{}_{v_2}\lambda_f}{{}_{v_2}\lambda_g}, \frac{{}_{v_2}\rho_f}{{}_{v_2}\rho_g}\right\} \le \max\left\{\frac{{}_{v_2}\lambda_f}{{}_{v_2}\lambda_g}, \frac{{}_{v_2}\rho_f}{{}_{v_2}\rho_g}\right\} \le {}_{v_2}\rho_g(f) \le \frac{{}_{v_2}\rho_f}{{}_{v_2}\lambda_g}.$$

3 Theorems

In this section we present the main results of the paper.

Theorem 1. *Let* $f(z_1,z_2)$ *and* $g(z_1,z_2)$ *be any two entire functions with* $0 \le {}_{v_2}\lambda_f \le {}_{v_2}\rho_f < \infty$ *and* $0 \le {}_{v_2}\lambda_g \le {}_{v_2}\rho_g < \infty$. *Then*

$$\max\left\{\left[\frac{{}_{v_2}\overline{\sigma}_f}{{}_{v_2}\tau_g}\right]^{\frac{1}{{}_{v_2}\lambda_g}}, \left[\frac{{}_{v_2}\sigma_f}{{}_{v_2}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_2}\lambda_g}}\right\} \le {}_{v_2}\sigma_g(f) \le \min\left\{\left[\frac{{}_{v_2}\overline{\tau}_f}{{}_{v_2}\tau_g}\right]^{\frac{1}{{}_{v_2}\lambda_g}}, \left[\frac{{}_{v_2}\sigma_f}{{}_{v_2}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_2}\rho_g}}, \left[\frac{{}_{v_2}\overline{\tau}_f}{{}_{v_2}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_2}\rho_g}}\right\}$$

and

$$\left[\frac{{}_{v_2}\overline{\sigma}_f}{{}_{v_2}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_2}\lambda_g}} \le {}_{v_2}\overline{\sigma}_g(f) \le \min\left\{\begin{array}{l}\left[\frac{{}_{v_2}\overline{\sigma}_f}{{}_{v_2}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_2}\rho_g}}, \left[\frac{{}_{v_2}\sigma_f}{{}_{v_2}\sigma_g}\right]^{\frac{1}{{}_{v_2}\rho_g}}, \left[\frac{{}_{v_2}\tau_f}{{}_{v_2}\tau_g}\right]^{\frac{1}{{}_{v_2}\lambda_g}}, \\ \left[\frac{{}_{v_2}\overline{\tau}_f}{{}_{v_2}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_2}\lambda_g}}, \left[\frac{{}_{v_2}\overline{\tau}_f}{{}_{v_2}\sigma_g}\right]^{\frac{1}{{}_{v_2}\rho_g}}, \left[\frac{{}_{v_2}\tau_f}{{}_{v_2}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_2}\rho_g}}\end{array}\right\}.$$

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Proof. From the definitions of ${}_{v_2}\sigma_f$ and ${}_{v_2}\overline{\sigma}_f$, we have for all sufficiently large values of r_1, r_2 that

$$M_f(r_1,r_2) \le \exp\left\{\left({}_{v_2}\sigma_f + \varepsilon\right)[r_1 r_2]^{{}_{v_2}\rho_f}\right\}, \tag{1}$$
$$M_f(r_1,r_2) \ge \exp\left\{\left({}_{v_2}\overline{\sigma}_f - \varepsilon\right)[r_1 r_2]^{{}_{v_2}\rho_f}\right\} \tag{2}$$

and also for a sequence of values of r_1, r_2 tending to infinity, we get that

$$M_f(r_1,r_2) \ge \exp\left\{\left({}_{v_2}\sigma_f - \varepsilon\right)[r_1 r_2]^{{}_{v_2}\rho_f}\right\}, \tag{3}$$
$$M_f(r_1,r_2) \le \exp\left\{\left({}_{v_2}\overline{\sigma}_f + \varepsilon\right)[r_1 r_2]^{{}_{v_2}\rho_f}\right\} . \tag{4}$$

Similarly from the definitions of ${}_{v_2}\sigma_g$ and ${}_{v_2}\overline{\sigma}_f$, it follows for all sufficiently large values of r_1, r_2 that

$$M_g(r_1,r_2) \le \exp\left\{({}_{v_2}\sigma_g + \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}.$$

Thus

$$[r_1 r_2] \le M_g^{-1}\left[\exp\left\{({}_{v_2}\sigma_g + \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}\right]$$

and

$$M_g^{-1}(r_1,r_2) \ge \left[\left(\frac{\log(r_1 r_2)}{({}_{v_2}\sigma_g + \varepsilon)}\right)^{\frac{1}{{}_{v_2}\rho_g}}\right]. \tag{5}$$

$$M_g(r_1,r_2) \ge \exp\left\{({}_{v_2}\overline{\sigma}_g - \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}.$$

Thus

$$[r_1 r_2] \ge M_g^{-1}\left[\exp\left\{({}_{v_2}\overline{\sigma}_g - \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}\right]$$

and

$$M_g^{-1}(r_1,r_2) \le \left[\left(\frac{\log(r_1 r_2)}{({}_{v_2}\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{{}_{v_2}\rho_g}}\right]. \tag{6}$$

and for a sequence of values of r_1, r_2 tending to infinity, we obtain that

$$M_g(r_1,r_2) \ge \exp\left\{({}_{v_2}\sigma_g - \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}.$$

Thus

$$[r_1 r_2] \ge M_g^{-1}\left[\exp\left\{({}_{v_2}\sigma_g - \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}\right]$$

and

$$M_g^{-1}(r_1,r_2) \le \left[\left(\frac{\log(r_1 r_2)}{({}_{v_2}\sigma_g - \varepsilon)}\right)^{\frac{1}{{}_{v_2}\rho_g}}\right]. \tag{7}$$

$$M_g(r_1,r_2) \le \exp\left\{({}_{v_2}\overline{\sigma}_g + \varepsilon)[r_1 r_2]^{{}_{v_2}\rho_g}\right\}.$$

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Thus

$$[r_1r_2] \leq M_g^{-1}\left[\exp\left\{\left({}_{v_2}\overline{\sigma}_g+\varepsilon\right)[r_1r_2]^{v_2\rho_g}\right\}\right]$$

and

$$M_g^{-1}(r_1,r_2) \geq \left[\left(\frac{\log(r_1r_2)}{\left({}_{v_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{v_2\rho_g}}\right]. \tag{8}$$

From the definitions of ${}_{v_2}\overline{\tau}_f$ and ${}_{v_2}\tau_f$, we have for all sufficiently large values of r_1,r_2 that

$$M_f(r_1,r_2) \leq \exp\left[\left({}_{v_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right], \tag{9}$$

$$M_f(r_1,r_2) \geq \exp\left[\left({}_{v_2}\tau_f-\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right] \tag{10}$$

and also for a sequence of values of r_1,r_2 tending to infinity, we get that

$$M_f(r_1,r_2) \geq \exp\left[\left({}_{v_2}\overline{\tau}_f-\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right], \tag{11}$$

$$M_f(r_1,r_2) \leq \exp\left[\left({}_{v_2}\tau_f+\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right]. \tag{12}$$

Similarly from the definitions of ${}_{v_2}\overline{\tau}_g$ and ${}_{v_2}\tau_g$, it follows for all sufficiently large values of r_1,r_2 that

$$M_g(r_1,r_2) \leq \exp\left[\left({}_{v_2}\overline{\tau}_g+\varepsilon\right)\cdot[r_1r_2]^{v_2\lambda_g}\right].$$

Thus

$$[r_1r_2] \leq M_g^{-1}\left[\exp\left[\left({}_{v_2}\overline{\tau}_g+\varepsilon\right)\cdot[r_1r_2]^{v_2\lambda_g}\right]\right]$$

and

$$M_g^{-1}(r_1,r_2) \geq \left[\left(\frac{\log(r_1r_2)}{\left({}_{v_2}\overline{\tau}_g+\varepsilon\right)}\right)^{\frac{1}{v_2\lambda_g}}\right]. \tag{13}$$

$$M_g(r_1,r_2) \geq \exp\left[\left({}_{v_2}\tau_g-\varepsilon\right)\cdot[r_1r_2]^{v_2\lambda_g}\right].$$

Thus

$$[r_1r_2] \geq M_g^{-1}\left[\exp\left[\left({}_{v_2}\tau_g-\varepsilon\right)\cdot[r_1r_2]^{v_2\lambda_g}\right]\right]$$

and

$$M_g^{-1}(r_1,r_2) \leq \left[\left(\frac{\log(r_1r_2)}{\left({}_{v_2}\tau_g-\varepsilon\right)}\right)^{\frac{1}{v_2\lambda_g}}\right]. \tag{14}$$

and for a sequence of values of r_1,r_2 tending to infinity, we obtain that

$$M_g(r_1,r_2) \geq \exp\left[\left({}_{v_2}\overline{\tau}_g-\varepsilon\right)\cdot[r_1r_2]^{v_2\lambda_g}\right],$$

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that is

$$[r_1r_2] \geq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\overline{\tau}_g - \varepsilon\right)\cdot [r_1r_2]^{\nu_2\lambda_g}\right]\right]$$

and

$$M_g^{-1}(r_1,r_2) \leq \left[\left(\frac{\log(r_1r_2)}{\left({}_{\nu_2}\overline{\tau}_g - \varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right]. \tag{15}$$

$$M_g(r_1,r_2) \leq \exp\left[\left({}_{\nu_2}\tau_g + \varepsilon\right)\cdot [r_1r_2]^{\nu_2\lambda_g}\right],$$

that is

$$[r_1r_2] \leq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\tau_g + \varepsilon\right)\cdot [r_1r_2]^{\nu_2\lambda_g}\right]\right]$$

and

$$M_g^{-1}(r_1,r_2) \geq \left[\left(\frac{\log(r_1r_2)}{\left({}_{\nu_2}\tau_g - \varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right]. \tag{16}$$

Now from (3) and in view of (13), we get for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\sigma_f - \varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]\right].$$

Thus

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\sigma_f - \varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]}{\left({}_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left({}_{\nu_2}\sigma_f - \varepsilon\right)}{\left({}_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}} \cdot [r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\lambda_g}}} \geq \left[\frac{\left({}_{\nu_2}\sigma_f - \varepsilon\right)}{\left({}_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 1, $\frac{\nu_2\rho_f}{\nu_2\lambda_g} \geq {}_{\nu_2}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\rho_g(f)}} \geq \left[\frac{{}_{\nu_2}\sigma_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}}$$

that is

$${}_{\nu_2}\sigma_g(f) \geq \left[\frac{{}_{\nu_2}\sigma_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}}. \tag{17}$$

Similarly from (2) and in view of (16), it follows for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\overline{\sigma}_f - \varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]\right],$$

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therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(_{v_2}\overline{\sigma}_f-\varepsilon\right)[r_1r_2]^{v_2\rho_f}\right]}{\left(_{v_2}\tau_g-\varepsilon\right)}\right)^{\frac{1}{v_2\lambda_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(_{v_2}\overline{\sigma}_f-\varepsilon\right)}{\left(_{v_2}\tau_g+\varepsilon\right)}\right]^{\frac{1}{v_2\lambda_g}} \cdot [r_1r_2]^{\frac{v_2\rho_f}{v_2\lambda_g}},$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\rho_f}{v_2\lambda_g}}} \geq \left[\frac{\left(_{v_2}\overline{\sigma}_f-\varepsilon\right)}{\left(_{v_2}\tau_g+\varepsilon\right)}\right]^{\frac{1}{v_2\lambda_g}}.$$

Since in view of Lemma 1, it follows that $\frac{v_2\rho_f}{v_2\lambda_g} \geq {}_{v_2}\rho_g(f)$. Also $\varepsilon\,(>0)$ is arbitrary, so we get from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\rho_g(f)}} \geq \left[\frac{_{v_2}\overline{\sigma}_f}{_{v_2}\tau_g}\right]^{\frac{1}{v_2\lambda_g}}$$

and

$$_{v_2}\sigma_g(f) \geq \left[\frac{_{v_2}\overline{\sigma}_f}{_{v_2}\tau_g}\right]^{\frac{1}{v_2\lambda_g}}. \tag{18}$$

Again in view of (14), we have from (9) for all sufficiently large values of r_1,r_2 that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left(_{v_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left(_{v_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right]}{\left(_{v_2}\tau_g-\varepsilon\right)}\right)^{\frac{1}{v_2\lambda_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left(_{v_2}\overline{\tau}_f+\varepsilon\right)}{\left(_{v_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{v_2\lambda_g}} \cdot [r_1r_2]^{\frac{v_2\lambda_f}{v_2\lambda_g}},$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\lambda_f}{v_2\lambda_g}}} \leq \left[\frac{\left(_{v_2}\overline{\tau}_f+\varepsilon\right)}{\left(_{v_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{v_2\lambda_g}}.$$

Since in view of Lemma 1, we get that $\frac{v_2\lambda_f}{v_2\lambda_g} \leq_{v_2} \rho_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\rho_g(f)}} \leq \left[\frac{_{v_2}\overline{\tau}_f}{_{v_2}\tau_g}\right]^{\frac{1}{v_2\lambda_g}}$$

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and

$$ {}_{\nu_2}\sigma_g(f) \leq \left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\tau_g}\right]^{\frac{1}{\nu_2\lambda_g}} . \tag{19}$$

Again in view of (6), we have from (1) for all sufficiently large values of r_1, r_2 that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\sigma_f+\varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]\right]$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\sigma_f+\varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\sigma_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}} ,$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}}} \leq \left[\frac{\left({}_{\nu_2}\sigma_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} . \tag{20}$$

As in view of Lemma 1, it follows that $\frac{\nu_2\rho_f}{\nu_2\rho_g} \leq {}_{\nu_2}\rho_g(f)$. Since $\varepsilon(>0)$ is arbitrary, we get from (20) that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\rho_g(f)}} \leq \left[\frac{{}_{\nu_2}\sigma_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

that is

$$ {}_{\nu_2}\sigma_g(f) \leq \left[\frac{{}_{\nu_2}\sigma_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}} . \tag{21}$$

Further in view of (6), we have from (9) for all sufficiently large values of r_1, r_2 that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]\right]$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}} ,$$

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and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\lambda_f}{v_2\rho_g}}} \leq \left[\frac{(v_2\overline{\tau}_f+\varepsilon)}{(v_2\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{v_2\rho_g}}.$$

Since in view of Lemma 1, we get that $\frac{v_2\lambda_f}{v_2\rho_g} \leq {}_{v_2}\rho_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\rho_g^{L^*}(f)}} \leq \left[\frac{v_2\overline{\tau}_f}{v_2\overline{\sigma}_g}\right]^{\frac{1}{v_2\rho_g}},$$

that is

$$_{v_2}\sigma_g^{L^*}(f) \leq \left[\frac{v_2\overline{\tau}_f}{v_2\overline{\sigma}_g}\right]^{\frac{1}{v_2\rho_g}}. \tag{22}$$

Thus the first part of the theorem follows from (17), (18), (19), (21) and (22).

Further from (2) and in view of (13), we get for all sufficiently large values of r_1,r_2 that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[(v_2\overline{\sigma}_f-\varepsilon)[r_1r_2]^{v_2\rho_f}\right]\right]$$

and

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[(v_2\overline{\sigma}_f-\varepsilon)[r_1r_2]^{v_2\rho_f}\right]}{(v_2\overline{\tau}_g+\varepsilon)}\right)^{\frac{1}{v_2\lambda_g}}\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{(v_2\overline{\sigma}_f-\varepsilon)}{(v_2\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{v_2\lambda_g}}\cdot[r_1r_2]^{\frac{v_2\rho_f}{v_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\rho_f}{v_2\lambda_g}}} \geq \left[\frac{(v_2\overline{\sigma}_f-\varepsilon)}{(v_2\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{v_2\lambda_g}}.$$

Now in view of Lemma 1, it follows that $\frac{v_2\rho_f}{v_2\lambda_g} \geq {}_{v_2}\rho_g(f)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\rho_g(f)}} \geq \left[\frac{v_2\overline{\sigma}_f}{v_2\overline{\tau}_g}\right]^{\frac{1}{v_2\lambda_g}}$$

and

$$_{v_2}\overline{\sigma}_g(f) \geq \left[\frac{v_2\overline{\sigma}_f}{v_2\overline{\tau}_g}\right]^{\frac{1}{v_2\lambda_g}}. \tag{23}$$

Also in view of (7), we get from (1) for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[(v_2\sigma_f+\varepsilon)[r_1r_2]^{v_2\rho_f}\right]\right],$$

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therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\sigma_f+\varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]}{\left({}_{\nu_2}\sigma_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\sigma_f+\varepsilon\right)}{\left({}_{\nu_2}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}}} \leq \left[\frac{\left({}_{\nu_2}\sigma_f+\varepsilon\right)}{\left({}_{\nu_2}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g(m,p)}}. \tag{24}$$

Again in view of Lemma 1, $\frac{\nu_2\rho_f}{\nu_2\rho_g} \leq {}_{\nu_2}\rho_g(f)$ and $\varepsilon(>0)$ is arbitrary, so we get from (24) that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\rho_g(f)}} \leq \left[\frac{\nu_2\sigma_f}{\nu_2\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

that is

$${}_{\nu_2}\overline{\sigma}_g(f) \leq \left[\frac{\nu_2\sigma_f}{\nu_2\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}}. \tag{25}$$

Likewise from (4) and in view of (6), it follows for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\overline{\sigma}_f+\varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]\right],$$

thefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\overline{\sigma}_f+\varepsilon\right)[r_1r_2]^{\nu_2\rho_f^{L^*}}\right]}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\overline{\sigma}_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}}} \leq \left[\frac{\left({}_{\nu_2}\overline{\sigma}_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}}. \tag{26}$$

Analogously, we get from (26) that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\rho_g(f)}} \leq \left[\frac{\nu_2\overline{\sigma}_f}{\nu_2\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

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and

$$_{v_2}\overline{\sigma}_g(f) \leq \left[\frac{_{v_2}\overline{\sigma}_f}{_{v_2}\overline{\sigma}_g}\right]^{\frac{1}{_{v_2}\rho_g}}, \tag{27}$$

since in view of Lemma 1, $\frac{_{v_2}\rho_f}{_{v_2}\rho_g} \leq {}_{v_2}\rho_g(f)$ and $\varepsilon(>0)$ is arbitrary.

Further in view of (15), we get from (9) for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left(_{v_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{_{v_2}\lambda_f}\right]\right],$$

thefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left(_{v_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{_{v_2}\lambda_f}\right]}{\left(_{v_2}\overline{\tau}_g-\varepsilon\right)}\right)^{\frac{1}{_{v_2}\lambda_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left(_{v_2}\overline{\tau}_f+\varepsilon\right)}{\left(_{v_2}\overline{\tau}_g-\varepsilon\right)}\right]^{\frac{1}{_{v_2}\lambda_g}} \cdot [r_1r_2]^{\frac{_{v_2}\lambda_f}{_{v_2}\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{_{v_2}\lambda_f}{_{v_2}\lambda_g}}} \leq \left[\frac{\left(_{v_2}\overline{\tau}_f+\varepsilon\right)}{\left(_{v_2}\overline{\tau}_g-\varepsilon\right)}\right]^{\frac{1}{_{v_2}\lambda_g}}.$$

As in view of Lemma 1, we get that $\frac{_{v_2}\lambda_f}{_{v_2}\lambda_g} \leq {}_{v_2}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{_{v_2}\rho_g^{L^*}(f)}} \leq \left[\frac{_{v_2}\overline{\tau}_f}{_{v_2}\overline{\tau}_g}\right]^{\frac{1}{_{v_2}\lambda_g}}$$

that is

$$_{v_2}\overline{\sigma}_g^{L^*}(f) \leq \left[\frac{_{v_2}\overline{\tau}_f}{_{v_2}\overline{\tau}_g}\right]^{\frac{1}{_{v_2}\lambda_g}}. \tag{28}$$

Similarly from (12) and in view of (14), it follows for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left(_{v_2}\tau_f+\varepsilon\right)[r_1r_2]^{_{v_2}\lambda_g}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left(_{v_2}\tau_f+\varepsilon\right)[r_1r_2]^{_{v_2}\lambda_f}\right]}{\left(_{v_2}\tau_g-\varepsilon\right)}\right)^{\frac{1}{_{v_2}\lambda_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left(_{v_2}\tau_f+\varepsilon\right)}{\left(_{v_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{_{v_2}\lambda_g}} \cdot [r_1r_2]^{\frac{_{v_2}\lambda_f}{_{v_2}\lambda_g}}$$

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and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\lambda_f}{v_2\lambda_g}}} \le \left[\frac{(v_2\tau_f+\varepsilon)}{(v_2\tau_g-\varepsilon)}\right]^{\frac{1}{v_2\lambda_g}} .$$

Also in view of Lemma 1, we get that $\frac{v_2\lambda_f}{v_2\lambda_g} \le {}_{v_2}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\rho_g(f)}} \le \left[\frac{v_2\tau_f}{v_2\tau_g}\right]^{\frac{1}{v_2\lambda_g}}$$

that is

$$_{v_2}\overline{\sigma}_g(f) \le \left[\frac{v_2\tau_f}{v_2\tau_g}\right]^{\frac{1}{v_2\lambda_g}} . \tag{29}$$

Again in view of (7), we get from (9) for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \le M_g^{-1}\left[\exp\left[(v_2\overline{\tau}_f+\varepsilon)[r_1r_2]^{v_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \le \left[\left(\frac{\log\exp\left[(v_2\overline{\tau}_f+\varepsilon)[r_1r_2]^{v_2\lambda_f}\right]}{(v_2\sigma_g-\varepsilon)}\right)^{\frac{1}{v_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \le \left[\frac{(v_2\overline{\tau}_f+\varepsilon)}{(v_2\sigma_g-\varepsilon)}\right]^{\frac{1}{v_2\rho_g}} \cdot [r_1r_2]^{\frac{v_2\lambda_f}{v_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\lambda_f^{L^*}}{v_2\rho_g}}} \le \left[\frac{(v_2\overline{\tau}_f+\varepsilon)}{v_2(\sigma_g-\varepsilon)}\right]^{\frac{1}{v_2\rho_g}} .$$

Since in view of Lemma 1, we get that $\frac{v_2\lambda_f}{v_2\rho_g} \le {}_{v_2}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\rho_g^{L^*}(f)}} \le \left[\frac{v_2\overline{\tau}_f}{v_2\sigma_g}\right]^{\frac{1}{v_2\rho_g}}$$

that is

$$_{v_2}\overline{\sigma}_g(f) \le \left[\frac{v_2\overline{\tau}_f}{v_2\sigma_g}\right]^{\frac{1}{v_2\rho_g}} . \tag{30}$$

Similarly from (12) and in view of (6), it follows for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \le M_g^{-1}\left[\exp\left[(v_2\tau_f+\varepsilon)[r_1r_2]^{v_2\lambda_f}\right]\right],$$

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therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\tau_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\tau_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}} \leq \left[\frac{\left({}_{\nu_2}\tau_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}}.$$

As in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\rho_g} \leq {}_{\nu_2}\rho_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\rho_g(f)}} \leq \left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

that is

$${}_{\nu_2}\overline{\sigma}_g(f) \leq \left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}. \tag{31}$$

Hence the second part of the theorem follows from (23), (25), (27), (28), (29), (30) and (31).

Theorem 2. *Let $f(z_1,z_2)$ and $g(z_1,z_2)$ be any two entire functions with $0 \leq {}_{\nu_2}\lambda_f \leq {}_{\nu_2}\rho_f < \infty$ and $0 \leq {}_{\nu_2}\lambda_g \leq {}_{\nu_2}\rho_g < \infty$. Then*

$$\max\left\{\begin{array}{c}\left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}},\left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\tau_g}\right]^{\frac{1}{\nu_2\lambda_g}},\left[\frac{{}_{\nu_2}\overline{\sigma}_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}},\\ \left[\frac{{}_{\nu_2}\sigma_f}{{}_{\nu_2}\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}},\left[\frac{{}_{\nu_2}\sigma_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}},\left[\frac{{}_{\nu_2}\overline{\sigma}_f}{{}_{\nu_2}\tau_g}\right]^{\frac{1}{\nu_2\lambda_g}}\end{array}\right\} \leq {}_{\nu_2}\overline{\tau}_g(f) \leq \left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

and

$$\max\left\{\left[\frac{{}_{\nu_2}\overline{\sigma}_f}{{}_{\nu_2}\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}},\left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}},\left[\frac{{}_{\nu_2}\overline{\sigma}_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}}\right\} \leq {}_{\nu_2}\tau_g(f) \leq \min\left\{\left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}},\left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}}\right\}.$$

Proof. We obtain from (11) and (13), for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\overline{\tau}_f-\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\overline{\tau}_f-\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\overline{\tau}_g+\varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right],$$

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that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left({}_{\nu_2}\overline{\tau}_f-\varepsilon\right)}{\left(\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\lambda_g}}} \geq \left[\frac{\left({}_{\nu_2}\overline{\tau}_f-\varepsilon\right)}{\left({}_{\nu_2}\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\lambda_g} \geq {}_{\nu_2}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\lambda_g(f)}} \geq \left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}}$$

and

$${}_{\nu_2}\overline{\tau}_g(f) \geq \left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}}. \tag{32}$$

Further we obtain from (10) and (16), for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\tau_f-\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\tau_f-\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\tau_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left({}_{\nu_2}\tau_f-\varepsilon\right)}{\left({}_{\nu_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\lambda_g}}} \geq \left[\frac{\left({}_{\nu_2}\tau_f-\varepsilon\right)}{\left({}_{\nu_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}}.$$

As in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\lambda_g} \geq {}_{\nu_2}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\lambda_g(f)}} \geq \left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\tau_g}\right]^{\frac{1}{\nu_2\lambda_g}}$$

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and

$$_{\nu_2}\overline{\tau}_g(f) \geq \left[\frac{_{\nu_2}\tau_f}{_{\nu_2}\tau_g}\right]^{\frac{1}{_{\nu_2}\lambda_g}}. \tag{33}$$

Now from (3) and in view of (5), we get for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(_{\nu_2}\sigma_f-\varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(_{\nu_2}\sigma_f-\varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]}{\left(_{\nu_2}\sigma_g+\varepsilon\right)}\right)^{\frac{1}{_{\nu_2}\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(_{\nu_2}\sigma_f-\varepsilon\right)}{\left(_{\nu_2}\sigma_g+\varepsilon\right)}\right]^{\frac{1}{_{\nu_2}\rho_g}} \cdot [r_1r_2]^{\frac{_{\nu_2}\rho_f}{_{\nu_2}\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{_{\nu_2}\rho_f}{_{\nu_2}\rho_g}}} \geq \left[\frac{\left(_{\nu_2}\sigma_f-\varepsilon\right)}{\left(_{\nu_2}\sigma_g+\varepsilon\right)}\right]^{\frac{1}{_{\nu_2}\rho_g}}. \tag{34}$$

Also from (2) and in view of (8), it follows for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(_{\nu_2}\overline{\sigma}_f-\varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(_{\nu_2}\overline{\sigma}_f-\varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]}{\left(_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{_{\nu_2}\rho_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(_{\nu_2}\overline{\sigma}_f-\varepsilon\right)}{\left(_{\nu_2}\overline{\sigma}_g+\varepsilon\right)}\right]^{\frac{1}{_{\nu_2}\rho_g}} \cdot [r_1r_2]^{\frac{_{\nu_2}\rho_f}{_{\nu_2}\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{_{\nu_2}\rho_f}{_{\nu_2}\rho_g}}} \geq \left[\frac{\left(_{\nu_2}\overline{\sigma}_f-\varepsilon\right)}{\left(_{\nu_2}\overline{\sigma}_g+\varepsilon\right)}\right]^{\frac{1}{_{\nu_2}\rho_g}}. \tag{35}$$

As in view of Lemma 1, $\frac{_{\nu_2}\rho_f}{_{\nu_2}\rho_g} \geq {_{\nu_2}\lambda_g}(f)$ and $\varepsilon\,(>0)$ is arbitrary, we get from (34) that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{_{\nu_2}\lambda_g(f)}} \geq \left[\frac{_{\nu_2}\sigma_f}{_{\nu_2}\sigma_g}\right]^{\frac{1}{_{\nu_2}\rho_g}}$$

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that is

$$_{\nu_2}\overline{\tau}_g(f) \geq \left[\frac{_{\nu_2}\sigma_f}{_{\nu_2}\sigma_g}\right]^{\frac{1}{_{\nu_2}\rho_g}} . \tag{36}$$

Simialrly, we get from (35) that

$$\limsup_{r_1,r_2\to\infty} \frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\lambda_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

that is

$$\overline{\tau}_g(f) \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} , \tag{37}$$

since in view of Lemma 1, $\frac{_{\nu_2}\rho_f}{_{\nu_2}\rho_g} \leq {_{\nu_2}\lambda_g}(f)$ and $\varepsilon(>0)$ is arbitrary.

Likewise from (3) and in view of (13), we get for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(_{\nu_2}\sigma_f - \varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(_{\nu_2}\sigma_f - \varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]}{\left(_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right)^{\frac{1}{_{\nu_2}\lambda_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(_{\nu_2}\sigma_f - \varepsilon\right)}{\left(_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{_{\nu_2}\lambda_g}} \cdot [r_1r_2]^{\frac{_{\nu_2}\rho_f}{_{\nu_2}\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{_{\nu_2}\rho_f}{_{\nu_2}\lambda_g}}} \geq \left[\frac{\left(_{\nu_2}\sigma_f - \varepsilon\right)}{\left(_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{_{\nu_2}\lambda_g}} .$$

Since in view of Lemma 1, we get that $\frac{_{\nu_2}\rho_f}{_{\nu_2}\lambda_g} \geq {_{\nu_2}\lambda_g}(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r_1,r_2\to\infty} \frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\lambda_g(f)}} \geq \left[\frac{_{\nu_2}\sigma_f}{_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{_{\nu_2}\lambda_g}}$$

that is

$$_{\nu_2}\overline{\tau}_g(f) \geq \left[\frac{_{\nu_2}\sigma_f}{_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{_{\nu_2}\lambda_g}} . \tag{38}$$

Further from (2) and in view of (16), it follows for a sequence of values of r_1, r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(_{\nu_2}\overline{\sigma}_f - \varepsilon\right)[r_1r_2]^{_{\nu_2}\rho_f}\right]\right],$$

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therefore

$$M_g^{-1}M_f(r_1,r_2)\geq\left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\overline{\sigma}_f-\varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]}{\left({}_{\nu_2}\tau_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2)\geq\left[\frac{\left({}_{\nu_2}\overline{\sigma}_f-\varepsilon\right)}{\left({}_{\nu_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}}\cdot[r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\lambda_g}}}\geq\left[\frac{\left({}_{\nu_2}\overline{\sigma}_f-\varepsilon\right)}{\left({}_{\nu_2}\tau_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}}.$$

As in view of Lemma 1, we get that $\frac{\rho_f^{\nu_2}}{\nu_2\lambda_g}\geq{}_{\nu_2}\lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\lambda_g(f)}}\geq\left[\frac{{}_{\nu_2}\overline{\sigma}_f}{{}_{\nu_2}\tau_g}\right]^{\frac{1}{\nu_2\lambda_g}},$$

that is

$${}_{\nu_2}\overline{\tau}_g(f)\geq\left[\frac{{}_{\nu_2}\overline{\sigma}_f}{{}_{\nu_2}\tau_g}\right]^{\frac{1}{\nu_2\lambda_g}}. \tag{39}$$

Again from (6) and (9), we have for all sufficiently large values of r_1,r_2 that

$$M_g^{-1}M_f(r_1,r_2)\leq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2)\leq\left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2)\leq\left[\frac{\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}}\cdot[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}}\leq\left[\frac{\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}}.$$

Since in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\rho_g}\leq{}_{\nu_2}\lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\lambda_g(f)}}\leq\left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

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that is

$$_{\nu_2}\overline{\tau}_g(f) \leq \left[\frac{_{\nu_2}\overline{\tau}_f}{_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}} . \tag{40}$$

Thus the first part of the theorem follows from (32), (33), (36), (37), (38), (39) and (40).

Further from (10) and in view of (13), we get for all sufficiently large values of r_1, r_2 that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(_{\nu_2}\tau_f - \varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(_{\nu_2}\tau_f - \varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left(_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right)^{\frac{1}{\nu_2\lambda_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(_{\nu_2}\tau_f - \varepsilon\right)}{\left(_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\lambda_g}}} \geq \left[\frac{\left(_{\nu_2}\tau_f - \varepsilon\right)}{\left(_{\nu_2}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\nu_2\lambda_g}} .$$

Since in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\lambda_g} \geq {}_{\nu_2}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\lambda_g(f)}} \geq \left[\frac{_{\nu_2}\tau_f}{_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}}$$

that is

$$_{\nu_2}\tau_g(f) \geq \left[\frac{_{\nu_2}\tau_f}{_{\nu_2}\overline{\tau}_g}\right]^{\frac{1}{\nu_2\lambda_g}} . \tag{41}$$

Again from (2) and in view of (5), we get for all sufficiently large values of r_1, r_2 that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(_{\nu_2}\overline{\sigma}_f - \varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(_{\nu_2}\overline{\sigma}_f - \varepsilon\right)[r_1r_2]^{\nu_2\rho_f}\right]}{\left(_{\nu_2}\sigma_g + \varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right],$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(_{\nu_2}\overline{\sigma}_f - \varepsilon\right)}{\left(_{\nu_2}\sigma_g + \varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\rho_f}{\nu_2\rho_g}}$$

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and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\rho_f}{v_2\rho_g}}} \geq \left[\frac{\left(v_2\overline{\sigma}_f-\varepsilon\right)}{\left(v_2\sigma_g+\varepsilon\right)}\right]^{\frac{1}{v_2\rho_g}} . \tag{42}$$

As in view of Lemma 1, $\frac{v_2\rho_f}{v_2\rho_g} \geq {}_{v_2}\lambda_g(f)$ and $\varepsilon(>0)$ is arbitrary, we get from (42) that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\lambda_g(f)}} \geq \left[\frac{v_2\overline{\sigma}_f}{v_2\sigma_g}\right]^{\frac{1}{v_2\rho_g}}$$

that is

$${}_{v_2}\tau_g(f) \geq \left[\frac{v_2\overline{\sigma}_f}{v_2\sigma_g}\right]^{\frac{1}{v_2\rho_g}} . \tag{43}$$

Again from (2) and in view of (13), we get for all sufficiently large values of r_1,r_2 that

$$M_g^{-1}M_f(r_1,r_2) \geq M_g^{-1}\left[\exp\left[\left(v_2\overline{\sigma}_f-\varepsilon\right)[r_1r_2]^{v_2\rho_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\left(\frac{\log\exp\left[\left(v_2\overline{\sigma}_f-\varepsilon\right)[r_1r_2]^{v_2\rho_f}\right]}{\left(v_2\overline{\tau}_g+\varepsilon\right)}\right)^{\frac{1}{v_2\lambda_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \geq \left[\frac{\left(v_2\overline{\sigma}_f-\varepsilon\right)}{\left(v_2\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{v_2\lambda_g}} \cdot [r_1r_2]^{\frac{v_2\rho_f}{v_2\lambda_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{v_2\rho_f}{v_2\lambda_g}}} \geq \left[\frac{\left(v_2\overline{\sigma}_f-\varepsilon\right)}{\left(v_2\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{v_2\lambda_g}} .$$

Since in view of Lemma 1, we get that $\frac{v_2\rho_f}{v_2\lambda_g} \geq {}_{v_2}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{v_2\lambda_g(f)}} \geq \left[\frac{v_2\overline{\sigma}_f}{v_2\overline{\tau}_g}\right]^{\frac{1}{v_2\lambda_g}}$$

that is

$${}_{v_2}\tau_g(f) \geq \left[\frac{v_2\overline{\sigma}_f}{v_2\overline{\tau}_g}\right]^{\frac{1}{v_2\lambda_g}} . \tag{44}$$

Moreover, we get from (7) and (9) for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left(v_2\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{v_2\lambda_f}\right]\right],$$

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therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\sigma_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)}{\left({}_{\nu_2}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}} \leq \left[\frac{\left({}_{\nu_2}\overline{\tau}_f+\varepsilon\right)}{\left({}_{\nu_2}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}}.$$

As in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\rho_g} \leq {}_{\nu_2}\lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\nu_2\lambda_g(f)}} \leq \left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

that is

$${}_{\nu_2}\tau_g(f) \leq \left[\frac{{}_{\nu_2}\overline{\tau}_f}{{}_{\nu_2}\sigma_g}\right]^{\frac{1}{\nu_2\rho_g}}. \tag{45}$$

Similarly, from (12) and in view of (6), it follows for a sequence of values of r_1,r_2 tending to infinity that

$$M_g^{-1}M_f(r_1,r_2) \leq M_g^{-1}\left[\exp\left[\left({}_{\nu_2}\tau_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]\right],$$

therefore

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\left(\frac{\log\exp\left[\left({}_{\nu_2}\tau_f+\varepsilon\right)[r_1r_2]^{\nu_2\lambda_f}\right]}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\nu_2\rho_g}}\right]$$

that is

$$M_g^{-1}M_f(r_1,r_2) \leq \left[\frac{\left({}_{\nu_2}\tau_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}} \cdot [r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}$$

and

$$\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\frac{\nu_2\lambda_f}{\nu_2\rho_g}}} \leq \left[\frac{\left({}_{\nu_2}\tau_f+\varepsilon\right)}{\left({}_{\nu_2}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\nu_2\rho_g}}.$$

Since in view of Lemma 1, we get that $\frac{\nu_2\lambda_f}{\nu_2\rho_g} \leq {}_{\nu_2}\lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r_1,r_2\to\infty}\frac{M_g^{-1}M_f(r_1,r_2)}{[r_1r_2]^{\lambda_g(f)}} \leq \left[\frac{{}_{\nu_2}\tau_f}{{}_{\nu_2}\overline{\sigma}_g}\right]^{\frac{1}{\nu_2\rho_g}}$$

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that is

$$_{v_2}\tau_g(f) \leq \left[\frac{_{v_2}\tau_f}{_{v_2}\overline{\sigma}_g}\right]^{\frac{1}{_{v_2}\rho_g}}. \quad (46)$$

Hence the second part of the theorem follows from (41), (43), (44), (45) and (46).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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