New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.174

Holder valuation and holder rigidity for right ring of fractions

Abdolghafoor Allahyari and Mohammad Hossein Hosseini

Department of Mathematics, Farhangian University, Birjand, Iran

Received: 8 October 2015, Accepted: 13 July 2016
Published online: 16 August 2017.

Abstract: The purpose of this article is to introduce the notion of (C_1,C_2)-Hölder Krull valuation on right ring of fractions (with respect to right denominator set S in a ring R). It is proved that if R is a ring satisfying in Hölder rigidity condition, and S a right permutable set of regular elements in R, then the right ring of fractions $R' = Q^r_\varphi(R)$ with respect to S satisfies in Hölder rigidity condition. This results provide an extension of the Garsia theorem (see [2]) for right ring of fractions.

Keywords: valuation, Hölder valuation, Hölder equivalent, right ring of fractions.

1 introduction and preliminaries

The theory of valuations may be viewed as a branch of topological algebra. The devolopment of valuation theory has spanned over more than a hundred years. First the notion of valuations on fields was introduced. Details of valuations on fields can be found in many monographs, e. g. Endler (see [1]), Ribenboim (see [7]), and Schilling (see [8]). Then Manis introduced the notion of valuations in the category of commutative rings and it can be found in Manis (see [6]), Huckaba (see [3]), and Knebusch and Zhang (see [4]). A group Γ is called an *ordered multiplicative group* if it has a total ordering $\leq$ which is compatible with the group structure, i. e. $\alpha \leq \beta$ ($\alpha,\beta \in \Gamma$), implies $\gamma\alpha \leq \gamma\beta$, $\alpha\gamma \leq \beta\gamma$, for all $\gamma \in \Gamma$ and $\beta^{-1} \leq \alpha^{-1}$. Let Γ be an ordered multiplicative group. A *Krull valuation* $|\quad|$ on ring R with values in Γ is a mapping $|\quad| : R \to \Gamma \cup \{0\}$ satisfying the conditions.

(i) For $a \in R, |a| = 0$ iff $a = 0$;
(ii) For $a,b \in R, |a+b| \leq Max\{|a|,|b|\}$;
(iii) For $a,b \in R, |ab| = |a||b|$.

with the properties $0.0 = 0$, $0.\alpha = \alpha.0 = 0, \alpha \in \Gamma$ and $0 < \alpha$ for all $\alpha \in \Gamma$.
Let Γ be an ordered multiplicative group and $C_1 \geq 1, C_2 \geq 1$.
A (C_1,C_2)-*Hölder Krull valuation* on ring R with values in Γ is a mapping $\|.\| : R \to \Gamma \cup \{0\}$ satisfying the conditions.

(i) For $a \in R, \|a\| = 0$ iff $a = 0$;
(ii) For $a,b \in R$, $\|a+b\| \leq C_2 Max\{\|a\|,\|b\|\}$;
(iii) For $a,b \in R$, $C_1^{-1}\|a\|\|b\| \leq \|ab\| \leq C_1\|a\|\|b\|$.

Remark. Note that $(1,1)$-Hölder Krull valuation on ring R is a classical Krull valuation on a ring R.

In this paper R is a noncommutative ring with unit element.A ring R' is said to be a *right ring of fraction* if there is a given ring homomorphism $\varphi : R \to R'$ such that.

* Corresponding author e-mail: g.allahyari@birjand.ac.ir

(a) φ is S-inverting ($\varphi(S) \subset U(R)$, where U(R) is set of unit elements of R).
(b) Every element of $R^{'}$ has the form $\varphi(a)\varphi(s)^{-1}$ for some $a \in R$ and $s \in S$.
(c) $Ker\varphi = \{r \in R | rs = 0 \text{ for some } s \in S\}$.

The multiplicative set $S \subset R$ is *right permutable* if for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$, also set $S \subset R$ is *right reversible*, for $a \in R$, if $s^{'}a = 0$ for some $s^{'} \in S$, then $as = 0$ for some $s \in S$.

If the multiplicative set $S \subset R$ is both right permutable and right reversible, we shall say that S is a *right denominator set*. The ring R has a *right ring of fractions* with respect to multiplicative set S iff S is a *right denominator set* (see [5]).Let S be the multiplicative set of all regular elements. We say that R is a *right ore* ring iff S is right permutable, iff RS^{-1} exists. In this case, we speak of RS^{-1} as *the classical right ring of quotients* of R, and denote it by $Q^r_{cl}(R)$. Let R be a domain and $S = R - \{0\}$. In this case, the right permutable condition on S may be re-expressed in the equivalent form: $aR \cap bR \neq 0$ for $a, b \in R - \{0\}$. This is called the (right) *ore condition* on R. Thus, the domain R is *right(resp. left) ore* iff R satisfies the right (resp. left) ore condition.

2 Krull valuation and (C_1, C_2)-Hölder Krull valuation for right ring of fractions

Definition 1. *Let $|.|_1$and $|.|_2$ be two valuations on ring R. Then we say that $|.|_1$and $|.|_2$ are (C_0, α)-Hölder equivalent (where $C_0 \geq 1, \alpha > 0$) if for all $x \in R$,*

$$C_0^{-1}|x|_1^{\alpha^{'}} \leq |x|_2 \leq C_0|x|_1^{\alpha^{'}}$$

where $\alpha^{'} = \alpha$ or $\alpha^{'} = \alpha^{-1}$.

Lemma 1. *Let $|.| : R \to \Gamma \cup \{0\}$ be a Krull valuation on ring R with right ring of fractions $R^{'} = Q^r_{\varphi}(R)$, where Γ is an ordered multiplicative group. Then $|.|_{\varphi} : R^{'} \to \Gamma \cup \{0\}$ by equation.*

$$|x|_{\varphi} = |\varphi(a)\varphi(b)^{-1}|_{\varphi} = |a||b|^{-1}, \text{ for } a \in R, b \in S \text{ is a Krull valuation on ring } R^{'} = Q^r_{\varphi}(R).$$

Proof. (i) Let $x \in R^{'}$ and $|x|_{\varphi} = |\varphi(a)\varphi(b)^{-1}|_{\varphi} = |a||b|^{-1} = 0$ for some $a \in R$, $b \in S$. Then $|a| = 0$ implies $a = 0$. Hence $x = \varphi(0)/\varphi(b) = 0$. Conversely, for $x \in R^{'} = Q^r_{\varphi}(R)$ if $x = 0$, then $|x|_{\varphi} = |0|_{\varphi} = |\varphi(0)\varphi(1)^{-1}| = |0||1|^{-1} = 0$.

(ii) For each $x, y \in R^{'}$, we have $|x+y|_{\varphi} = |\varphi(a)/\varphi(b) + \varphi(c)/\varphi(d)|_{\varphi}$ for some $a, c \in R$ and $b, d \in S$. From $bS \cap dR \neq \emptyset$, there exist $d_1 \in S$ and $b_1 \in R$ such that $bd_1 = db_1$. Thus,

$$\begin{aligned}
|x+y|_{\varphi} =& |(\varphi(a)\varphi(d_1)/(\varphi(b)\varphi(d_1)) + (\varphi(c)\varphi(b_1)/(\varphi(d)\varphi(b_1))|_{\varphi} \\
=& |\varphi(ad_1 + cb_1)/\varphi(bd_1)|_{\varphi} \\
=& |ad_1 + cb_1||bd_1|^{-1} \\
=& |ad_1 + cb_1||db_1|^{-1} \\
\leq& Max\{|ad_1||bd_1|^{-1}, |cb_1||db_1|^{-1}\} \\
=& Max\{|a||d_1||d_1|^{-1}|b|^{-1}, |c||b_1||b_1|^{-1}|d|^{-1}\} \\
=& Max\{|a||b|^{-1}, |c||d|^{-1}\} \\
=& Max\{|x|_{\varphi}, |y|_{\varphi}\}.
\end{aligned}$$

Hence $|x+y|_{\varphi} \leq Max\{|x|_{\varphi}, |y|_{\varphi}\}$.

(iii) For each $x,y \in R'$, we have $xy = (\varphi(a)/\varphi(b))(\varphi(c)/\varphi(d))$ for some $a,c \in R$ and $b,d \in S$. From $bR \cap cS \neq \emptyset$, we get elements $r \in R$, $s \in S$ such that $br = cs \in S$, this implies that $c^{-1}br = s$. Therefore, $ds = dc^{-1}br, b^{-1}c = rs^{-1}$. Thus,

$$xy = \varphi(a)\varphi(b^{-1})\varphi(c)\varphi(d^{-1}) = \varphi(a)\varphi(r)\varphi(s^{-1})\varphi(d^{-1}) = \varphi(ar)(\varphi(ds))^{-1}.$$

Therefore,

$$|xy|_\varphi = |\varphi(ar)/\varphi(ds)|_\varphi = |ar||ds|^{-1} = |ar||dc^{-1}br|^{-1} = |a||r||r|^{-1}|b|^{-1}|c||d|^{-1} = |a||b|^{-1}|c||d|^{-1} = |x|_\varphi|y|_\varphi.$$

Consequently, $|.|_\varphi$ is Krull valuation on $R' = Q^r_\varphi(R)$.

Lemma 2. *Let $|.| : R \to \Gamma \cup \{0\}$ be a (C_1,C_2)-Hölder Krull valuation on ring R with right ring of fractions $R' = Q^r_\varphi(R)$, where $C_1 \geq 1$, $C_2 \geq 1$, and Γ is an ordered multiplicative group. Then $|.|_\varphi : R' = Q^r_\varphi(R) \to \Gamma \cup \{0\}$ by equation: $|x|_\varphi = |\varphi(a)\varphi(b)^{-1}|_\varphi = |a||b|^{-1}$ for $a \in R$, $b \in S$, is $(C_1^4, C_1^2C_2)$-Hölder Krull valuation on ring R'.*

Proof. (i) let $x \in R' = Q^r_\varphi(R)$ and $|x|_\varphi = 0$. Then $|\varphi(a)\varphi(b)^{-1}|_\varphi = |a||b|^{-1} = 0$, for $a \in R, b \in S$. Therefore, $|a| = 0$, it implies that $a = 0$. Consequently, $x = \varphi(0)/\varphi(b) = 0$. Conversely, let $x \in R' = Q^r_\varphi(R)$ and $x = 0$. Then

$$|x|_\varphi = |0|_\varphi = |\varphi(0)\varphi(1)^{-1}|_\varphi = |0||1|^{-1} = 0.$$

(ii) For each $x,y \in R' = Q^r_\varphi(R)$, we have $|x+y|_\varphi = |\varphi(a)/\varphi(b) + \varphi(c)/\varphi(d)|$ for some $a,c \in R$ and $b,d \in S$. From $bS \cap dR \neq \emptyset$, there exist $d_1 \in S$ and $b_1 \in R$ such that $bd_1 = db_1 \in S$. Thus, $\varphi(b)\varphi(d_1) = \varphi(d)\varphi(b_1)$. Therefore,

$$\begin{aligned}
|x+y|_\varphi =& |(\varphi(a)\varphi(d_1) + \varphi(c)\varphi(b_1))/(\varphi(b)\varphi(d_1))|_\varphi \\
=& |\varphi(ad_1 + cb_1/\varphi(bd_1)|_\varphi = |ad_1 + cb_1||bd_1|^{-1} \\
=& |ad_1 + cb_1||db_1|^{-1} \leq C_2 Max\{|ad_1||bd_1|^{-1}, |cb_1||db_1|^{-1}\} \\
\leq& C_2 Max\{C_1|a||d_1|C_1|d_1|^{-1}|b|^{-1}, C_1|c||b_1|C_1|b_1|^{-1}|d|^{-1}\} \\
=& C_2C_1^2 Max\{|a||b|^{-1}, |c||d|^{-1}\} = C_1^2C_2 Max\{|x|_\varphi, |y|_\varphi\}.
\end{aligned}$$

(iii) For each $x,y \in R'$, we have $xy = (\varphi(a)/\varphi(b))(\varphi(c)/\varphi(d))$, for some $a,c \in R$ and $b,d \in S$. From $bR \cap cS \neq \emptyset$, there exist $r \in R$ and $s \in S$ such that $br = cs \in S$ implies $c^{-1}br = s$. Therefore, $ds = dc^{-1}br$ and $b^{-1}c = rs^{-1}$. Thus,

$$xy = \varphi(a)\varphi(b^{-1})\varphi(c)\varphi(d^{-1}) = \varphi(a)\varphi(r)\varphi(s^{-1})\varphi(d^{-1}) = \varphi(ar)(\varphi(ds))^{-1}.$$

Therefore,

$$\begin{aligned}
|xy|_\varphi =& |\varphi(ar)/\varphi(ds)|_\varphi = |ar||ds|^{-1} \\
=& |ar||dc^{-1}br|^{-1} \geq C_1^{-1}|a||r|(C_1^{-1}|br|^{-1}|dc^{-1}|^{-1}) \\
\geq& C_1^{-2}|a||r|C_1^{-1}|r|^{-1}|b|^{-1}C_1^{-1}|c||d|^{-1} \\
\geq& C_1^{-4}|a||b|^{-1}|c||d|^{-1} = C_1^{-4}|x|_\varphi|y|_\varphi.
\end{aligned}$$

Thus,

$$|xy|_\varphi \geq C_1^{-4}|x|_\varphi|y|_\varphi.$$

Also, we have

$$\begin{aligned}|xy|_\varphi =&|ar||dc^{-1}br|^{-1} \leq C_1|a||r|(C_1|br|^{-1}|dc^{-1}|^{-1})\\ \leq&C_1^2|a||r|C_1|r|^{-1}|b|^{-1}C_1|c||d|^{-1}\\ \leq&C_1^4|a||b|^{-1}|c||d|^{-1} = C_1^4|x|_\varphi|y|_\varphi.\end{aligned}$$

Therefore,

$$C_1^{-4}|x|_\varphi|y|_\varphi \leq |xy|_\varphi \leq C_1^4|x|_\varphi|y|_\varphi.$$

Consequently, $|.|_\varphi$ is $(C_1^4, C_1^2C_2)$-Hölder Krull valuation on ring $R^{'} = Q^r_\varphi(R)$.

Definition 2. *Let R be a ring. We say that R satisfies in Hölder rigidity condition if for every (C_1,C_2)-Hölder Krull valuation $|\ .\ |$ on R, there exists a classical Krull valuation $\|\ .\ \|$ on R such that $\|\ .\ \|$ is (C_0,α)-Hölder equivalent (where $C_0 \geq 1, \alpha > 0$) to $|\ .\ |$.*

Theorem 1. *Let R be a ring satisfying in Hölder rigidity condition, and S a right permutable set of regular elements in R. Then the right ring of fractions $R^{'} = Q^r_\varphi(R)$ with respect to S satisfies in Hölder rigidity condition.*

Proof. Let $\|.\|_\varphi : R' \to \Gamma \cup \{0\}$ be (C_1,C_2)-Hölder Krull valuation on right ring of fractions $R^{'} = Q^r_\varphi(R)$, where Γ is an ordered abelian multiplicative group, $C_1 \geq 1$ and $C_2 \geq 1$. We define $\|.\| : R \to \Gamma \cup \{0\}$ by equation: $\|a\| = \|\varphi(a)\|_\varphi$, for all $a \in R$. Thus, we have.

(i) let $a \in R$ and $a = 0$. Then $\|0\| = \|\varphi(0)\|_\varphi = \|0\|_\varphi = 0$. Conversely, let $a \in R$ and $\|a\| = 0$. Then $\|\varphi(a)\|_\varphi = 0$ implies $\varphi(a) = 0$. Hence there exists $s \in S$ such that $as = 0$. Therefore, $a = 0$ (s is regular element).

(ii) For each $a,b \in R$, we have

$$\|a+b\| = \|\varphi(a+b)\|_\varphi = \|\varphi(a)+\varphi(b)\|_\varphi \leq C_2Max\{\|\varphi(a)\|_\varphi, \|\varphi(b)\|_\varphi\} = C_2Max\{\|a\|,\|b\|\}.$$

(iii) For each $a,b \in R$, we have

$$C_1^{-1}\|a\|\|b\| = C_1^{-1}\|\varphi(a)\|_\varphi\|\varphi(b)\|_\varphi \leq \|\varphi(a)\varphi(b)\|_\varphi (= \|\varphi(ab)\|_\varphi = \|ab\|) \leq C_1\|\varphi(a)\|_\varphi\|\varphi(b)\|_\varphi = C_1\|a\|\|b\|.$$

Therefore, $\|.\|$ is (C_1,C_2)-Hölder Krull valuation on R. Since R satisfies in Hölder rigidity condition, hence there exists a classical Krull valuation $|.|$ on R such that (C_0,α)-Hölder equivalent (where $C_0 \geq 1$, $\alpha > 0$) to (C_1,C_2)-Hölder Krull valuation $\|.\|$ on ring R. Now by Lemma 2, the mapping

$$|.|_\varphi : R^{'} = Q^r_\varphi(R) \to \Gamma \cup \{0\} \quad \text{by} \quad |x|_\varphi = |\varphi(a)\varphi(b)^{-1}|_\varphi = |a||b|^{-1}$$

(for $a \in R$, $b \in S$) is a Krull valuation on ring $R^{'} = Q^r_\varphi(R)$. On the other hand, for each $a \in R$, we have

$$C_0^{-1}|a|^{\alpha^{'}} \leq \|a\| \leq C_0|a|^{\alpha^{'}}$$

where $\alpha' = \alpha$ or $\alpha' = \alpha^{-1}$. Therefore, for each $x = \varphi(a)/\varphi(b)$, for some $a \in R, b \in S$, we have

$$\begin{aligned}\|x\|_\varphi =& \|\varphi(a)\varphi(b)^{-1}\|_\varphi \leq C_1\|\varphi(a)\|_\varphi\|\varphi(b)\|_\varphi^{-1}(= C_1\|a\|\|b\|^{-1})\\ \leq& C_1C_0|a|^{\alpha'}C_0|b|^{-\alpha'}(= C_1C_0^2(|a||b|^{-1})^{\alpha'} = C_1C_0^2|x|_\varphi^{\alpha'}).\end{aligned}$$

on the other hand,

$$\begin{aligned}\|x\|_\varphi =& \|\varphi(a)\varphi(b)^{-1}\|_\varphi \geq C_1^{-1}\|\varphi(a)\|_\varphi\|\varphi(b)\|_\varphi^{-1}(= C_1^{-1}\|a\|\|b\|^{-1})\\ \geq& C_1^{-1}C_0^{-1}|a|^{\alpha'}C_0^{-1}(|b|^{-1})^{\alpha'} = C_1^{-1}C_0^{-2}(|a||b|^{-1})^{\alpha'} = C_1^{-1}C_0^{-2}|x|_\varphi^{\alpha'}.\end{aligned}$$

Therefore,

$$(C_1C_0^2)^{-1}|x|_\varphi^{\alpha'} \leq \|x\|_\varphi \leq C_1C_0^2|x|_\varphi^{\alpha'}.$$

Hence $|.|_\varphi : R' \to \Gamma \cup \{0\}$ is $(C_1C_0^2, \alpha)$-Hölder equivalent to (C_1, C_2)-Hölder Krull valuation $\|.\|_\varphi$ on ring R'. Therefore, R' satisfies in Hölder rigidity condition.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] O. Endler, Valuation Theory, New York, Springer-Verlag, (1972).
[2] E. Munoz Garcia, Hölder absolute values are equivalent to classicalones, Proc. Amer. Math. Soc, 127(7), 1967-1971, (1999).
[3] J. A. Huckaba, Commutative Rings with Zero Divisors, New York, Marcel Dekker, Inc (1988).
[4] M. Knebusch, D. Zhang, Manis Valuations and Prufer Extensions, I. A new chapter in commutative algebra. Lecture Notes in Math. 1791. Berlin, Springer-Verlage, (2002).
[5] T. Y. Lam, A First Course in Noncommutative Rings, Springer-Verlag, (1991).
[6] M. Manis, Valuations on a commutative ring, Proc. Amer. Math. Soc, 20, 193-198, (1969).
[7] P. Ribenboim, The theorie des Valuations, Montreal, Less presses de I Universite de Montreal (1964).
[8] O. F. G. Schilling, The theory of Valuations, New York, Amer. Math. Soc, Thomas, J. J, (1978). Set theory, New York, Academic press, Inc (1952).