Universal Journal of Mathematics and Applications, 2 (1) (2019) 11-13
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: DOI: http://dx.doi.org/10.32323/ujma.469745

A Characterization of Left Regularity

Peter R. Fuchs[1]

[1] *Inst. for Algebra, Johannes Kepler University, Austria*

Article Info

Keywords: *Anshel-Clay near-ring, Integral near-ring, Left regular, Regular*
2010 AMS: *16E50, 16Y30, 20M17*
Received: *12 October 2018*
Accepted: *6 December 2018*
Available online: *20 March 2019*

Abstract

We show that a zero-symmetric near-ring N is left regular if and only if N is regular and isomorphic to a subdirect product of integral near-rings, where each component is either an Anshel-Clay near-ring or a trivial integral near-ring. We also show that a zero-symmetric near-ring is regular without nonzero nilpotent elements if and only if the multiplicative semigroup of N is a union of disjoint groups.

1. Introduction

A (right) near-ring is an algebraic system $(N,+,\cdot)$ such that (1) $(N,+)$ is a (not necessarily abelian) group, (2) $(N,\cdot)$ is a semigroup and (3) the multiplication $\cdot$ is right distributive over the addition $+$. From (3) we obtain that $0x=0$ for all $x\in N$. The near-ring of constant functions on a group $(G,+)$ shows that in general $x0\neq 0$ in a near-ring. N is called zero-symmetric, if $x0=0$ for all $x\in N$. A near-ring N is called regular, if for all $x\in N$ there exists an element $y\in N$ such that $x=xyx$. N is called left (right) regular, if for all $x\in N$ there exists $y\in N$ such that $x=yx^2$ $(x=x^2y)$. N is called integral, if N has no nonzero divisors of zero. A zero-symmetric integral near-ring N is called trivial, if $xy=x$ for all $x,y\in N, y\neq 0$. The set $N-\{0\}$ shall be denoted by N^*. For this and other terminology we refer to [1]. In the next section we define Anshel-Clay near-rings and characterize them in the class of nontrivial integral near-rings. Then we show that a zero-symmetric near-ring N is left regular, if and only if N is regular and isomorphic to a subdirect product of near-rings, which are either trivial integral near-rings or Anshel-Clay near-rings. We also prove for an arbitrary zero-symmetric near-ring N that the multiplicative semigroup $(N,\cdot)$ is a union of disjoint groups, if and only N is regular without nonzero nilpotent elements.

2. Left regular near-rings

Definition 2.1. [2] *A near-ring N is called Anshel-Clay near-ring (ACN), if $N*$ is a disjoint union of subsets $A_i, i\in I$, where I is an index set, such that the following conditions hold:*

1. $|A_i|\geq 2$ *for all* $i\in I$.
2. $(A_i,\cdot)$ *is a group with neutral element* 1_i *for all* $i\in I$.
3. *For all* $i,j\in I$, *the mapping* $x\mapsto 1_jx$ *for* $x\in A_i$ *is a group isomorphism from* $(A_i,\cdot)$ *onto* $(A_j,\cdot)$.
4. *Each* $1_i, i\in I$, *is a right identity of* N.

As we shall see in the next result, condition 3 follows from the other conditions, so when we say that N is an ACN, we mean that N satisfies conditions $1,2,4$. Anshel-Clay near-rings have been defined in [2], but they occured implicitely in previous papers on planar and strongly uniform near-rings, see for example [3], [4], [5] and [6]. In [2] and in [7] these near-rings have been used to coordinatise certain noncommutative spaces.

Theorem 2.2. *Let N be an ACN. Then*

1. $A_i=\{n\in N^*\mid 1_in=n\}$
2. $A_i=1_iN^*$
3. *For all* $i,j\in I, h_{ij}:A_i\to A_j, h_{ij}(x):=1_jx$ *for* $x\in A_i$ *is a group isomorphism.*

***Email address and ORCID number*:** **peter.fuchs@jku.at, 0000-0001-9165-3688 (P. R. Fuchs)**

Proof. 1. Since 1_i is the identity of the group $A_i, A_i \subseteq \{n \in N^* | 1_i n = n\}$. Conversely, let $n \in N^*$ such that $1_i n = n$. Since $N^* = \bigcup_{j \in I} A_j$, $n \in A_j$ for some $j \in I$. Suppose that $i \neq j$ and let n^{-1} denote the inverse of n in A_j. Then $1_j = nn^{-1} = (1_i n)n^{-1} = 1_i(nn^{-1}) = 1_i 1_j = 1_i$, since 1_j is a right identity of N. Thus $1_i = 1_j$, a contradiction, since $i \neq j$ implies $A_i \cap A_j = \varnothing$. It follows that $\{n \in N^* | 1_i n = n\} \subseteq A_i$.
2. From 1. we have that $A_i \subseteq 1_i N^*$. Conversely, if $n = 1_i m \in 1_i N^*$, then $1_i n = 1_i(1_i m) = 1_i^2 m = 1_i m = n$, hence from 1. $1_i N^* \subseteq A_i$.
3. Let $x, y \in A_i$. Since 1_j is a right identity of $N, h_{ij}(xy) = 1_j xy = 1_j x 1_j y = h(x)h(y)$, so h_{ij} is a group homomorphism. Now suppose that $1_j x = 1_j y$, for some elements $x, y \in A_i$. Then $1_i(1_j x) = 1_i(1_j y)$. Since $1_i 1_j = 1_i$, we obtain $1_i x = 1_i y$. By 1. it follows that $x = y$, so h_{ij} is injective. If x is an arbitrary element of A_j, then $1_i x \in A_i$ by 2., hence $h_{ij}(1_i x) = 1_j(1_i x) = (1_j 1_i)x = 1_j x = x$, which shows that h_{ij} is an isomorphism. □

A near-field is a near-ring with identity, where every nonzero element is invertible.

Theorem 2.3. *1. Every ACN is a zero-symmetric, nontrivial integral near-ring.*
2. Let N be an ACN. Then N is a near-field, if and only if N has an identity, if and only if I is a one element set.

Proof. 1. If $x0 \neq 0$ for some $x \in N$, then $x0 \in A_i$ for some $i \in I$, since $N^* = \bigcup_{i \in I} A_i$. From $(x0)^2 = x(0x)0 = x0$ we obtain $x0 = 1_i$. Thus $1_i n = (x0)n = x(0n) = x0 = 1_i$ for all $n \in N^*$. By 2. of Theorem 2.2, it follows that $A_i = 1_i N^* = \{1_i\}$, which contradicts condition 1 in the definition of an ACN. It follows that N is zero-symmetric. Now suppose $xy = 0$ for some elements $x, y \in N$. If $y \neq 0$, then $y \in A_i$ for some $i \in I$. If y^{-1} is the inverse of y in A_i, then $0 = 0y^{-1} = (xy)y^{-1} = x(yy^{-1}) = x1_i = x$, thus N is integral. If N is a trivial integral near-ring, then $xy = x$ for all $y \neq 0$, hence $A_i = 1_i N^* = \{1_i\}$ for all $i \in I$, a contradiction.
2. If N is an ACN with identity 1, then $1 = 1_i$ for all $i \in I$, since each 1_i is a right identity of N. Thus N is a near-field. □

Next we characterize which nontrivial integral near-rings are Anshel-Clay near-rings.

Theorem 2.4. *For a zero-symmetric, nontrivial integral near-ring N, the following are equivalent:*

1. *N is an ACN*
2. *$\forall n \in N^* : Nn = N$*
3. *N is left regular*
4. *N is regular*

Proof. Let N be an ACN and let $n \in N^*$. Then $n \in A_i$ for some $i \in I$. Since 1_i is a right identity of N, $N = N1_i = Nn^{-1}n \subseteq Nn$, hence $Nn = N$. Next, suppose $Nx = N$ for all $x \in N, x \neq 0$. Then $Nx^2 = (Nx)x = Nx = N$, for all $x \neq 0$, hence there exists an element $y \in N$, such that $x = yx^2$, thus N is left regular. That 3. implies 4. has been shown in [8], Proposition 1. Finally we show that 4. implies 1. If $0 \neq e \in N$ is idempotent, then for all $n \in N$ we have $(ne - n)e = ne^2 - ne = ne - ne = 0$. Since N is integral it follows that $ne = n$, hence each idempotent is a right identity of N. Now suppose that N is regular and let $n \in N$. Then there exists an element $x \in N$ such that $n = nxn$. Then nx is idempotent, hence $n = n(nx) = n^2 x$. It follows that N is regular and right regular. By [9], Theorem 4.3, $N^* = \bigcup_{i \in I} A_i$, where A_i is a group with identity 1_i for $i \in I$ and $A_i \cap A_j = \varnothing$ if $i \neq j$. As we have seen before, each 1_i is a right identity of N. Now we can show like in Theorem 2.2, No. 3, that $h_{ij} : A_i \to A_j$, $h_{ij}(x) = 1_j x$ for $x \in A_i$ is a group isomorphism. If $A_i = \{1_i\}$ for all $i \in I$, then $(N^*, \cdot)$ is a band since each $1_i, i \in I$, is a right identity of N. Since this contradicts our assumption that N is a nontrivial integral near-ring, it follows that $|A_i| \geq 2$ for all $i \in I$, hence N is an ACN. □

An idempotent e of a near-ring N is called right semi-central in N, if $eN = eNe$. It is easy to show that e is right semi-central in N if and only if $en = ene$ for all $n \in N$. N is called right semi-central, if every idempotent e of N is right semi-central in N (see [10]). Let N be an integral near-ring and $i, n \in N$. i is called a left identity of n, if $in = n$. Note that if $n \neq 0$ has a left identity i, then i is uniquely determined, since $i_1 n = n = i_2 n$ implies $(i_1 - i_2)n = 0$, hence $i_1 = i_2$.

Theorem 2.5. *For a zero-symmetric regular near-ring N the following are equivalent:*

1. *N has no nonzero nilpotent elements.*
2. *N is right semi-central.*
3. *N is isomorphic to a subdirect product of Anshel-Clay near-rings and trivial integral near-rings.*
4. *N is left regular.*

Proof. That 1. implies 2. has been shown in [10], Cor. 7. Conversely, suppose that there exists an element $n \in N, n \neq 0, n^2 = 0$. Since N is regular, $n = nxn$ for some $x \in N$. Then $e := nx$ is idempotent and $n = en = ene$, since e is semi-central by assumption, so $n = ne = n^2 x = 0$, a contradiction, which shows the equivalence of 1. and 2. Next we show that 1. implies 3. By [11], N is isomorphic to a subdirect product of integral near-rings $N_i, i \in I$. Since N is regular, each N_i is also regular. Therefore, if N_i is a nontrivial integral near-ring, then N_i is an ACN by Theorem 2.4. Since each ACN is integral by Theorem 2.3 it follows that 3. implies 1. Since 4. implies 1. is clear, it remains to show that 3. implies 4. Suppose that N is isomorphic to a subdirect product of near-rings $N_i, i \in I$, where each N_i is an ACN or a trivial integral near-ring. Let $n \in N$. We have to show that there exists an element $x \in N$ such that $n = xn^2$. Since N is regular, there exists $y \in N$ such that $n = nyn$. N is isomorphic to a subdirect product of the near-rings N_i, so $n = (n_i)_{i \in I}$, $y = (y_i)_{i \in I}$, for some $n_i, y_i \in N_i$, $i \in I$. Then $n_i = n_i y_i n_i$ and $e_i := n_i y_i$ is an idempotent for all $i \in I$. Since $(n_i - n_i e_i)e_i = 0_i$ and each N_i is integral, we obtain $n_i = n_i e_i = n_i^2 y_i$. Let $x := ny^2$. Then for all $i \in I$, $n_i^2 x_i = n_i(n_i^2 y_i)y_i = n_i^2 y_i = n_i$ and $n_i x_i n_i = (n_i^2 y_i)(y_i n_i) = n_i y_i n_i = n_i$, hence $n^2 x = n = nxn$. Now fix an element $i \in I$ and suppose that N_i is an ACN. Then there exists an index set J_i such that $N_i = \{0_i\} \cup \bigcup_{j \in J_i} A_j$, using the terminology of Definition 2.1 Suppose $n_i \neq 0$. Since $n_i = n_i x_i n_i$, $n_i x_i$ is a left identity for n_i. There exists an element $j \in J_i$, such that $n_i \in A_j$. But then 1_j is also a left identity for n_i, hence by the uniqueness of the left identity, $n_i x_i = 1_j$. Note that x_i is also an element of A_j. This follows from Theorem 2.2, since $n_i \in A_j$ and $1_j x_i = 1_j n_i y_i^2 = (1_j n_i) y_i^2 = n_i y_i^2 = x_i$. Therefore we obtain that $x_i n_i = 1_j = n_i x_i$, hence $n_i^2 x_i = n_i = n_i x_i n_i = x_i n_i^2$. This equation also holds if $n_i = 0$, so it holds for all $i \in I$, where N_i is an ACN. Since the previous equation is obviously true for all those $i \in I$, where N_i is a trivial integral near-ring, we conclude that $n^2 x = n = xn^2$. Thus N is left regular. □

In [12], the equivalence of 1. and 4. has been shown with a different proof. From Theorem 2.5 and [9], Theorem 4.3 we also obtain

Theorem 2.6. *For a zero-symmetric near-ring N, the following are equivalent:*

1. *N is regular without nonzero nilpotent elements.*
2. *The multiplicative semigroup of N is a union of disjoint groups.*

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