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Filtering of Multidimensional Stationary Processes with Missing Observations

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Abstract

The problem of the mean-square optimal linear estimation of linear functionals which depend on the unknown values of a multidimensional continuous time stationary stochastic process from observations of the process with a stationary noise is considered. Formulas for calculating the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals are derived under the condition of spectral certainty, where spectral densities of the signal and the noise processes are exactly known. The minimax (robust) method of estimation is applied in the case of spectral uncertainty, where spectral densities of the processes are not known exactly, while some sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and minimax spectral characteristics of the optimal estimates are derived for some special sets of admissible spectral densities.

1. Introduction

The problem of estimation of the unknown values of stochastic processes is of constant interest in the theory and applications of stochastic processes. The formulation of the interpolation, extrapolation and filtering problems for stationary stochastic sequences with known spectral densities and reducing the estimation problems to the corresponding problems of the theory of functions belongs to Kolmogorov [1]. Effective methods of solution of the estimation problems for stationary stochastic processes were developed by Wiener [2] and Yaglom [3, 4]. Further results are presented in the books by Rozanov [5], Hannan [6], Box et. al [7], Brockwell and Davis [8].
The crucial assumption of most of the methods developed for estimating the unobserved values of stochastic processes is that the spectral densities of the involved stochastic processes are exactly known. However, in practice, complete information on the spectral densities is impossible in most cases. In this situation, one finds the parametric or nonparametric estimate of the unknown spectral density and then apply one of the traditional estimation methods provided that the selected density is the true one. This procedure can result in significant increasing of the value of the error of estimate as Vastola and Poor [9] have demonstrated with the help of some examples. To avoid this effect one can search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the errors of estimates. The paper by Grenander [10] was the first one where this approach to extrapolation problem for stationary processes was proposed. Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by Kassam and Poor [11]. In the papers by Franke [12], [13] Franke and Poor [14] the minimax extrapolation and filtering problems for stationary sequences were investigated with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities. In the papers by Moklyachuk [15, 16] the extrapolation, interpolation and filtering problems for functionals which depend on the unknown values of stationary processes and sequences are investigated. The estimation problems for functionals which depend on the unknown values of multidimensional stationary stochastic processes is the aim of the investigation by Moklyachuk and Masyutka [17, 18]. In their book Moklyachuk and Golichenko [19] presented results of investigation of the interpolation, extrapolation and filtering problems for periodically correlated stochastic sequences. In the papers by Luz and Moklyachuk [20], Luz2016 results of an investigation of the estimation problems for functionals which depend on the unknown values of stochastic sequences with stationary increments are described. Prediction problem

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for stationary sequences with missing observations is investigated in papers by Bondon [21, 22], Cheng, Miamee and Pourahmadi [23], Cheng and Pourahmadi [24], Kasahara, Pourahmadi and Inoue [25], Pourahmadi, Inoue and Kasahara [26], Pelagatti [27]. In papers by Moklyachuk and Sidei [28] - [31] an approach is developed to an investigation of the interpolation, extrapolation and filtering problems for stationary stochastic processes with missing observations.
In this article, we deal with the problem of the mean-square optimal linear estimation of the functional

$$A\vec{\xi}=\int_{R^s}\vec{a}(t)^{\top}\vec{\xi}(-t)dt,$$

which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^-\backslash S$, $S=\bigcup\limits_{l=1}^{s}[-M_l-N_l,-M_l]$, $R^s=[0,\infty)\backslash S^+$, $S^+=\bigcup\limits_{l=1}^{s}[M_l,\,M_l+N_l]$. The case of spectral certainty, as well as the case of spectral uncertainty, are considered. Formulas for calculating the spectral characteristic and the mean-square error of the optimal linear estimate of the functional are derived under the condition of spectral uncertainty, where the spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities are not exactly known while a set of admissible spectral densities is given, the minimax method is applied. Formulas for determination the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functional are proposed for some specific classes of admissible spectral densities.

2. Hilbert space projection method of filtering

Let $\vec{\xi}(t)=\{\xi_k(t)\}_{k=1}^{T},t\in\mathbb{R}$, and $\vec{\eta}(t)=\{\eta_k(t)\}_{k=1}^{T},t\in\mathbb{R}$, be uncorrelated mean square continuous multidimensional stationary stochastic processes with zero first moments, $E\vec{\xi}(t)=\vec{0}$, $E\vec{\eta}(t)=\vec{0}$, absolutely continuous spectral functions and spectral density matrices which satisfy the minimality condition

$$\int_{-\infty}^{\infty}(b(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}\overline{b(\lambda)}d\lambda<\infty, \tag{2.1}$$

where $b(\lambda)=\sum\limits_{l=1}^{s}\int\limits_{-M_l-N_l}^{-M_l}\vec{\alpha}(t)e^{it\lambda}dt$ is a nontrivial function of the exponential type. Under this condition the error-free estimate of the process $\vec{\xi}(t)+\vec{\eta}(t)$ is impossible (see, for example, Rozanov [5]).
Suppose that we have observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^-\backslash S$, where

$$S=\bigcup_{l=1}^{s}[-M_l-N_l,-M_l],\; R^s=[0,\infty)\backslash S^+,\; S^+=\bigcup_{l=1}^{s}[M_l,\,M_l+N_l]$$

.
The main purpose of this article is to find the mean-square optimal linear estimate of the functional

$$A\vec{\xi}=\int_{R^s}\vec{a}(t)^{\top}\vec{\xi}(-t)dt,$$

which depends on the unknown values of the process $\vec{\xi}(t)$.
We will assume that the function $\vec{a}(t)$ satisfies the condition

$$\sum_{k=1}^{T}\int_{R^s}|a_k(t)|dt<\infty. \tag{2.2}$$

This condition ensures that the functional $A_s\vec{\xi}$ has a finite second moment.
It follows from the spectral decompositions of the processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ (see Gikhman and Skorokhod [32])

$$\vec{\xi}(t)=\int_{-\infty}^{\infty}e^{it\lambda}Z_{\xi}(d\lambda),\qquad \vec{\eta}(t)=\int_{-\infty}^{\infty}e^{it\lambda}Z_{\eta}(d\lambda),$$

where $Z_{\xi}(d\lambda)$ and $Z_{\eta}(d\lambda)$ are vector valued orthogonal stochastic measures, that the functional $A\vec{\xi}$ can be represented in the form

$$A\vec{\xi}=\int_{-\infty}^{\infty}(A(\lambda))^{\top}Z_{\xi}(d\lambda),\quad A(\lambda)=\int_{R^s}\vec{a}(t)e^{-it\lambda}dt.$$

Consider the Hilbert space $H=L_2(\Omega,\mathscr{F},P)$ generated by random variables ξ with zero mathematical expectations, $E\xi=0$, finite variations, $E|\xi|^2<\infty$, and inner product $(\xi,\eta)=E\xi\overline{\eta}$. Denote by $H^s(\xi+\eta)$ the closed linear subspace generated by elements $\{\xi_k(t)+\eta_k(t):t\in\mathbb{R}^-\backslash S,k=\overline{1,T}\}$ in the Hilbert space $H=L_2(\Omega,\mathscr{F},P)$.
Let $L_2(F+G)$ be the Hilbert space of complex-valued functions $\vec{a}(\lambda)=\{a_k(\lambda)\}_{k=1}^{T}$ such that

$$\int_{-\infty}^{\infty}\vec{a}(\lambda)^{\top}(F(\lambda)+G(\lambda))\overline{\vec{a}(\lambda)}d\lambda=\int_{-\infty}^{\infty}\sum_{k,l=1}^{T}a_k(\lambda)\overline{a_l(\lambda)}(f_{kl}(\lambda)+g_{kl}(\lambda))d\lambda<\infty.$$

Denote by $L_2^s(F+G)$ the subspace of $L_2(F+G)$ generated by functions

$$e^{it\lambda}\delta_k,\ \delta_k=\{\delta_{kl}\}_{l=1}^{T},\ k=\overline{1,T},\ t\in\mathbb{R}^-\backslash S.$$

Denote by $\hat{A}_s\vec{\xi}$ the optimal linear estimate of the functional $A_s\vec{\xi}$ from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ and denote by $\Delta(F,G)=E\left|A_s\vec{\xi}-\hat{A}_s\vec{\xi}\right|^2$ the mean-square error of the estimate $\hat{A}_s\vec{\xi}$.

The mean-square optimal linear estimate $\hat{A}_s\vec{\xi}$ of the functional $A_s\vec{\xi}$ is of the form

$$\hat{A}\vec{\xi}=\int_{-\infty}^{\infty}(h(\lambda))^{\top}(Z_{\xi}(d\lambda)+Z_{\eta}(d\lambda)),$$

where $h(\lambda)=\{h_k(\lambda)\}_{k=1}^{T}\in L_2^s(F+G)$ is the spectral characteristic of the estimate, and the mean-square error $\Delta(h;F,G)$ of the estimate is determined by formula

$$\Delta(h;F,G)=E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}(A(\lambda)-h(\lambda))^{\top}F(\lambda)\overline{(A(\lambda)-h(\lambda))}d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}(h(\lambda))^{\top}G(\lambda)\overline{h(\lambda)}d\lambda.$$

Since we suppose that the spectral densities of the stationary processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ are known, we can apply the method of orthogonal projections in the Hilbert spaces proposed by A. N. Kolmogorov [1] in order to find the optimal estimate. According to this method, the optimal linear estimation of the functional $A\vec{\xi}$ is a projection of the element $A\vec{\xi}$ of the space H on the subspace $H^s(\xi+\eta)$. The estimate is determined by two conditions:

$$1)\hat{A}\vec{\xi}\in H^s(\xi+\eta),$$
$$2)A\vec{\xi}-\hat{A}\vec{\xi}\perp H^s(\xi+\eta).$$

Under the second condition, the spectral characteristic $h(\lambda)$ of the optimal linear estimate $\hat{A}\vec{\xi}$ satisfies the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[(A(\lambda))^{\top}F(\lambda)-(h(\lambda))^{\top}(F(\lambda)+G(\lambda))\right]e^{-it\lambda}d\lambda=0,\quad t\in\mathbb{R}^-\backslash S. \tag{2.3}$$

Consider the function $(C(\lambda))^{\top}=(A(\lambda))^{\top}F(\lambda)-(h(\lambda))^{\top}(F(\lambda)+G(\lambda))$ and its Fourier transform

$$\vec{\mathbf{c}}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C(\lambda)e^{-it\lambda}d\lambda,\quad t\in\mathbb{R}.$$

It follows from relation (2.3), that the function $\mathbf{c}(t)$ can be nonzero only on the set $U=S\cup[0,\infty)$. Hence, the function $C(\lambda)$ is of the form

$$C(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}\vec{\mathbf{c}}(t)e^{it\lambda}dt+\int_{0}^{\infty}\vec{\mathbf{c}}(t)e^{it\lambda}dt,$$

and the spectral characteristic of the estimate $\hat{A}\vec{\xi}$ is of the form

$$(h(\lambda))^{\top}=(A(\lambda))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-(C(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}.$$

It follows from the first condition, $\hat{A}\vec{\xi}\in H^s(\xi+\eta)$, which determines the estimate of the functional $A\vec{\xi}$, that for any $t\in U$ the following relation holds true

$$\int_{-\infty}^{\infty}\left((A(\lambda))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-(C(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}\right)e^{-it\lambda}d\lambda=0. \tag{2.4}$$

Let us define the following operators in the space $L_2(U)$

$$(\mathbf{B}\mathbf{x})(t)=\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{R}\mathbf{x})(t)=\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u+t)}d\lambda du+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{Q}\mathbf{x})(t)=\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)e^{i\lambda(u-t)}d\lambda du+$$

$$+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)e^{i\lambda(u-t)}d\lambda du,$$

$\vec{\mathbf{x}}(t) \in L_2(U), \quad t \in U.$

The equality (2.4) can be represented in the form

$$\int_{-\infty}^{\infty}\int_{R^s} \vec{a}(u)^\top F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)}dud\lambda - \int_{-\infty}^{\infty}\left(\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}\vec{\mathbf{c}}(t)^\top(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}du\right)d\lambda$$
$$-\int_{-\infty}^{\infty}\int_{0}^{\infty}\vec{\mathbf{c}}(t)^\top(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}dud\lambda = 0, \quad t \in U. \quad (2.5)$$

Let $\vec{\mathbf{a}}(t)$ be a function such that

$\vec{\mathbf{a}}(t) = \vec{0},\ t \in S, \quad \vec{\mathbf{a}}(t) = \vec{a}(t), t \in R^s \quad \vec{\mathbf{a}}(t) = \vec{0},\ t \in S^+.$

Making use of the introduces above notations, we can represent equality (2.5) in terms of linear operators in the space $L_2(U)$

$(\mathbf{Ra})(t) = (\mathbf{Bc})(t), \quad t \in U.$

Assume that the operator $\mathbf{B}$ is invertible (see paper by Salehi [33] for more details). Then the function $\vec{\mathbf{c}}(t)$ can be found and it is calculated by the formula

$\vec{\mathbf{c}}(t) = (\mathbf{B}^{-1}\mathbf{Ra})(t), \quad t \in U.$

The spectral characteristic $h(\lambda)$ of the estimate $\hat{A}\vec{\xi}$ is calculated by the formula

$$\begin{aligned}(h(\lambda))^\top &= (A(\lambda))^\top F(\lambda)(F(\lambda)+G(\lambda))^{-1} - (C(\lambda))^\top(F(\lambda)+G(\lambda))^{-1},\\ C(\lambda) &= \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt + \int_{0}^{\infty}(\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt.\end{aligned} \quad (2.6)$$

The mean-square error of the estimate $\hat{A}\vec{\xi}$ is calculated by the formula

$$\Delta(h;F,G) = \frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top G(\lambda)+(C(\lambda))^\top)(F(\lambda)+G(\lambda))^{-1}F(\lambda)(F(\lambda)+G(\lambda))^{-1}((A(\lambda))^\top G(\lambda)+(C(\lambda))^\top)^*d\lambda+$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top F(\lambda)-(C(\lambda))^\top)(F(\lambda)+G(\lambda))^{-1}G(\lambda)(F(\lambda)+G(\lambda))^{-1}((A(\lambda))^\top G(\lambda)+(C(\lambda))^\top)^*d\lambda =$$
$$= \langle(\mathbf{Ra})(t),(\mathbf{B}^{-1}\mathbf{Ra})(t)\rangle + \langle(\mathbf{Qa})(t),\vec{\mathbf{a}}(t)\rangle, \quad (2.7)$$

where

$$\langle\vec{a}(t),\vec{b}(t)\rangle = \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}a_k(t)\overline{b_k(t)}dt + \int_{0}^{\infty}a_k(t)\overline{b_k(t)}dt$$

is the inner product in the space $L_2(U)$.
The obtained results can be summarized in the form of theorem.

Theorem 2.1. *Let $\vec{\xi}(t)$ and $\vec{\eta}(t)$ be uncorrelated multidimensional stationary stochastic processes with the spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (2.1). Let condition (2.2) be satisfied and let the operator* $\mathbf{B}$ *be invertible. The spectral characteristic $h(\lambda)$ and the mean-square error $\Delta(h;F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$, $t \in \mathbb{R}^-\setminus S$ are calculated by formulas (2.6), (2.7).*

3. Minimax-robust method of filtering

In the previous sections, we deal with the filtering problem under the condition that we know spectral densities of the processes. In this case, we derived formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the introduced functionals. In the case of spectral uncertainty, where full information on spectral densities is impossible while it is known that spectral densities belong to some specified classes of admissible densities, the minimax method of filtering is reasonable. This method gives us a procedure of finding estimates which minimize the maximum values of the mean-square errors of the estimates for all spectral densities from the given class of admissible spectral densities. For the description of the minimax method, we propose the following definitions (see book by Moklyachuk and Masytka [18] for more details).

Definition 3.1. *For a given class of spectral densities $D = D_F \times D_G$ the spectral densities $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ are called least favorable in class D for the optimal linear filtering of the functional $A\vec{\xi}$ if the following relation holds true*

$$\Delta\left(F^0,G^0\right) = \Delta\left(h\left(F^0,G^0\right);F^0,G^0\right) = \max_{(F,G)\in D_F\times D_G}\Delta\left(h\left(F,G\right);F,G\right).$$

Definition 3.2. *For a given class of spectral densities $D=D_F\times D_G$ the spectral characteristic $h^0(\lambda)$ of the optimal linear filtering of the functional $A\vec{\xi}$ is called minimax-robust if there are satisfied conditions*

$$h^0(\lambda)\in H_D=\bigcap_{(F,G)\in D_F\times D_G}L_2^s(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta(h;F,G)=\max_{(F,G)\in D}\Delta\left(h^0;F,G\right).$$

From the introduced definitions and formulas derived above, we can obtain the following statement.

Lemma 3.3. *Spectral densities $F^0(\lambda)\in D_F$, $G^0(\lambda)\in D_G$ satisfying the minimality condition (2.1) are the least favorable in the class $D=D_F\times D_G$ for the optimal linear filtering of the functional $A\vec{\xi}$, if the Fourier coefficients of the functions*

$$(F^0(\lambda)+G^0(\lambda))^{-1},\quad F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1},\quad F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}G^0(\lambda)$$

determine operators $\mathbf{B}^0,\mathbf{R}^0,\mathbf{Q}^0$, which give a solution of the constrained optimization problem

$$\max_{(F,G)\in D_F\times D_G}(\langle(\mathbf{Ra})(t),(\mathbf{B}^{-1}\mathbf{Ra})(t)\rangle+\langle(\mathbf{Qa})(t),\vec{\mathbf{a}}(t)\rangle)=\langle(\mathbf{R}^0\mathbf{a})(t),((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)\rangle+\langle(\mathbf{Q}^0\mathbf{a})(t),\vec{\mathbf{a}}(t)\rangle. \tag{3.1}$$

The minimax spectral characteristic $h^0=h(F^0,G^0)$ is determined by formula (2.6) if $h(F^0,G^0)\in H_D$.

For more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics we observe that the least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ and the minimax spectral characteristic $h^0=h(F^0,G^0)$ form a saddle point of the function $\Delta(h;F,G)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta\left(h^0;F,G\right)\leq\Delta\left(h^0;F^0,G^0\right)\leq\Delta\left(h;F^0,G^0\right),\ \forall h\in H_D,\forall F\in D_F,\forall G\in D_G,$$

hold true if $h^0=h(F^0,G^0)$, $h(F^0,G^0)\in H_D$, where (F^0,G^0) is a solution of the constrained optimization problem

$$\sup_{(F,G)\in D_F\times D_G}\Delta\left(h(F^0,G^0);F,G\right)=\Delta\left(h(F^0,G^0);F^0,G^0\right). \tag{3.2}$$

The linear functional $\Delta\left(h\left(F^0,G^0\right);F,G\right)$ is calculated by the formula

$$\Delta\left(h\left(F^0,G^0\right);F,G\right)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)(F^0(\lambda)+G^0(\lambda))^{-1}F(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*d\lambda+$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)(F^0(\lambda)+G^0(\lambda))^{-1}G(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}((A(\lambda))^\top G^0(\lambda)-(C^0(\lambda))^\top)^*d\lambda,$$
$$C^0(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt+\int_{0}^{\infty}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt.$$

The constrained optimization problem (3.2) is equivalent to the unconstrained optimization problem (see book by Pshenichnyj [34])

$$\Delta_D(F,G)=-\Delta(h(F^0,G^0);F,G)+\delta((F,G)\,|D_F\times D_G\,)\to\inf, \tag{3.3}$$

where $\delta((F,G)\,|D_F\times D_G\,)$ is the indicator function of the set $D=D_F\times D_G$. Solution of the problem (3.3) is characterized by the condition $0\in\partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point (F^0,G^0), namely, the set of all continuous linear functionals Λ on $L_1\times L_1$ satisfying the inequality $\Delta_D(F,G)-\Delta_D(F^0,G^0)\geq\Lambda(F,G)-\Lambda(F^0,G^0)$. This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities D (see books by Ioffe and Tihomirov [35], Pshenichnyj [34], Rockafellar [36]).
Note, that the form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application of the Lagrange method of indefinite multipliers for finding solution of the problem (3.2). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see books by Moklyachuk [37, 15], Moklyachuk and Masyutka [18] for additional details).

4. Least favorable spectral densities in the class $D=D_0\times D_{1\delta}$

Consider the problem of minimax filtering of the functional $A\vec{\xi}$ in the case where spectral densities of the processes belong to the following classes of admissible spectral densities $D=D_0\times D_{1\delta}$,

$$D_0^1=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}\,F(\lambda)d\lambda=p\right.\right\},$$

$$D_{1\delta}^1=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathrm{Tr}(G(\lambda)-G_1(\lambda))|d\lambda\leq\delta\right.\right\};$$

$$D_0^2=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}f_{kk}(\lambda)d\lambda=p_k,k=\overline{1,T}\right.\right\},$$
$$D_{1\delta}^2=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{kk}(\lambda)-g_{kk}^1(\lambda)\right|d\lambda\leq\delta_k,k=\overline{1,T}\right.\right\};$$
$$D_0^3=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle B_1,F(\lambda)\rangle d\lambda=p\right.\right\},$$
$$D_{1\delta}^3=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\langle B_2,G(\lambda)-G_1(\lambda)\rangle|d\lambda\leq\delta\right.\right\},$$
$$D_0^4=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\lambda)d\lambda=P\right.\right\},$$
$$D_{1\delta}^4=\left\{G(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{ij}(\lambda)-g_{ij}^1(\lambda)\right|d\lambda\leq\delta_i^j,i,j=\overline{1,T}\right.\right\},$$

where $G_1(\lambda)$ is a known and fixed spectral density matrix, $\delta,p,\delta_k,p_k,k=\overline{1,T}$, $\delta_i^j,i,j=\overline{1,T}$, are given numbers, P,B_1,B_2 are given positive definite Hermitian matrices.
The classes $D_{1\delta}$ describe the "δ-neighborhood" models in the space L_1 of a given bounded spectral density matrix $G_1(\lambda)$.
From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.
For the first pair $D_0^1\times D_{1\delta}^1$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=\alpha^2(F^0(\lambda)+G^0(\lambda))^2, \tag{4.1}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=\beta^2\gamma(\lambda)(F^0(\lambda)+G^0(\lambda))^2, \tag{4.2}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda))\right|d\lambda=\delta, \tag{4.3}$$

where α^2,β^2 are Lagrange multipliers, $|\gamma(\lambda)|\leq 1$ and

$$\gamma(\lambda)=\mathrm{sign}\,(\mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda)))\quad\text{if}\quad\mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda))\neq 0.$$

For the second pair $D_0^2\times D_{1\delta}^2$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\alpha_k^2\delta_{kl}\right\}_{k,l=1}^T(F^0(\lambda)+G^0(\lambda)), \tag{4.4}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\beta_k^2\gamma_k(\lambda)\delta_{kl}\right\}_{k,l=1}^T(F^0(\lambda)+G^0(\lambda)), \tag{4.5}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\right|d\lambda=\delta_k,\;k=\overline{1,T}, \tag{4.6}$$

where α_k^2,β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, $|\gamma_k(\lambda)|\leq 1$ and

$$\gamma_k(\lambda)=\mathrm{sign}\,(g_{kk}^0(\lambda)-g_{kk}^1(\lambda))\quad\text{if}\quad g_{kk}^0(\lambda)-g_{kk}^1(\lambda)\neq 0,\;k=\overline{1,T}.$$

For the third pair $D_0^3\times D_{1\delta}^3$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=\alpha^2(F^0(\lambda)+G^0(\lambda))B_1^\top(F^0(\lambda)+G^0(\lambda)), \tag{4.7}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=\beta^2\gamma'(\lambda)(F^0(\lambda)+G^0(\lambda))B_2^\top(F^0(\lambda)+G^0(\lambda)), \tag{4.8}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left\langle B_2,G^0(\lambda)-G_1(\lambda)\right\rangle\right|d\lambda=\delta, \tag{4.9}$$

where α^2,β^2 are Lagrange multipliers, $|\gamma'(\lambda)|\leq 1$ and

$$\gamma'(\lambda)=\mathrm{sign}\left\langle B_2,G^0(\lambda)-G_1(\lambda)\right\rangle\quad\text{if}\quad\left\langle B_2,G^0(\lambda)-G_1(\lambda)\right\rangle\neq 0.$$

For the fourth pair $D_0^4\times D_{1\delta}^4$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)-(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\vec{\alpha}\cdot\vec{\alpha}^*(F^0(\lambda)+G^0(\lambda)), \tag{4.10}$$

$$((A(\lambda))^{\top}F^0(\lambda)-(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}F^0(\lambda)+(C^0(\lambda))^{\top})=(F^0(\lambda)+G^0(\lambda))\left\{\beta_{ij}\gamma_{ij}(\lambda)\right\}_{i,j=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{4.11}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|d\lambda=\delta_i^j,\ i,j=\overline{1,T}, \tag{4.12}$$

where $\vec{\alpha},\beta_{ij}$ are Lagrange multipliers, $\left|\gamma_{ij}(\lambda)\right|\leq 1$ and

$$\gamma_{ij}(\lambda)=\frac{g_{ij}^0(\lambda)-g_{ij}^1(\lambda)}{\left|g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|}\quad \text{if}\quad g_{ij}^0(\lambda)-g_{ij}^1(\lambda)\neq 0,\ i,j=\overline{1,T}.$$

Thus, the following statement holds true.

Theorem 4.1. *The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_0\times D_{1\delta}$ for the optimal linear filtering of the functional $A\vec{\xi}$ are determined by relations (4.1) – (4.3) for the first pair $D_0^1\times D_{1\delta}^1$ of sets of admissible spectral densities; by relations (4.4) – (4.6) for the second pair $D_0^2\times D_{1\delta}^2$ of sets of admissible spectral densities; by relations (4.7) – (4.9) for the third pair $D_0^3\times D_{1\delta}^3$ of sets of admissible spectral densities; by relations (4.10) – (4.12) for the fourth pair $D_0^4\times D_{1\delta}^4$ of sets of admissible spectral densities; the minimality condition (2.1), the constrained optimization problem (3.1) and restrictions on densities from the corresponding classes $D_0\times D_{1\delta}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (2.6).*

Corollary 4.2. *Assume that the spectral density $G(\lambda)$ is known. Let the function $F^0(\lambda)+G(\lambda)$ satisfy the minimality condition (2.1). The spectral density $F^0(\lambda)$ is the least favorable in the classes D_0^k, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (4.1), (4.4), (4.7), (4.10), respectively, and the pair $(F^0(\lambda),G(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

Corollary 4.3. *Assume that the spectral density $F(\lambda)$ is known. Let the function $F(\lambda)+G^0(\lambda)$ satisfy the minimality condition (2.1). The spectral density $G^0(\lambda)$ is the least favorable in the classes $D_{1\delta}^k$, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (4.2) – (4.3), (4.5) – (4.6), (4.8) – (4.9), (4.11) – (4.12), respectively, and the pair $(F(\lambda),G^0(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

5. Least favorable spectral densities in the class $D=D_{2\delta}\times D_{\varepsilon}$

Consider the problem of filtering of the functional $A\vec{\xi}$ in the case where spectral densities of the processes belong to the class of admissible spectral densities $D_{2\delta}\times D_{\varepsilon}$,

$$D_{2\delta}^1=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathrm{Tr}(F(\lambda)-F_1(\lambda))|^2d\lambda\leq\delta\right.\right\};$$

$$D_{\varepsilon}^1=\left\{G(\lambda)\left|\mathrm{Tr}\,G(\lambda)=(1-\varepsilon)\mathrm{Tr}\,G_1(\lambda)+\varepsilon\mathrm{Tr}\,W(\lambda),\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}\,G(\lambda)d\lambda=q\right.\right\};$$

$$D_{2\delta}^2=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{kk}(\lambda)-f_{kk}^1(\lambda)\right|^2d\lambda\leq\delta_k,k=\overline{1,T}\right.\right\};$$

$$D_{\varepsilon}^2=\left\{G(\lambda)\left|g_{kk}(\lambda)=(1-\varepsilon)g_{kk}^1(\lambda)+\varepsilon w_{kk}(\lambda),\frac{1}{2\pi}\int_{-\infty}^{\infty}g_{kk}(\lambda)d\lambda=q_k,k=\overline{1,T}\right.\right\};$$

$$D_{2\delta}^3=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\langle B_1,F(\lambda)-F_1(\lambda)\rangle|^2d\lambda\leq\delta\right.\right\};$$

$$D_{\varepsilon}^3=\left\{G(\lambda)\left|\langle B_2,G(\lambda)\rangle=(1-\varepsilon)\langle B_2,G_1(\lambda)\rangle+\varepsilon\langle B_2,W(\lambda)\rangle,\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle B_2,G(\lambda)\rangle d\lambda=q\right.\right\};$$

$$D_{2\delta}^4=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{ij}(\lambda)-f_{ij}^1(\lambda)\right|^2d\lambda\leq\delta_i^j,i,j=\overline{1,T}\right.\right\},$$

$$D_{\varepsilon}^4=\left\{G(\lambda)\left|G(\lambda)=(1-\varepsilon)G_1(\lambda)+\varepsilon W(\lambda),\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\lambda)d\lambda=Q\right.\right\},$$

where $F_1(\lambda),G_1(\lambda)$ are known and fixed spectral densities, $W(\lambda)$ is unknown spectral density, $q,\delta,q_k,\delta_k,k=\overline{1,T}$, $\delta_i^j,i,j=\overline{1,T}$, are given numbers, Q,B_1,B_2 are given positive definite Hermitian matrices.
The classes $D_{2\delta}$ describe the "δ-neighborhood" models in the space L_2 of the given bounded spectral density $F_1(\lambda)$, the classes D_{ε} describe the "ε-contamination" models of spectral densities.
From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.
For the first pair $D_{2\delta}\times D_{\varepsilon}^1$, we have equations

$$((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})^*((A(\lambda))^{\top}G^0(\lambda)+(C^0(\lambda))^{\top})=\alpha^2\mathrm{Tr}(F^0(\lambda)-F_1(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{5.1}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}\left(F^0(\lambda)-F_1(\lambda)\right)\right|^2 d\lambda=\delta, \tag{5.2}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=(\beta^2+\gamma(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{5.3}$$

where α^2,β^2 are Lagrange multipliers, $\gamma(\lambda)\leq 0$ and $\gamma(\lambda)=0$ if $\mathrm{Tr}\,F^0(\lambda)>(1-\varepsilon)\mathrm{Tr}\,G_1(\lambda)$.
For the second pair $D^2_{2\delta}\times D^2_{\varepsilon}$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\alpha_k^2(f^0_{kk}(\lambda)-f^1_{kk}(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{5.4}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f^0_{kk}(\lambda)-f^1_{kk}(\lambda)\right|^2 d\lambda=\delta_k,\ k=\overline{1,T}, \tag{5.5}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{(\beta_k^2+\gamma_k(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{5.6}$$

where α_k^2,β_k^2 are Lagrange multipliers, $\gamma_k(\lambda)\leq 0$ and $\gamma_k(\lambda)=0$ if $g^0_{kk}(\lambda)>(1-\varepsilon)g^1_{kk}(\lambda)$.
For the third pair $D^3_{2\delta}\times D^3_{\varepsilon}$, we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=\alpha^2\left\langle B_1,F^0(\lambda)-F_1(\lambda)\right\rangle(F^0(\lambda)+G^0(\lambda))^2, \tag{5.7}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left\langle B_1,F^0(\lambda)-F_1(\lambda)\right\rangle\right|^2 d\lambda=\delta, \tag{5.8}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=(\beta^2+\gamma'(\lambda))(F^0(\lambda)+G^0(\lambda))B_2^\top(F^0(\lambda)+G^0(\lambda)), \tag{5.9}$$

where α^2,β^2 are Lagrange multipliers, $\gamma'(\lambda)\leq 0$ and $\gamma'(\lambda)=0$ if $\langle B_2,G^0(\lambda)\rangle>(1-\varepsilon)\langle B_2,G_1(\lambda)\rangle$.
For the fourth pair $D^4_{2\delta}\times D^4_{\varepsilon}$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))\left\{\alpha_{ij}(f^0_{ij}(\lambda)-f^1_{ij}(\lambda))\right\}_{i,j=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{5.10}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f^0_{ij}(\lambda)-f^1_{ij}(\lambda)\right|^2 d\lambda=\delta_i^j,\ i,j=\overline{1,T}, \tag{5.11}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)=(F^0(\lambda)+G^0(\lambda))(\vec{\beta}\cdot\vec{\beta}^*+\Gamma(\lambda))(F^0(\lambda)+G^0(\lambda)), \tag{5.12}$$

where $\vec{\beta},\alpha_{ij}$ are Lagrange multipliers, $\Gamma(\lambda)\leq 0$ and $\Gamma_3(\lambda)=0$ if $G^0(\lambda)>(1-\varepsilon)G_1(\lambda)$.
Thus, the following statement holds true.

Theorem 5.1. *The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_{2\delta}\times D_{\varepsilon}$ for the optimal linear filtering of the functional $A_s\vec{\xi}$ are determined by relations (5.1) – (5.3) for the first pair $D^1_{2\delta}\times D^1_{\varepsilon}$ of sets of admissible spectral densities; by relations (5.4) – (5.6) for the second pair $D^2_{2\delta}\times D^2_{\varepsilon}$ of sets of admissible spectral densities; by relations (5.7) – (5.9) for the third pair $D^3_{2\delta}\times D^3_{\varepsilon}$ of sets of admissible spectral densities; by relations (5.10) – (5.12) for the fourth pair $D^4_{2\delta}\times D^4_{\varepsilon}$ of sets of admissible spectral densities; the minimality condition (2.1), the constrained optimization problem (3.1) and restrictions on densities from the corresponding classes $D_{2\delta}\times D_{\varepsilon}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A_s\vec{\xi}$ is determined by the formula (2.6).*

Corollary 5.2. *Assume that the spectral density $G(\lambda)$ is known. Let the function $F^0(\lambda)+G(\lambda)$ satisfy the minimality condition (2.1). The spectral density $F^0(\lambda)$ is the least favorable in the classes $D^k_{2\delta}$, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (5.1) – (5.2), (5.4) – (5.5), (5.7) – (5.8), (5.10) – (5.11), respectively, and the pair $(F^0(\lambda),G(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

Corollary 5.3. *Assume that the spectral density $F(\lambda)$ is known. Let the function $F(\lambda)+G^0(\lambda)$ satisfy the minimality condition (2.1). The spectral density $G^0(\lambda)$ is the least favorable in the classes D^k_{ε}, $k=\overline{1,4}$ for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (5.3), (5.6), (5.9), (5.12), respectively, and the pair $(F(\lambda),G^0(\lambda))$ is a solution of the optimization problem (3.1). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (2.6).*

6. Conclusion

In the article, we propose methods of the mean-square optimal linear filtering of functionals which depend on the unknown values of the multidimensional stationary stochastic process based on observations of the process with an additive stationary stochastic noise process. The case of spectral certainty, as well as the case of spectral uncertainty, are considered. In the case of spectral certainty, where the spectral density matrices of the stationary processes are exactly known, we apply a method based on orthogonal projections in a Hilbert space and derive formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the functionals. In the case of spectral uncertainty, where the spectral density matrices of the stationary processes are not exactly known while some sets of admissible spectral density matrices are given, we apply the minimax-robust method of estimation. This method allows us to find estimates that minimize the maximum values of the mean-square errors of estimates for all spectral density matrices from a given class of admissible spectral density matrices and derive relations which determine the least favourable spectral density matrices. These least favourable spectral density matrices are solutions of the optimization problem $\Delta_D(F,G) = -\Delta(h(F^0,G^0);F,G) + \delta((F,G)\,|D_F \times D_G\,) \to \inf$, which is characterized by the condition $0 \in \partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point (F^0,G^0). The form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application of the Lagrange method of indefinite multipliers for finding a solution to the optimization problem. The complexity of the problem is determined by the complexity of calculation of the subdifferential of the convex functional $\Delta_D(F,G)$. Making use of the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine the least favourable spectral densities in some special classes of spectral densities. These are: classes D_0 of densities with the moment restrictions, classes $D_{1\delta}$ which describe the "δ-neighborhood" models in the space L_1 of a given bounded spectral density, classes $D_{2\delta}$ which describe the "δ-neighborhood" models in the space L_2 of a given bounded spectral density, classes D_ε which describes the "ε-contamination" models of spectral densities.

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