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MORE ON TRANSLATIONALLY SLOWLY VARYING SEQUENCES

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Abstract. We define and study an equivalence relation in the class $\mathsf{Tr}(\mathsf{SV_s})$ of translationally slowly varying positive real sequences and its relations with selection principles and game theory. We also prove a game-theoretic result for translationally rapidly varying sequences.

1. Introduction

Throughout the paper $\mathbb{N}$ will denote the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{S}$ the set of sequences of positive real numbers.

The theory of regular variation, including in particular slow variation, was initiated in 1930 by J. Karamata [8]. Nowadays this branch of asymptotic analysis of divergent processes is known as *Karamata's theory of regular variation*. Another kind of variation, called *rapid variation*, was introduced and first studied in 1970 by de Haan [7]. These two theories are developed for functions and sequences and have various applications in several mathematical disciplines: number theory, differential and difference equations, probability theory, q-calculus, and so on. For more information about the theory of regular variation and the theory of rapid variation we refer the reader to the book [1]. In this article we are interested in two classes of sequences related to slow and rapid variations.

We recall first the definitions of slowly and rapidly varying sequences.

Definition 1.1. ([1, 2, 12]) A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is *slowly varying* (respectively, *rapidly varying*) if for each $\lambda > 0$ (respectively, $\lambda > 1$) the following is satisfied:

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = 1, \tag{1.1}$$

(respectively,

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = \infty), \tag{1.2}$$

where for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer part of x.

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The classes of slowly varying and rapidly varying sequences are denoted by $\mathsf{SV_s}$ and $\mathsf{R_{s,\infty}}$, respectively.

In what follows we work with the following two classes of sequences.

Definition 1.2. ([3, 11]) A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is *translationally slowly varying* (respectively, *translationally rapidly varying*) if for each $\lambda \geq 1$ the following asymptotic condition is satisfied:

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = 1 \tag{1.3}$$

(respectively,

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = \infty). \tag{1.4}$$

$\mathsf{Tr}(\mathsf{SV_s})$ denotes the class of translationally slowly varying sequences, and $\mathsf{Tr}(\mathsf{R_{s,\infty}})$ denotes the class of translationally rapidly varying sequences (see [2, 3, 4, 5]).

Observe that $R_{s,\infty} \cap \mathsf{Tr}(\mathsf{SV_s}) \neq \emptyset$, $R_{s,\infty} \setminus \mathsf{Tr}(\mathsf{SV_s}) \neq \emptyset$, $\mathsf{Tr}(\mathsf{SV_s}) \setminus \mathsf{R_{s,\infty}} \neq \emptyset$, and $\mathsf{Tr}(\mathsf{R_{s,\infty}}) \subset \mathsf{R_{s,\infty}}$.

In this paper we define and study a new equivalence relation in the class $\mathsf{Tr}(\mathsf{SV_s})$, in particular its relations with selection principles and game theory. We also provide a game-theoretic result concerning the class $\mathsf{Tr}(\mathsf{R_{s,\infty}})$.

2. Results

We begin this section with definitions of concepts we use in this article.

Definition 2.1. Sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ from $\mathbb{S}$ are *mutually translationally slowly equivalent*, denoted by

$$c_n \overset{ts}{\sim} d_n, \text{ as } n \to \infty,$$

if

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{d_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{d_{[n+\lambda]}}{c_n} = 1 \tag{2.1}$$

hold for each $\lambda \geq 1$.

Definition 2.2. Sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ from $\mathbb{S}$ are *mutually translationally rapidly equivalent*, denoted by

$$c_n \overset{tr}{\sim} d_n, \text{ as } n \to \infty,$$

if

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{d_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{d_{[n+\lambda]}}{c_n} = \infty \tag{2.2}$$

hold for each $\lambda \geq 1$.

Theorem 2.1. *Let sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ be elements from $\mathbb{S}$. If $c_n \overset{ts}{\sim} d_n$, as $n \to \infty$, then $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$ and $\mathbf{d} \in \mathsf{Tr}(\mathsf{SV_s})$.*

Proof. For $\lambda \geq 1$ we have

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = \lim_{n\to\infty} \left(\frac{c_{n+1}}{c_n}\right)^{[\lambda]}$$

if the limit on the right side exists. Further, since $c_n \overset{ts}{\sim} d_n$, we have

$$\lim_{n\to\infty} \frac{c_{n+2}}{c_n} = \lim_{n\to\infty} \left(\frac{c_{n+2}}{d_{n+1}} \cdot \frac{d_{n+1}}{c_n}\right) = \lim_{n\to\infty} \frac{c_{n+2}}{d_{n+1}} \cdot \lim_{n\to\infty} \frac{d_{n+1}}{c_n} = 1.$$

Therefore

$$1 = \lim_{n\to\infty} \left(\frac{c_{n+2}}{c_{n+1}} \cdot \frac{c_{n+1}}{c_n} \right) = \lim_{k\to\infty} \left(\frac{c_{k+1}}{c_k} \right)^2 ,$$

hence

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1.$$

This means that

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = 1 \ \text{ for each } \lambda \geq 1 \ ,$$

i.e. $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$.

Similarly we prove $\mathbf{d} \in \mathsf{Tr}(\mathsf{SV_s})$. □

In a similar way, by suitable modifications in the proof, we prove the following result.

Theorem 2.2. *Let* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *and* $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ *be sequences in* $\mathbb{S}$. *If* $c_n \overset{tr}{\sim} d_n$, *as* $n \to \infty$, *then* $\mathbf{c} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$ *and* $\mathbf{d} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$.

Theorem 2.3. *Relation* $\overset{ts}{\sim}$ *is an equivalence relation on* $\mathsf{Tr}(\mathsf{SV_s})$.

Proof. 1. (Reflexivity) Let $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$. Then $\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{c_n} = 1$ for each $\lambda \geq 1$, that is $c_n \overset{ts}{\sim} c_n$ as $n \to \infty$, and so reflexivity holds.

2.(Symmetry) It follows from the definition of relation $\overset{ts}{\sim}$.

3. (Transitivity) Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ and $\mathbf{e} = (e_n)_{n\in\mathbb{N}}$ be elements from $\mathsf{Tr}(\mathsf{SV_s})$ such that $c_n \overset{ts}{\sim} d_n$, $n \to \infty$, and $d_n \overset{ts}{\sim} e_n$, $n \to \infty$. Then we have

$$\lim_{n\to\infty} \frac{c_{n+2}}{e_n} \quad = \quad \lim_{n\to\infty} \frac{c_{n+2}}{d_{n+1}} \cdot \lim_{n\to\infty} \frac{d_{n+1}}{e_n} = 1.$$

We conclude

$$1 = \lim_{n\to\infty} \left(\frac{c_{n+2}}{e_{n+1}} \cdot \frac{e_{n+1}}{e_n} \right).$$

Because of $\mathbf{e} \in \mathsf{Tr}(\mathsf{SV_s})$, we obtain

$$\lim_{n\to\infty} \frac{c_{n+1}}{e_n} = 1.$$

It follows from here that for each $\lambda \geq 1$ it holds

$$\lim_{n\to\infty} \frac{c_{[n+\lambda]}}{e_n} = 1.$$

In a similar way one proves

$$\lim_{n\to\infty} \frac{e_{[n+\lambda]}}{c_n} = 1, \ \lambda \geq 1,$$

which means $c_n \overset{ts}{\sim} e_n$. □

Remark. Let a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ belong to the class $\mathsf{Tr}(\mathsf{SV_s})$ and let $\mathbf{d} = (d_n)_{n\in\mathbb{N}} \in \mathbb{S}$ be such that $c_n \overset{ts}{\sim} d_n$. Then

$$\lim_{n\to\infty} \frac{c_n}{d_n} = \lim_{n\to\infty} \left(\frac{c_n}{c_{n+1}} \cdot \frac{c_{n+1}}{d_n} \right) = 1$$

and we conclude that sequences $\mathbf{c}$ and $\mathbf{d}$ are strongly asymptotically equivalent (see, for instance. [1, 6]), i.e. $\lim_{n\to\infty} \frac{c_n}{d_n} = 1$.

Recall the definition of selection principles, which we need in what follows (see [9, 10]).

Definition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be subfamilies of the set $\mathbb{S}$. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i \in \{2, 3, 4\}$, denotes the following selection hypotheses: for each sequence $(A_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

(1) $\alpha_2(\mathcal{A}, \mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is infinite for each $n \in \mathbb{N}$;
(2) $\alpha_3(\mathcal{A}, \mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is infinite for infinitely many $n \in \mathbb{N}$;
(3) $\alpha_4(\mathcal{A}, \mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is nonempty for infinitely many $n \in \mathbb{N}$,

where Im denotes the image of the corresponding sequence.

The following infinitely long game is related to α_2 (see [9, 10]).

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of $\mathbb{S}$. The symbol $\mathsf{G}_{\alpha_2}(\mathcal{A}, \mathcal{B})$ denotes the following infinitely long game for two players, I and II, who play a round for each natural number n. In the first round I chooses an arbitrary element $\mathbf{A_1} = (A_{1,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$, and II chooses a subsequence $y_{r_1} = (A_{1,r_1(j)})_{j\in\mathbb{N}}$ of the sequence A_1. At the k^{th} round, $k \geq 2$, I chooses an arbitrary element $A_k = (A_{k,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and II chooses a subsequence $y_{r_k} = (A_{k,r_k(j)})_{j\in\mathbb{N}}$ of the sequence A_k, such that $\mathrm{Im}(r_{k(j)}) \cap \mathrm{Im}(r_{p(j)}) = \emptyset$ is satisfied, for each $p \leq k-1$. II wins a play

$$A_1, y_{r_1}; \ldots; A_k, y_{r_k}; \ldots$$

if and only if all elements from $Y = \bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} A_k, r_{k(j)}$, with respect to second index, form a subsequence $\mathbf{y} = (y_m)_{m\in\mathbb{N}} \in \mathcal{B}$.

A strategy σ for the player II is a *coding strategy* if II remembers only the most recent move by I and by II before deciding how to play the next move.

Observe, that if II has a winning strategy in the game $\mathsf{G}_{\alpha_2}(\mathcal{A}, \mathcal{B})$, then the selection principle $\alpha_2(\mathcal{A}, \mathcal{B})$ is true. Also, $\alpha_2(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_3(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B})$.

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. Then we define

$$[\mathbf{c}]_{ts} = \{\mathbf{d} = (d_n)_{n\in\mathbb{N}} \in \mathbb{S} : c_n \overset{ts}{\sim} d_n, n \to \infty\} \tag{2.3}$$

as the equivalence class of $\mathbf{c}$ in $\mathsf{Tr}(\mathsf{SV_s})$.

Theorem 2.4. *For a fixed element $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$, the player* II *has a winning coding strategy in the game $\mathsf{G}_{\alpha_2}([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$,*

Proof. (1^{st} round): Let σ be the strategy of the player II. The player I chooses a sequence $\mathbf{x_1} = (x_{1,n})_{n\in\mathbb{N}} \in [\mathbf{c}]_{ts}$ arbitrary. Then the player II chooses the subsequence $\sigma(\mathbf{x_1}) = (x_{1,k_1(n)})_{n\in\mathbb{N}}$ of the sequence $\mathbf{x_1}$, where $\mathrm{Im}(k_1)$ is the set of natural numbers greater of or equal to $n_1 \in \mathbb{N}$ which are divisible by 2 and not divisible by 2^2, and $1 - \frac{1}{2} \leq \frac{c_n}{x_{m,n}} \leq 1 + \frac{1}{2}$ holds for each $n \geq n_1$.

(m^{th} round, $m \geqslant 2$): The player I chooses a sequence $\mathbf{x_m} = (x_{m,n})_{n\in\mathbb{N}} \in [\mathbf{c}]_{ts}$. Then the player II chooses the subsequence

$$\sigma(\mathbf{x_m}, (x_{m-1,k_{m-1}(n)})_{n\in\mathbb{N}}) = (x_{m,k_m(n)})_{n\in\mathbb{N}}$$

of the sequence $\mathbf{x_m}$, so that $\mathrm{Im}(k_m)$ is the set of natural numbers greater of or equal to $n_m \in \mathbb{N}$, which are divisible by 2^m, and not divisible by 2^{m+1}, and $1 - \frac{1}{2^m} \leq \frac{c_n}{x_{m,n}} \leq 1 + \frac{1}{2^m}$ holds for each $n \geq n_m$.

Consider now the set $Y = \bigcup_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}} x_{m,k_m(n)}$ in $\mathbb{S}$ indexed by the second index $k_m(n)$. This set we can consider as the subsequence of the sequence $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ given by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction $\mathbf{y} \in \mathbb{S}$. Also, the intersection of $\mathbf{y}$ and $\mathbf{x_m}$, $m \in \mathbb{N}$, is an infinite set.

Let us prove that $y_m \overset{ts}{\sim} c_m$, as $m \to \infty$. Let $\varepsilon > 0$. Let m be the smallest natural number such that $\frac{1}{2^m} \leq \varepsilon$. For each $k \in \{1, 2, ..., m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $1 - \varepsilon \leq \frac{c_i}{x_{k,n}} \leq 1 + \varepsilon$ for each $n \geq n_k^*$. Set $n^* = \max\{n_1^*, n_2^*, \ldots, n_{m-1}^*\}$. For each $i \geq n^*$ we have $1 - \varepsilon \leq \frac{c_i}{y_i} \leq 1 + \varepsilon$. Therefore, $\lim_{n\to\infty} \frac{c_i}{y_i} = 1$. It follows

$$\lim_{i\to\infty} \frac{c_{i+1}}{y_i} = \lim_{i\to\infty} \left(\frac{c_{i+1}}{c_i} \cdot \frac{c_i}{y_i} \right) = 1$$

because $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$. In a similar way we prove

$$\lim_{i\to\infty} \frac{y_{i+1}}{c_i} = 1.$$

One concludes that for each $\lambda \geq 1$

$$\lim_{i\to\infty} \frac{y_{[i+\lambda]}}{c_i} = \lim_{i\to\infty} \frac{c_{[i+\lambda]}}{y_i} = 1$$

i.e. $\mathbf{y} = (y_i)_{i\in\mathbb{N}} \in [\mathbf{c}]_{ts}$. The theorem is proved. □

Corollary 2.5. *The selection principle $\alpha_2([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$ holds for each fixed element $\mathbf{c} \in \mathsf{Tr}(\mathsf{SV_s})$. Consequently, $\alpha_3([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$ and $\alpha_4([\mathbf{c}]_{ts}, [\mathbf{c}]_{ts})$ also hold.*

We end the paper by proving a result about mutually translationally rapidly equivalent sequences.

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. Then we define

$$[\mathbf{c}]_{tr} = \{\mathbf{d} = (d_n)_{n\in\mathbb{N}} \in \mathbb{S} : c_n \overset{tr}{\sim} d_n, n \to \infty\}. \tag{2.4}$$

Theorem 2.6. *The player* II *has a winning coding strategy in the game $\mathsf{G}_{\alpha_2}([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$, for any fixed element $\mathbf{c} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$.*

Proof. Let σ be the strategy of II.

(m^{th} round, $m \geq 1$): The player I chooses a sequence $\mathbf{x_m} = (x_{m,n})_{n\in\mathbb{N}} \in [\mathbf{c}]_{tr}$. Then the player II chooses the subsequence

$$\sigma(\mathbf{x_m}, (x_{m-1,k_{m-1}(n)})_{n\in\mathbb{N}}) = (x_{m,k_m(n)})_{n\in\mathbb{N}}$$

of the sequence $\mathbf{x_m}$, so that $\mathrm{Im}(k_m)$ is the set of natural numbers greater of or equal to n_m, which are divisible with 2^m, and not divisible with 2^{m+1}, $n_m \in \mathbb{N}$, and $\frac{c_{n+1}}{x_{m,n}} \geq 2^m$ and $\frac{x_{m,n+1}}{c_n} \geq 2^m$ for each $n \geq n_m$. Let $\lambda \geq 1$. Since $\mathbf{c} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$, we have $\frac{c_{n+1}}{c_n} \geq 1$ for sufficiently large n. Then

$$\frac{c_{[n+\lambda]}}{x_{m,n}} = \frac{c_{[n+\lambda]}}{c_{[n+\lambda]-1}} \cdot \frac{c_{[n+\lambda]-1}}{c_{[n+\lambda]-2}} \cdots \frac{c_{n+1}}{x_{m,n}} \geq 2^m$$

for each $n \geq n_m$. Since $x_{m,n} \overset{tr}{\sim} c_n$, as $n \to \infty$, we have $\mathbf{x_m} \in \mathsf{Tr}(\mathsf{R_{s,\infty}})$ (Theorem 2.2). In a similar way we prove $\frac{x_{m,[n+\lambda]}}{c_n} \geq 2^m$ for all $n \geq n_m$.

Form the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} x_{m,k_m(n)}$ of positive real numbers indexed by the second index. This set is a subsequence of the sequence $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ defined by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

Evidently, $\mathbf{y} \in \mathbb{S}$ and the intersection of $\mathbf{y}$ and $\mathbf{x_m}$, $m \in \mathbb{N}$, is an infinite set.

We prove $y_m \overset{tr}{\sim} c_m$, as $m \to \infty$. Let $M > 0$. Choose the smallest $m \in \mathbb{N}$ such that $2^m > M$. For each $k \in \{1, 2, ..., m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $\frac{c_{[n+\lambda]}}{x_{k,n}} \geq M$ and $\frac{x_{k,[n+\lambda]}}{c_n} \geq M$ for each $\lambda \geq 1$ and each $n \geq n_k^*$. Let $n^* = \max\{n_1^*, \dots, n_{m-1}^*\}$. Therefore, the inequalities $\frac{c_{[i+\lambda]}}{y_i} \geq M$ and $\frac{y_{[i+\lambda]}}{c_i} \geq M$ hold for each $\lambda \geq 1$ and each $i \geq n^*$. As M was arbitrary, one concludes $y_i \overset{tr}{\sim} c_i$, as $i \to \infty$. In other words, $\mathbf{y} \in [\mathbf{c}]_{tr}$. □

Corollary 2.7. *The selection principle $\alpha_2([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$ holds for each fixed element $\mathbf{c} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$, and thus $\alpha_3([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$ and $\alpha_4([\mathbf{c}]_{tr}, [\mathbf{c}]_{tr})$ hold.*

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