Sakarya University Journal of Science

ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University | http://www.saujs.sakarya.edu.tr/

Title: Coefficient İnequalities For Janowski Type Close-To-Convex Functions Associated With Ruscheweyh Derivative Operator

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Recieved: 2019-01-10 00:00:00

Accepted: 2019-01-28 00:00:00

Article Type: Research Article
Volume: 23
Issue: 5
Month: October
Year: 2019
Pages: 714-717

How to cite
Öznur Özkan Kılıç; (2019), Coefficient İnequalities For Janowski Type Close-To-Convex Functions Associated With Ruscheweyh Derivative Operator. Sakarya University Journal of Science, 23(5), 714-717, DOI: 10.16984/saufenbilder.511321
Access link
http://www.saujs.sakarya.edu.tr/issue/44066/511321


Sakarya University Journal of Science 23(5), 714-717, 2019

# Coefficient Inequalities for Janowski Type Close-to-Convex Functions Associated with Ruscheweyh Derivative Operator

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## ABSTRACT

The aim of this paper is to introduce a new subclasses of the Janowski type close-to-convex functions defined by Ruscheweyh derivative operator and obtain coefficient bounds belonging to this class.

**Keywords:** Univalent Function, Subordination, Close-to-Convex Function, Ruscheweyh Derivative Operator

## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ denote the subclasses of $\mathcal{A}$ which are univalent in $\Delta$.

An analytic function $f$ is subordinate to an analytic function $F$, written as $f \prec F$ or

$f(z) \prec F(z)$, if there exists a Schwarz function

$\omega: \Delta \longrightarrow \Delta$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F\big(\omega(z)\big)$. In particular, if $F$ is univalent in $\Delta$, we have the following equivalence:

$f(z) \prec F(z) \iff [f(0) = F(0) \wedge f(\Delta) = F(\Delta)].$

The Hadamard product or convolution of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

for $z \in \Delta$.

In 1975, Ruscheweyh [5] introduced a linear operator $\mathcal{D}^{\delta} : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\mathcal{D}^{\delta} f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z)$$

$$= z + \sum_{n=2}^{\infty} \varphi_n(\delta)\, a_n z^n$$

with

$$\varphi_n(\delta) = \frac{(\delta+1)_{n-1}}{(n-1)!}$$

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for $\delta > -1$ and $(a)_n$ is Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & if\ n = 0 \\ a(a+1)\cdots(a+n-1) & if\ n \in \mathbb{N} \end{cases}$$

for $a \in \mathbb{C}$ and $\mathbb{N} = \{1,2,3,\dots\}$.

Notice that

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = zf'(z)$$

and

$$\mathcal{D}^m f(z) = \frac{z\left(z^{m-1}f(z)\right)^m}{m!}$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+m)}{\Gamma(m+1)(n-1)!} a_n z^n$$

for all $\delta = m \in \mathbb{N}_0 = \{0,1,2,\dots\}$.

In geometric function theory, various subclasses defined by Ruscheweyh derivative operator were studied.

Let $\mathcal{S}^*$ and $\mathcal{C}$ be the usual subclasses of functions which members are univalent, starlike and convex in $\Delta$, respectively. We also denote $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ the class of starlike functions of order $\alpha$ and the class of convex functions of order $\alpha$, for $0 \le \alpha < 1$, respectively. Note that $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{C} = \mathcal{C}(0)$.

In 1973, Janowski [2] introduced the classes by $\mathcal{S}^*(A,B)$ and $\mathcal{C}(A,B)$

$$\mathcal{S}^*(A,B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\mathcal{C}(A,B) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

for $-1 \le B < A \le 1,\ z \in \Delta$. Note that $\mathcal{S}^*(\alpha) = \mathcal{S}^*(1-2\alpha,-1)$, $\mathcal{S}^* = \mathcal{S}^*(1,-1)$ and $\mathcal{C}(\alpha) = \mathcal{C}(1-2\alpha,-1)$, $\mathcal{C} = \mathcal{C}(1,-1)$.

A function $f \in \mathcal{A}$ is said to be close-to-star if and only if there exists $g \in \mathcal{S}^*$ such that $\Re\{f(z)/g(z)\} > 0$ for all $z \in \Delta$. Also, a function $f \in \mathcal{A}$ is said to be close-to-convex if and only if there exists $g \in \mathcal{C}$ such that $\Re\{f'(z)/g'(z)\} > 0$ for all $z \in \Delta$. The classes of close-to-star and close-to-convex functions denote by $\mathcal{CS}^*$ and $\mathcal{CC}$, respectively. The class of close-to-star functions was introduced by Reade in [4] and the class of close-to-convex functions was introduced by Kaplan in [3]. Similarly, we denote by $\mathcal{CS}^*(\gamma)$ and $\mathcal{CC}(\gamma)$ the classes of close-to-star functions of order $\gamma$ and close-to-convex functions of order $\gamma$, for $0 \le \gamma < 1$, respectively. Note that $\mathcal{CS}^* = \mathcal{CS}^*(0)$ and $\mathcal{CC} = \mathcal{CC}(0)$.

The class of Janowski type close-to-starlike functions in $\Delta$, denoted by $\mathcal{CS}^*(A,B)$, is defined by

$$\mathcal{CS}^*(A,B) = \left\{ f \in \mathcal{A} : \frac{f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, g \in \mathcal{S}^* \right\}$$

for $-1 \le B < A \le 1,\ z \in \Delta$. Similarly, the class of Janowski type close-to-convex functions in $\Delta$, denoted by $\mathcal{CC}(A,B)$, is defined by

$$\mathcal{CC}(A,B) = \left\{ f \in \mathcal{A} : \frac{f'(z)}{g'(z)} \prec \frac{1+Az}{1+Bz},\ g \in \mathcal{C} \right\}$$

for $-1 \le B < A \le 1,\ z \in \Delta$. The classes are introduced by Reade [4] in 1955.

**Definition 1.1.** The class of Janowski type functions defined by Ruscheweyh derivative operator in $\Delta$, denoted by $\mathcal{J}_{\mathcal{R}}(\delta,\beta,A,B)$, is defined by

$$\mathcal{J}_{\mathcal{R}}(\delta,\beta,A,B) = \left\{ f \in \mathcal{A} : \frac{\mathcal{D}^\delta f(z)}{\mathcal{D}^\beta g(z)} \prec \frac{1+Az}{1+Bz},\ g \in \mathcal{S}^* \right\}$$

for $\delta,\beta > -1, -1 \le B < A \le 1,\ z \in \Delta$.

We need the following lemma to obtain our results.

**Lemma 1.2.** [1] If the function $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

is analytic in $\Delta$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}$$

then $|p_n| \leq A - B,$ for $n \in \mathbb{N}, -1 \leq B < A \leq 1.$

## 2. MAIN RESULTS AND THEIR CONSEQUENCES

We begin by finding the estimates on the coefficient $|a_n|$ for functions in the class $\mathcal{J}_{\mathcal{R}}(\delta, \beta, A, B)$.

**Theorem 2.1.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{J}_{\mathcal{R}}(\delta, \beta, A, B)$, then

$$|a_n| \leq \frac{n\,\varphi_n(\beta) + (A-B)\sum_{m=1}^{n-1} m\,\varphi_m(\beta)}{\varphi_n(\delta)}. \qquad (2.1)$$

**Proof.** Let $f(z) \in \mathcal{J}_{\mathcal{R}}(\delta, \beta, A, B)$. Then, there are analytic functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, $\omega$ is a Schwarz function and

$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ as in Lemma 1.2 such that

$$\frac{\mathcal{D}^{\delta} f(z)}{\mathcal{D}^{\beta} g(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} = p(z) \qquad (2.2)$$

for $z \in \Delta$. Then (2.2) can be written as

$$\mathcal{D}^{\delta} f(z) = p(z).\mathcal{D}^{\beta} g(z)$$

or

$$z + \sum_{n=2}^{\infty} \varphi_n(\delta)\, a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \sum_{m=1}^{n} \varphi_{n-m+1}(\beta)\, b_{n-m+1}\, p_{m-1}$$

Equating the coefficients of like powers of $z$, we obtain

$$\varphi_2(\delta)\, a_2 = \varphi_2(\beta)\, b_2 + p_{1,}$$

$$\varphi_3(\delta)\, a_3 = \varphi_2(\beta)\, b_2\, p_1 + \varphi_3(\beta)\, b_3 + p_{2,}$$

and

$$\varphi_n(\delta)\, a_n = \varphi_n(\beta)\, b_n + \varphi_{n-1}(\beta)\, b_{n-1}\, p_1 + \varphi_{n-2}(\beta)\, b_{n-2}\, p_{2+\cdots+} p_{n-1}.$$

By using Lemma 1.2 and $g \in \mathcal{S}^*$, we get

$$\varphi_n(\delta)|a_n| \leq n\,\varphi_n(\beta) + (A-B)\sum_{m=1}^{n-1} m\,\varphi_m(\beta)$$

and this inequality is equivalent to (2.1).

**Corollary 2.2.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{CS}^*(A, B)$, then

$|a_n| \leq n + \frac{(A-B)(n-1)n}{2}.$

**Proof.** In Theorem 2.1, we take $\delta = 0,\ \beta = 0.$

**Corollary 2.3.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{CS}^*(\gamma)$, then

$|a_n| \leq n + (1-\gamma)(n-1)n.$

**Proof.** In Theorem 2.1, we take $\delta = 0,\ \beta = 0,$

$A = 1 - 2\gamma, B = -1.$

**Corollary 2.4.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{CS}^*$, then

$|a_n| \leq n^2.$

**Proof.** In Theorem 2.1, we take $\delta = 0,\ \beta = 0,$

$A = 1, B = -1.$

**Corollary 2.5.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{CC}(A, B)$, then

$|a_n| \leq 1 + \frac{(A-B)(n-1)}{2}.$

**Proof.** In Theorem 2.1, we take $\delta = 1,\ \beta = 0.$

**Corollary 2.6.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{CC}(\gamma)$, then

$$|a_n| \leq 1 + (1-\gamma)(n-1).$$

**Proof.** In Theorem 2.1, we take $\delta = 1,\ \beta = 0,$

$A = 1 - 2\gamma, B = -1.$

**Corollary 2.7.** If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{CC}$, then

$$|a_n| \leq \text{n}.$$

**Proof.** In Theorem 2.1, we take $\delta = 1,\ \beta = 0,$

$A = 1, B = -1.$

We note that Results in Corollary 2.4 and Corollary 2.7 were proved by Reade in 1955.

( See [4] )

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