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On Generalized Tribonacci Octonions

Arzu Özkoç Öztürk*[1]

Abstract

In this paper, we introduce generalized tribonacci octonion sequence which is a generalization of the third order recurrence relations. We investigate many identities which are created by using generalized tribonacci sequence. We get different results for these classes of octonions, comprised recurrence relation, summation formulas, Binet formula, norm value and generating function.

Keywords: Tribonacci octonion, Binet formula, Generalized Tribonacci sequence

1. INTRODUCTION

Tribonacci numbers which are described as

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \qquad (1.1)$$

for $n \geq 4$, with initial conditions $T_1 = 1$, $T_2 = 1$ and $T_3 = 2$.

The generalized tribonacci sequence is the generalization of the sequences tribonacci, Padovan, Narayana and third order Jacobsthal sequences. The generalized Tribonacci sequence V_n defined as

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, n \geq 3,$$

where $V_0 = a$, $V_1 = b$, $V_2 = c$ are arbitrary integers and r, s, t are real numbers. Its characteristic equation is $x^3 - rx^2 - sx - t = 0$ and it has one real and two conjugate complex roots of it are $\alpha = \frac{r}{3} + A + B, \omega_1 = \frac{r}{3} + \varepsilon A + \varepsilon^2 B$ and $\omega_2 = \frac{r}{3} + \varepsilon^2 A + \varepsilon B$ where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{\frac{1}{3}},$$

$$B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{\frac{1}{3}},$$

with $\Delta = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}$ and $\varepsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

It has Binet formula

$$V_n = \frac{P\alpha^n}{(\alpha - \omega_1)(\alpha - \omega_2)} - \frac{Q\omega_1^n}{(\alpha - \omega_1)(\omega_1 - \omega_2)}$$

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$+\frac{R\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}$ (1.2)

where $P = c - (\omega_1 + \omega_2)b + \omega_1\omega_2 a$, $Q = c - (\alpha + \omega_2)b + \alpha\omega_2 a$ and $R = c - (\alpha + \omega_1)b + \alpha\omega_1 a$.

In Clifford algebra, octonions are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. For α_i and $\beta_i \in \mathbb{R}$ $(i = 0,1,\cdots,7)$, the field of octo-nion

$$\alpha = \sum_{s=0}^{7} \alpha_s e_s \text{ and } \beta = \sum_{s=0}^{7} \beta_s e_s$$

is an eight-dimensional non-commutative and non-associative (but satisfy a weaker form of a associativity) algebra generated by eight base elements $e_0, e_1, \cdots, e_6$ and e_7 which satisfy the non-commutative and non-associative multip-lication rules. Afterwards the Fibonacci octonion numbers are given in [1]. For $n \geq 0$ the Fibonacci octonion numbers that are given for the n-th classic Fibonacci F_n number are defined by the following recurrence relation:

$$O_n = \sum_{s=0}^{7} F_{n+s} e_s.$$

Also the sum and subtract of m and n are

$$m \pm n = \sum_{s=0}^{7} (\alpha_s \pm \beta_s) e_s$$

where $m \in \mathbb{Q}$ can be written as, respectively.

$$\overline{m} = \alpha_0 - \sum_{s=0}^{7} \alpha_s e_s$$

is the conjugate of m and this operation provides $\overline{\overline{m}} = m, \overline{m+n} = \overline{m} + \overline{n}$ and $\overline{m \cdot n} = \overline{m} \cdot \overline{n}$ for all $m, n \in \mathbb{Q}$. The norm of an octonion is defined

$Nr^2(m \cdot n) = Nr^2(m) Nr^2(n)$ and

$$(m \cdot n)^{-1} = m^{-1} \cdot n^{-1}.$$

Octonions are alternative but not commutative and not associative, $m \cdot (m \cdot n) = m^2 \cdot n$, $(m \cdot n) \cdot n = m \cdot n^2, (m \cdot n) \cdot m = m \cdot (n \cdot m) = m \cdot n \cdot m$

where $\cdot$ is the product on the octonions.

We give a definition of generalized tribonacci sequence which is a generalization of tribonacci numbers. Also we consider generalized tribonacci octonions which contain tribonacci, Padovan, Narayana octonions. In our work, we introduce for finding special identities of generalized tribonacci octonions. We motivate by their results in [2], [3] and [4]. In [3], they studied Horadam octonions. In [5], they considered Padovan and Pell-Padovan quater-nions which is the third order quaternions. In our mind, with generalized tribonacci sequence V_n helped us for construct the recurrence relations of the generalized tribonacci octonions. The generalized tribonacci octonions $O_{V,n}$ are the example of the third order octonions with $n \geq 3$,

$$O_{V,n} = \sum_{k=0}^{7} V_{n+k} e_k \tag{1.3}$$

where V_n is the $n-$th generalized tribonacci sequence. Generalized tribonacci sequence was examined in detail [6], [7] and it was shown that this sequence is used to generalize all the third order linear recurrence relations on octonions.

Note that, the second order linear recurence octonion sequences for example in [8], they defined modified Pell and Modified $k-$Pell octonions and in [9], authors studied Pell octonions. Moreover in [10], $(p,q)-$Fibonacci octonions are obtained which is the same results in [3] but the initial conditions are $F_0(p,q) = 0$ and $F_1(p,q) = 1$, also in [11]. Another octonionic sequence was devoted to studying Jacobsthal and Jacobsthal-Lucas octonions in [12]. Furthermore in [13], they derived third-order Jacobsthal quaternions, now we expand all third order recurrence octonion sequences in one recurrence relations. New identities and relations were introduced in [14], which is on tribonacci quaternions. Also in [15], authors considered the bicomplex generalized tribonacci quaternions. Generalized tribonacci octonions were studied in [16], we expect to find octonions in a new third order recurrence concepts.

2. GENERALİZED TRİBONACCİ OCTONİONS

The generalized tribonacci octonions $O_{V,n}$ are the example of sequences defined by a recurrence relation for $n \geq 3$,

$$O_{V,n} = rO_{V,n-1} + sO_{V,n-2} + tO_{V,n-3} \qquad (2.1)$$

with the initial conditions of

$$O_{V,0} = \sum_{k=0}^{7} V_k e_k = V_0 e_0 + V_1 e_1 + \ldots + V_7 e_7$$

$$O_{V,1} = \sum_{k=0}^{7} V_{1+k} e_k = V_1 e_0 + V_2 e_1 + \ldots + V_8 e_7$$

and

$$O_{V,2} = \sum_{k=0}^{7} V_{2+k} e_k = V_2 e_0 + V_3 e_1 + \ldots + V_9 e_7$$

Theorem 2.1 Binet Formula for the generalized Tribonacci octonions $O_{V,n}$ are

$$O_{V,n} = \frac{P\alpha^* \alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q w_1^* w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R w_2^* w_2^n}{(\alpha - w_2)(w_1 - w_2)}$$

where

$$\alpha^* = e_0 + \alpha e_1 + \alpha^2 e_2 + \cdots + \alpha^7 e_7$$

$$w_1^* = e_0 + w_1 e_1 + w_1^2 e_2 + \cdots + w_1^7 e_7$$

$$w_2^* = e_0 + w_2 e_1 + w_2^2 e_2 + \cdots + w_2^7 e_7.$$

Proof. Using the definition of the (1.3) and Binet formula for the generalized tribonacci numbers (1.2), we have

$$O_{V,n} = V_n e_0 + V_{n+1} e_1 + \cdots + V_{n+7} e_7$$

$$= \left(\frac{P\alpha^* \alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q w_1^* w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R w_2^* w_2^n}{(\alpha - w_2)(w_1 - w_2)} \right) e_0$$

$$+ \left(\frac{P\alpha^* \alpha^{n+1}}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q w_1^* w_1^{n+1}}{(\alpha - w_1)(w_1 - w_2)} + \frac{R w_2^* w_2^{n+1}}{(\alpha - w_2)(w_1 - w_2)} \right) e_1 + \cdots$$

$$+ \left(\frac{P\alpha^* \alpha^{n+7}}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q w_1^* w_1^{n+7}}{(\alpha - w_1)(w_1 - w_2)} + \frac{R w_2^* w_2^{n+7}}{(\alpha - w_2)(w_1 - w_2)} \right) e_7.$$

Then make some arrangement,

$$O_{V,n} = \frac{P}{(\alpha - w_1)(\alpha - w_2)} (\alpha^n e_0 + \alpha^{n+1} e_1 + \cdots + \alpha^{n+7} e_7)$$

$$- \frac{Q}{(\alpha - w_1)(w_1 - w_2)} (w_1^n e_0 + w_1^{n+1} e_1 + \cdots + w_1^{n+7} e_7)$$

$$+ \frac{R}{(\alpha - w_2)(w_1 - w_2)} (w_2^n e_0 + w_2^{n+1} e_1 + \cdots + w_2^{n+7} e_7),$$

so

$$O_{V,n} = \frac{P\alpha^* \alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q w_1^* w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R w_2^* w_2^n}{(\alpha - w_2)(w_1 - w_2)}.$$

Theorem 2.2 [16] (Generating Function) The generating function for $O_{V,n}$ is

$$\sum_{n=0}^{\infty} O_{V,n} x^n = \frac{\left\{ \begin{array}{c} O_{V,0} + [O_{V,1} - rO_{V,0}]x \\ +[O_{V,2} - rO_{V,1} - sO_{V,0}]x^2 \end{array} \right\}}{1 - rx - sx^2 - tx^3}.$$

Proof. To compute generating function $O_{V,n}$ $\sum_{n=0}^{\infty} O_{V,n} x^n$

$$= O_{V,0} + O_{V,1} x + O_{V,2} x^2 + \cdots + O_{V,n} x^n + \cdots$$

then using the equations of $-rx \sum_{n=0}^{\infty} O_{V,n} x^n$, $-sx^2 \sum_{n=0}^{\infty} O_{V,n} x^n x^n$ and $-tx^3 \sum_{n=0}^{\infty} O_{V,n} x^n$,

$$\left\{ \begin{array}{c} \sum_{n=0}^{\infty} O_{V,n} x^n - rx \sum_{n=0}^{\infty} O_{V,n} x^n \\ -sx^2 \sum_{n=0}^{\infty} O_{V,n} x^n x^n - tx^3 \sum_{n=0}^{\infty} O_{V,n} x^n \end{array} \right\}$$

$$= O_{V,0} + (O_{V,1} - rO_{V,0})x + (O_{V,2} - rO_{V,1} - sO_{V,0})x^2$$

$+(O_{V,3} - rO_{V,2} - sO_{V,1} - tO_{V,0})x^3$

$+ \cdots + (O_{V,n} - rO_{V,n-1} - sO_{V,n-2} - tO_{V,n-3})x^n$

$+ \cdots$

So,

$$\sum_{n=0}^{\infty} O_{V,n}x^n(1 - rx - sx^2 - tx^3)$$

$$= O_{V,0} + [O_{V,1} - rO_{V,0}]x + [O_{V,2} - rO_{V,1} - sO_{V,0}]x^2$$

we get the result.

Then we can give the following theorem relative to summation formulas.

Theorem 2.3 The sum of the first n −terms of the octonion sequence $O_{V,n}$ is given by;

$$\sum_{l=0}^{n} O_{V,l} = \frac{\begin{Bmatrix}(r+s-1)O_{V,0} + (r-1)O_{V,1} - O_{V,2} \\ +tO_{V,n-2} + (s+t)O_{V,n-1} + (r+s+t)O_{V,n}\end{Bmatrix}}{r+s+t-1}$$

Proof. Note that, applying (2.1), we deduce that

$$n = 3 \Rightarrow O_{V,3} = rO_{V,2} + sO_{V,1} + tO_{V,0}$$

$$n = 4 \Rightarrow O_{V,4} = rO_{V,3} + sO_{V,2} + tO_{V,1}$$

$$\cdots \qquad (2.2)$$

$$n = n-1 \Rightarrow O_{V,n-1} = rO_{V,n-2} + sO_{V,n-3} + tO_{V,n-4}$$

$$n = n \Rightarrow O_{V,n} = rO_{V,n-1} + sO_{V,n-2} + tO_{V,n-3}.$$

If we sum of both sides of (2.2), then we obtain

$$O_{V,3} + \cdots + O_{V,n}$$

$$= r\sum_{l=0}^{n} O_{V,l} + s\sum_{l=0}^{n} O_{V,l} + t\sum_{l=0}^{n} O_{V,l}. \quad (2.3)$$

If we make necessary regulations, (2.3) becomes

$$(r+s+t-1)\sum_{l=0}^{n} O_{V,l}$$

$$= \begin{Bmatrix}(r+s-1)O_{V,0} + (r-1)O_{V,1} \\ +tO_{V,n-2} + (s+t)O_{V,n-1} + (r+s+t)O_{V,n}\end{Bmatrix}$$

as we claimed.

We formulate the norm value for the generalized tribonacci octonions.

Theorem 2.4 The norm value for generalized tribonacci octonions $O_{h,n}(x)$ is given by

$$Nr^2(O_{V,n}) = P^2\alpha^{2n}(w_1 - w_2)^2\underline{\alpha} + Q^2w_1^{2n}(\alpha - w_2)^2\underline{w_1} + R^2w_2^{2n}(\alpha - w_1)^2\underline{w_2} + 2RP(w_1 - w_2)(\alpha - w_1)(\alpha w_2)^n\underline{\alpha w_2} - 2PQ(\alpha w_1)^n(w_1 - w_2)(\alpha - w_2)\underline{\alpha w_1} - 2RQ(\alpha - w_1)(\alpha - w_2)(w_1w_2)^n\underline{w_1w_2}$$

where

$$\underline{\alpha} = 1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^{10} + \alpha^{12} + \alpha^{14}$$

$$\underline{w_1} = 1 + w_1^2 + w_1^4 + w_1^6 + w_1^8 + w_1^{10} + w_1^{12} + w_1^{14}$$

$$\underline{w_2} = 1 + w_2^2 + w_2^4 + w_2^6 + w_2^8 + w_2^{10} + w_2^{12} + w_2^{14}$$

$$\underline{\alpha w_1} = 1 + \alpha w_1 + (\alpha w_1)^2 + (\alpha w_1)^3 + (\alpha w_1)^4 + (\alpha w_1)^5 + (\alpha w_1)^6 + (\alpha w_1)^7$$

$$\underline{\alpha w_2} = 1 + \alpha w_2 + (\alpha w_2)^2 + (\alpha w_2)^3 + (\alpha w_2)^4 + (\alpha w_2)^5 + (\alpha w_2)^6 + (\alpha w_2)^7$$

$$\underline{w_1w_1} = \begin{Bmatrix}1 + w_1w_2 + (w_1w_2)^2 + (w_1w_2)^3 \\ +(w_1w_2)^4 + (w_1w_2)^5 \\ +(w_1w_2)^6 + (w_1w_2)^7\end{Bmatrix}.$$

Proof. Note that by the norm definition,

$$Nr^2(O_{V,n}) = \sum_{l=0}^{7} V_{n+l}^2$$

also

$$V_n = \frac{P\alpha^n(w_1 - w_2) - Qw_1^n(\alpha - w_2) + Rw_2^n(\alpha - w_1)}{(\alpha - w_1)(w_1 - w_2)(\alpha - w_2)}$$

where

$\delta = (\alpha - w_1)(w_1 - w_2)(\alpha - w_2)$.

Then

$$\delta^2 V_n^2$$
$$= P^2\alpha^{2n}(w_1 - w_2)^2 - 2PQ\alpha^n w_1^n(w_1 - w_2)(\alpha - w_2)$$
$$+Q^2 w_1^{2n}(\alpha - w_2)^2 + 2RP\alpha^n(w_1 - w_2)w_2^n(\alpha - w_1)$$
$$-2RQw_1^n w_2^n(\alpha - w_2)(\alpha - w_1) + R^2 w_2^{2n}(\alpha - w_1)^2$$

so

$$\delta^2 Nr^2(O_{V,n})$$
$$= P^2\alpha^{2n}(w_1 - w_2)^2 - 2PQ\alpha^n w_1^n(w_1 - w_2)$$
$$(\alpha - w_2) + Q^2 w_1^{2n}(\alpha - w_2)^2 + 2RP\alpha^n$$
$$(w_1 - w_2)w_2^n(\alpha - w_1) - 2RQw_1^n w_2^n(\alpha - w_2)(\alpha - w_1)$$
$$+R^2 w_2^{2n}(\alpha - w_1)^2 + \cdots + P^2\alpha^{2n+14}(w_1 - w_2)^2$$
$$+Q^2 w_1^{2n+14}(\alpha - w_2)^2 + R^2 w_2^{2n+14}(\alpha - w_1)^2$$
$$+2RP\alpha^{n+7}(w_1 - w_2)w_2^{n+7}(\alpha - w_1) - 2PQ\alpha^{n+7}w_1^{n+7}$$
$$(w_1 - w_2)(\alpha - w_2) - 2RQw_1^{n+7}w_2^{n+7}(\alpha - w_2)(\alpha - w_1).$$

Moreover, we done extra calculations so the result is clear.

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