On Mochizuki-Trooshin Theorem for Sturm-Liouville Operators

İbrahim ADALAR

Sivas Cumhuriyet University Zara Veysel Dursun Colleges of Applied Sciences Zara/Sivas, TURKEY

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Abstract. In this paper, the inverse spectral problems of Sturm-Liouville operators are considered. Some new uniqueness theorems and analogies of the Mochizuki-Trooshin Theorem are proved.

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1. INTRODUCTION

We consider the classical Sturm-Liouville problem $L = L(q(x), h, H)$

$$-y'' + q(x)y = \lambda y$$

(1)

$$y'(0) - hy(0) = 0$$

(2)

$$y'(1) + Hy(1) = 0$$

(3)

where $h, H \in \mathbb{R}$, $\lambda$ is a spectral parameter and $q(x) \in L_1(0,1)$. The spectrum of such problems consists of countable many real eigenvalues, which have no finite limit point.

The inverse spectral problem for $L$ is to determine the potential function $q(x)$ from some given data. The first result on this area is given by Ambarzumian [1]. Borg [2] showed that generally a single spectrum is insufficient to determine the potential. Levinson [9] showed that if the potential $q(x)$ is symmetric, $q(x) = q(1-x)$, then it is determined uniquely by a single spectrum. Later Gelfand and Levitan [3] proved that the eigenvalues and normalizing coefficients uniquely determine the potential $q(x)$. Hochstadt and Lieberman [7] proved that a single spectra and the potential on the interval $[1/2,1]$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$. 

* Corresponding author. Email address: iadalar@cumhuriyet.edu.tr
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In 2001, Mochizuki and Trooshin [5] proved a uniqueness theorem for interior spectral data of the Sturm-Liouville operator. They used similar techniques in [7]. This kind of problems for the Sturm-Liouville operator were formulated and studied in [12-19].

Together with $L$, we consider a boundary value problem $	ilde{L} = L(q(x), h, H)$ of the same form but with a different coefficient $q$. We agree that if a certain symbol $s$ denotes an object related to $L$, then $\tilde{s}$ will denote an analogous object related to $\tilde{L}$. The eigenvalues and the corresponding eigenfunctions of the problem $L$ are denoted by $\lambda_n$ and $\varphi_n(x) = \varphi(x, \lambda_n)$, respectively.

The statement of Mochizuki and Trooshin theorem is as following:

**Theorem 1.1.** [5] If for every $n = 0, 1, 2, \ldots$ we have

$$
\lambda_n = \tilde{\lambda}_n, \quad \frac{\varphi'_n(1/2)}{\varphi_n(1/2)} = \frac{\tilde{\varphi}'_n(1/2)}{\tilde{\varphi}_n(1/2)}
$$

then $q(x) = \tilde{q}(x)$ almost everywhere on $[0, 1]$.

The purpose of the present study is to prove some analogies of this theorem and new uniqueness theorems for inverse Sturm-Liouville problems.

In the second section, we give some preliminaries. Section 3 contains new uniqueness theorems and alternative proofs for Mochizuki-Trooshin theorem and Levinson’s theorem.

**2. PRELIMINARIES**

We shall first mention some known results which will be needed later. Let $\varphi(x, \lambda)$ be the solution of equation (1) satisfying the initial conditions,

$$
\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h.
$$

We need specifically to focus on the properties of $\varphi(1/2, \lambda)$. It is known that, [4,8,17,18] for each $x \in [0, 1]$, $\varphi(x, \lambda)$ and $\varphi'(x, \lambda)$ are entire functions of $\lambda$ and there exist some constants $c_1, c_2 > 0$ such that $\varphi(1/2, \lambda)$ and $\varphi'(1/2, \lambda)$ are all bounded by $c_1 \exp(c_2 |\lambda|^{1/2})$. For $|\lambda| \to \infty$ uniformly with respect to $x \in [0, 1],

$$
\varphi(x, \lambda) = \cos \rho x + O\left(\frac{\exp \tau x}{\rho}\right) \quad \varphi'(x, \lambda) = -\rho \sin \rho x + O(\exp \tau x).
$$

Here $\rho = \sqrt{\lambda}$ and $\tau = |\text{Im} \rho|$. The function

$$
\omega(\lambda) = \varphi'(1, \lambda) + H\varphi(1, \lambda)
$$

is entire in $\lambda$ and it has an at most countable set of zeros, $\{\lambda_n\}$. Denote

$$
G_\delta = \{\rho : |\rho - k\pi| \geq \delta, k = 0, \pm1, \pm2, \ldots\}, \delta > 0.
$$

We have that [8]
\[|\omega(\lambda)| \geq C_\delta |\rho| \exp \tau \] 

(7)

for \( \rho \in G_\delta, \ |\rho| \geq \rho^* \) and sufficiently large \( \rho^* \). The Weyl \( m_- \) function is defined by:

\[ m_-(a, \lambda) = -\frac{\varphi(a, \lambda)}{\varphi'(a, \lambda)} \]

where \( a \in [0, 1] \). The following Marchenko’s uniqueness theorem [6] is also necessary for our analysis.

**Theorem 2.1.** [6] The Weyl \( m_-(a, \lambda) \) function uniquely determines \( h \) as well as \( q(x) \) almost everywhere on \([0, a]\).

### 3. UNIQUENESS THEOREMS

Here we provide an alternative proof for Mochizuki and Trooshin theorem.

**Proof of the Theorem 1.1.** Consider the initial-value problems:

\[-\varphi'' + q(x)\varphi = \lambda \varphi \]
\[\varphi(0) = 1, \varphi'(0) = h \]  

(8)

and

\[-\tilde{\varphi}'' + \tilde{q}(x)\tilde{\varphi} = \lambda \tilde{\varphi} \]
\[\tilde{\varphi}(0) = 1, \tilde{\varphi}'(0) = h. \]  

(9)

The functions \( \varphi(x, \lambda) \) and \( \varphi'(x, \lambda) \) satisfy

\[\tilde{\varphi}(0, \lambda)\varphi'(0, \lambda) - \varphi(0, \lambda)\tilde{\varphi}'(0, \lambda) = 0. \]

Multiplying (8) by \( \tilde{\varphi}(x, \lambda) \) and (9) by \( \varphi(x, \lambda) \), subtracting, and integrating from 0 to 1/2, we obtain

\[ f(\lambda) = \int_0^{1/2} \left( q(x) - \tilde{q}(x) \right) \varphi(x, \lambda)\tilde{\varphi}(x, \lambda) dx = \tilde{\varphi}(1/2, \lambda)\varphi'(1/2, \lambda) - \varphi(1/2, \lambda)\tilde{\varphi}'(1/2, \lambda). \]  

(10)

The conditions of the theorem imply

\[ f(\lambda_0) = 0. \]

Define \( h(\lambda) = \frac{f(\lambda)}{\omega(\lambda)} \), which is an entire function. From the asymptotics (6) and (7) for \( f(\lambda) \) and \( \omega(\lambda) \), we see that

\[ h(\lambda) = O\left( \frac{1}{|\rho|} \right) \]

for large \( |\rho| \). Thus, by Liouville’s theorem, we obtain for all \( \lambda \),
\[ h(\lambda) = 0 \]

or

\[ f(\lambda) = 0. \]

From (10), we have that

\[ \frac{\varphi(1/2, \lambda)}{\varphi'(1/2, \lambda)} = \frac{\tilde{\varphi}(1/2, \lambda)}{\tilde{\varphi}'(1/2, \lambda)} \]

and hence

\[ m(1/2, \lambda) = \tilde{m}(1/2, \lambda). \]

By Theorem 2.1, we prove \( q(x) = \tilde{q}(x) \) almost everywhere on \([0, 1/2]\).

To prove that \( q(x) = \tilde{q}(x) \) almost everywhere on \([1/2, 1]\), we will consider the supplementary problem \( L : \)

\[ -y'' + q(1-x)y = \lambda y \]

\[ y'(0) - Hy(0) = 0 \]

\[ y'(1) + hy(1) = 0. \]

Since \( \varphi_n(1-x) = \tilde{\varphi}_n(x) \), the assumption conditions in Theorem 1.1 are still satisfied. If we repeat the above arguments then this yields \( q(1-x) = \tilde{q}(1-x) \) on \([0, 1/2]\), that is \( q(x) = \tilde{q}(x) \) almost everywhere on \([1/2, 1]\). This completes the proof. \( \square \)

By the remark to proof of Theorem 1, we have proved the following result:

**Corollary 3.1.** Let \( f(\lambda) = 0 \) for all \( \lambda \). If for every \( n = 0, 1, 2, \ldots \) we have

\[ \lambda_n = \tilde{\lambda}_n, \]

then \( q(x) = \tilde{q}(x) \) almost everywhere on \([0, 1]\).

Let \( L_0 : \)

\[ -y'' + q(x)y = \lambda y \]

\[ y'(0) - hy(0) = 0 \]

\[ y'(1) + hy(1) = 0. \]

Here we provide an alternative proof for the following Levinson's theorem [9].

**Theorem 3.2.** [9] If \( q(x) = q(1-x) \) then the function \( q(x) \) and \( h \) are uniquely determined by the spectrum of problem \( L_0. \)
Proof. Applying the same arguments as that in the proof of Theorem 1.1, we can see that
\[
f(\lambda) = 2 \int_0^{1/2} (q(x) - \tilde{q}(x))\phi(x, \lambda)\phi(x, \lambda)dx = 0
\]
and hence
\[
f(\lambda_n) = 2 \int_0^{1/2} (q(x) - \tilde{q}(x))\phi(x, \lambda_n)\phi(x, \lambda_n)dx = 0.
\]
We obtain for all \(\lambda_n\),
\[
f(\lambda) = \tilde{\varphi}(1/2, \lambda)\varphi'(1/2, \lambda) - \varphi(1/2, \lambda)\tilde{\varphi}'(1/2, \lambda) = 0
\]
Thus we arrive at
\[
m_n(1/2, \lambda) = m_n(1/2, \lambda).
\]
By Theorem 2.1, the proof is complete. \(\square\)

Let us consider the following Sturm-Liouville problems
\[
-y'' + q(x)y = \lambda y
\]
\[
y(0) = y(1/2) = 0
\]
\[
y(0) = y'(1/2) = 0.
\]

Let \(\{\mu_n\}_{n=0}^{\infty}\) and \(\{\nu_n\}_{n=0}^{\infty}\) be the spectra of the problems (11), (12) and (11), (13), respectively. Consider the problem: given three spectra \(\{\lambda_n\}_{n=0}^{\infty}\), \(\{\mu_n\}_{n=0}^{\infty}\) and \(\{\nu_n\}_{n=0}^{\infty}\) determine \(q(x)\). Knowledge of \(\{\mu_n\}_{n=0}^{\infty}\) and \(\{\nu_n\}_{n=0}^{\infty}\) is equivalent to the knowledge of \(q(x)\) on \([0,1/2]\). Thus this problem is the Hochstadt-Lieberman problem in [7]. Now consider the problem: given \(\{\lambda_n\}_{n=0}^{\infty}\subset\{\nu_n\}_{n=0}^{\infty} \cup \{\mu_n\}_{n=0}^{\infty}\) determine \(q(x)\). In this case, only spectra \(\{\lambda_n\}_{n=0}^{\infty}\) uniquely determine the potential \(q(x)\) on the whole \([0,1]\). We can give the following uniqueness theorem.

**Theorem 3.3.** Let \(\{\lambda_n\}_{n=0}^{\infty}\subset\{\nu_n\}_{n=0}^{\infty} \cup \{\mu_n\}_{n=0}^{\infty}\) and \(\{\lambda_n\}_{n=0}^{\infty}\subset\{\nu_n\}_{n=0}^{\infty} \cup \{\mu_n\}_{n=0}^{\infty}\). If for every \(n = 0,1,\ldots\) we have
\[
\lambda_n = \lambda_n,
\]
then \(q(x) = \tilde{q}(x)\) almost everywhere on \([0,1]\).

**Proof.** As in the proof of Theorem 1.1, we can show that
\[
f(\lambda) = \int_0^{1/2} (q(x) - \tilde{q}(x))\phi(x, \lambda)\phi(x, \lambda)dx = \tilde{\varphi}(1/2, \lambda)\phi'(1/2, \lambda) - \varphi(1/2, \lambda)\tilde{\varphi}'(1/2, \lambda).
\]
To prove, as in the Corollary 3.1, it suffices to show that \(f(\lambda) = 0\) for all \(\lambda\). The assumptions of the theorem imply that
\[
\varphi_n(1/2, \lambda_n) = 0 \text{ or } \varphi_n'(1/2, \lambda_n) = 0 \text{ and } \tilde{\varphi}_n(1/2, \lambda_n) = 0 \text{ or } \tilde{\varphi}_n'(1/2, \lambda_n) = 0.
\]
Hence, we have
\[ f(\lambda_n) = 0. \]
Thus, repeating the proof Theorem 1.1, we arrive at
\[ f(\lambda) = 0, \]
which implies that
\[ m_\sim(1/2, \lambda) = \sim m_\sim(1/2, \lambda) \]
and \( q(x) = \tilde{q}(x) \) almost everywhere on \([0, 1/2]\). The supplementary problem \( L_\sim \) in proof of Theorem 1.1 completes the proof. \( \square \)

Let us define
\[
g(\rho) = \int_0^{1/2} (q(x) - \tilde{q}(x)) \rho(x, \lambda) \tilde{\phi}(x, \lambda) dx = \tilde{\phi}(1/2, \lambda) \phi'(1/2, \lambda) - \phi(1/2, \lambda) \phi'(1/2, \lambda) \quad (14)\]
where \( \rho = \sqrt{\lambda} \). The asymptotics (6) imply that the entire function \( g(\rho) \) is a function of exponential type \( \leq 1 \). As shown by the above discussion, let \( g(\rho) = 0 \) then only spectra \( \{\lambda_n\}_{n=0}^\infty \) uniquely determine the potential \( q(x) \) on \([0,1]\). We now consider the problem: If the zeros of an entire function of exponential type are known to include a given sequence of positive real numbers what can be said about growth of the function. The first result of this type is given by Carlson's Theorem. This theorem [11, p.153] says, if \( g \) is entire function of exponential type \( < \pi \) and vanishes on the positive integers then \( g \) vanishes everywhere. This idea has been further developed by Rubel [10, p.422]:

**Theorem 3.4. [10]** Let \( \rho = t + i\tau \) and \( \Omega = \{\rho_n : \rho_{n+1} - \rho_n \geq \gamma > 0, \rho_n > 0, n \in \mathbb{Z}^+\} \). In order to each entire function \( g(\rho) \) satisfying
\[
g(\rho) = O(1) \exp(a|\rho|), \quad a < \infty \quad (15)
g(i\tau) = O(1) \exp(b|\tau|), \quad b < \delta \quad (16)
g(\rho_n) = 0 \quad (17)
\]
vanish identically, it is sufficient that
\[
\inf_{\rho > 1} \lim_{k \to \infty} \sup_{\rho_k \leq \rho} \sum_{\rho_k, \rho_p} \frac{1}{\rho_n} = L(\Omega) \geq \frac{\delta}{\pi}. \quad (18)
\]
Here, \( L(\Omega) \) is the logarithmic block density of \( \Omega \).

We turn repeat that equation (14). From asymptotics (6), the entire function
\[
g(\rho) = \int_0^{1/2} (q(x) - \tilde{q}(x)) \rho(x, \lambda) \tilde{\phi}(x, \lambda) dx
\]
satisfies (15) and (16). Also we have that
\[ \rho_{n+1} - \rho_n > 0 \]
where \( \sqrt{\lambda_n} = \rho_n \). In this case, we can give a uniqueness theorem by using Theorem 3.4.

**Theorem 3.5.** Let \( \Lambda \subset \mathbb{N} \cup \{0\} \) be a subset of nonnegative integer numbers and let \( \Omega := \{ \lambda_n \}_{n \in \Lambda} \) be a part of the spectrum of \( L \) such that the numbers \( \sqrt{\lambda_n} = \rho_n \) satisfy (18) for function \( g(\rho) \). If for \( n \in \Lambda \), we have

\[ \lambda_n = \tilde{\lambda}_n, \quad \frac{\varphi'_n(1/2)}{\varphi_n(1/2)} = \frac{\tilde{\varphi}'_n(1/2)}{\tilde{\varphi}_n(1/2)} \]

then \( q(x) = \tilde{q}(x) \) almost everywhere on \([0,1]\).

**Proof.** As in the proof of Theorem 1, we obtain

\[ g(\rho) = \int_0^{1/2} (q(x) - \tilde{q}(x)) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda) dx = \varphi(1/2, \lambda) \varphi'(1/2, \lambda) - \varphi(1/2, \lambda) \tilde{\varphi}'(1/2, \lambda). \]

The assumptions of the theorem imply

\[ g(\rho_n) = 0, \quad n \in \Lambda. \]

By the Theorem 3.4, we have that

\[ g(\rho) = 0 \]

on the whole \( \rho \)-plane. Thus, \( \varphi(x, \lambda) \) and \( \tilde{\varphi}(x, \lambda) \) satisfy

\[ \tilde{\varphi}(1/2, \lambda) \varphi'(1/2, \lambda) - \varphi(1/2, \lambda) \tilde{\varphi}'(1/2, \lambda) = 0 \]

and hence

\[ m_-(1/2, \lambda) = \tilde{m}_-(1/2, \lambda). \]

By the Theorem 2.1, we prove \( q(x) = \tilde{q}(x) \) almost everywhere on \([0,1/2]\). Repeating the supplementary problem in the last part of proof of Theorem 1.1, we can show that \( g(\rho) = 0 \) on the whole \( \rho \)-plane, which implies that \( q(x) = \tilde{q}(x) \) on \([1/2,1]\) and consequently, \( q(x) = \tilde{q}(x) \) almost everywhere on \([0,1]\). This completes the proof.

Let us consider the Sturm-Liouville problem \( L \) for \( q(x) \in L_2(0,1) \). Horvath [15, 19, p.268] proved Hochstadt-Lieberman type an uniqueness theorem by using simple closedness properties of the exponential system corresponding to the known eigenvalues. We can give the following uniqueness theorem with same arguments in [15] for Mochizuki-Trooshin type theorem.
Theorem 3.6. Let $\Lambda \subset \mathbb{N} \cup \{0\}$ be a subset of nonnegative integer numbers and let $\Omega := \{\lambda_n\}_{n \in \Lambda}$ be a part of the spectrum of $L$ such that the system of functions $\{\cos 2\rho_n x\}_{n \in \Lambda}$ is complete in $L^2(0,1/2)$. If for $n \in \Lambda$, we have

$$\lambda_n = \bar{\lambda}_n, \quad \frac{\varphi'_n(1/2)}{\varphi_n(1/2)} = \frac{\varphi'_n(1/2)}{\varphi_n(1/2)}$$

then $q(x) = \tilde{q}(x)$ almost everywhere on $[0,1]$.

Proof. As in the proof of Theorem 1, we can show that

$$f(\lambda) = \int_0^{1/2} \left( q(x) - \tilde{q}(x) \right) p(x,\lambda) \tilde{p}(x,\lambda) dx = \varphi(1/2,\lambda) \varphi'(1/2,\lambda) - \varphi(1/2,\lambda) \tilde{\varphi}'(1/2,\lambda) .$$

Hence, we have that

$$f(\lambda_n) = 0, \quad n \in \Lambda. \quad (19)$$

The following representation holds [4,6,8]

$$\varphi(x,\lambda) = \cos \rho x + \int_0^\lambda K(x,t) \cos \rho dt$$

where $K(x,t)$ is a continuous function which does not depend on $\lambda$. Hence,

$$\varphi(x,\lambda) \tilde{\varphi}(x,\lambda) = \frac{1}{2} \left( 1 + \cos 2\rho x + \int_0^\lambda K_1(x,t) \cos \rho dt \right) \quad (20)$$

where $K_1(x,t)$ is a continuous function which does not depend on $\lambda$. From (19) and (20), we have

$$\int_0^{1/2} \left[ \phi(x) + \int_0^{1/2} K_1(x,t) \phi(t) dt \right] \cos 2\rho_n x dx + \int_0^{1/2} \phi(x) dx = 0, \quad n \in \Lambda,$$

where $\phi(x) = q(x) - \tilde{q}(x)$. By the Riemann-Lebesgue lemma,

$$\int_0^{1/2} \phi(x) dx = 0.$$

By the completeness of the functions $\{\cos 2\rho_n x\}_{n \in \Lambda}$, we have

$$\phi(x) + \int_0^{1/2} K_1(x,t) \phi(t) dt = 0.$$
Since this homogeneous integral equation has only the trivial solution it follows that and $q(x) = \tilde{q}(x)$ almost everywhere on $[0, 1/2]$. The supplementary problem $\tilde{L}$ in proof of Theorem 1.1 completes the proof. □

REFERENCES