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Some applications of Cohen-Macaulay injective dimension

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Abstract

Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R, M a finitely generated R-module with finite projective dimension and N an arbitrary R-module with finite Cohen-Macaulay injective dimension. In this paper, we show that the generalized local cohomology $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is zero for every i larger than the Cohen-Macaulay injective dimension of N. As applications, we obtain new characterizations of Gorenstein and regular local rings.

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Local cohomology is an important tool in algebraic geometry and in commutative algebra. A generalization of local cohomoloy was first introduced in the local case by Herzog in his habilitation [14] and then continued by many authors. One basic theme of local cohomology theory is investigating vanishing and nonvanishing properties of (generalized) local cohomology (e.g. [1, 3, 5, 14, 21, 25, 26, 29]).

Sazeedeh in [25, Theorem 3.1] proved that if N is a Gorenstein injective R-module over a Gorenstein ring R, then $\mathrm{H}^i_{\mathfrak{a}}(N) = 0$ for all $i > 0$. Also, by using the notion of strongly cotorsion modules, he in [26, Corollary 3.6] improved this result by showing that if R is a commutative Noetherian ring with finite Krull dimension, M is a finitely generated R-module with finite projective dimension and N is an R-module of finite Gorenstein injective dimension t, then the generalizing local cohomology $\mathrm{H}^i_{\mathfrak{a}}(M, N) = 0$ for all $i > t$. Recently, by employing the tool of spectral sequences, K. Divaani-Aazar and A. Hajikarimi in [5, Lemma 2.9] have given a similar result without the assumption that

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R has finite Krull dimension. Motivated by these results, we provide a new vanishing result for generalized local cohomology for modules with finite Cohen-Macaulay injective dimension and find some applications of this result.

Throughout this article, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and C is a fixed semidualizing R-module (see Definition 2.1). For unexplained concepts and notations, we refer the reader to [4] and [22]. For two R-modules M and N, the *i-th generalized local cohomology* of M and N with respect to $\mathfrak{a}$ is defined by $\mathrm{H}^i_{\mathfrak{a}}(M,N) = \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$. Clearly, when $M = R$ we get the usual local cohomology functor $\mathrm{H}^i_{\mathfrak{a}}(-)$. It should be noted that if M is finitely generated and $N \to \mathbf{E}$ is an injective resolution of N, then the i-th generalized local cohomology module of M and N with respect to $\mathfrak{a}$ is the i-th cohomology module of the complex $\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(\mathbf{E}))$, where $\Gamma_{\mathfrak{a}}(-) = \mathrm{H}^0_{\mathfrak{a}}(-)$ denotes the $\mathfrak{a}$-torsion functor. M is $\mathfrak{a}$-*torsion* precisely when $\Gamma_{\mathfrak{a}}(M) = M$, that is, if and only if each element of M is annihilated by some power of $\mathfrak{a}$.

Let $\mathcal{X}$ be a class of R-modules and M an R-module. An $\mathcal{X}$-*coresolution* of M is an exact sequence of the form $\mathbf{X}^+ = 0 \to M \to X^0 \to \cdots \to X^{n-1} \to X^n \to \cdots$, where X^i is in $\mathcal{X}$ for $i \geqslant 0$. Furthermore, we call this exact sequence a *coproper* $\mathcal{X}$-*coresolution* of M if $\mathrm{Hom}_R(\mathbf{X}^+, X)$ is exact for all $X \in \mathcal{X}$. The $\mathcal{X}$-*injective dimension* of M is defined as $\mathcal{X}\text{-id}_R M = \inf\{\sup\{n \mid X^n \neq 0\} \mid X^+ \text{ is an } \mathcal{X}\text{-coresolution of } M\}$. For each positive integer i, we denote $\mathcal{X}^{\perp_i} := \{M \mid \mathrm{Ext}^i_R(X,M) = 0 \text{ for any } X \in \mathcal{X}\}$, $\mathcal{X}^{\perp} := \cap_{i>0}\mathcal{X}^{\perp_i}$.

The structure of the paper is summarized below. In Section 2, we mainly present a vanishing theorem for generalized local cohomology (see Theorem 2.7). Section 3 consists of three applications. One of them, Theorem 3.1, states that a local ring R having a dualizing module is regular if and only if $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for every finitely generated R-module M and every $i > \mathrm{CMid}_R N$ if and only if any $\mathfrak{a}$-torsion R-module has finite injective dimension. The second, Corollary 3.3, shows that a local ring R is Gorenstein if and only if its residue field k has copure injective dimension at most $\dim R$ if and only if every $\mathfrak{m}$-torsion module has a coproper Gorenstein injective coresolution of length $\dim R$ in which each term is $\mathfrak{m}$-torsion. The third, Theorem 3.5, claims that a local ring R is Cohen-Macaulay provided that there exists a non-zero cofinite R-module N with $\mathrm{CMid}_R N$ finite and $\dim_R N = \dim R$.

1. Vanishing of generalized local cohomology

We begin with the notion of a semidualizing module, which is a common generalization of a dualizing module and a free module of rank one. The following definitions are taken from [18].

1.1. Definition. A finitely generated R-module C is *semidualzing* if the natural homothety homomorphism $R \to \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}^{\geqslant 1}_R(C,C) = 0$. Furthermore, C is *dualizing* if it has finite injective dimension. We set $\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C,I) \mid I \text{ is injective}\}$. Modules in $\mathcal{I}_C(R)$ are called C-*injective*.

Next, we recall from [16] the definition of a C-Gorenstein injective R-module.

1.2. Definition. An R-module M is called *C-Gorenstein injective* if:

(I1) $\mathrm{Ext}_R^{\geqslant 1}(\mathrm{Hom}_R(C,I),M)=0$ for all injective R-modules I.

(I2) There exist injective R-modules $I_0, I_1, \cdots$ together with an exact sequence:

$$\cdots \to \mathrm{Hom}_R(C,I_1) \to \mathrm{Hom}_R(C,I_0) \to M \to 0,$$

and also, this sequence stays exact when we apply to it $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J),-)$ for any injective R-module J.

1.3. Remark. Injective and C-injective modules are C-Gorenstein injective. The R-Gorenstein injective modules are just Gorenstein injective modules defined by E. E. Enochs and O. M. G. Jenda in [7]. We write $\mathcal{GI}_C(R)$ for the class of all C-Gorenstein injective modules. For convenience, we set $\mathcal{GI}_C\text{-id}_R M = \mathcal{GI}_C(R)\text{-id}_R M$, and $\mathrm{Gid}_R M = \mathcal{GI}_R\text{-id}_R M$ if $C=R$. Following [15], the *Cohen-Macaulay injective dimension* of an R-module M is defined as $\mathrm{CMid}_R M = \inf\{\mathrm{Gid}_{R\ltimes C} M \mid C$ is a semidualizing R-module$\}$, where $R \ltimes C$ denotes the trivial extension of R by C (see [16, Definition 1.2]). Holm and Jrgensen in [16, Theorem 2.16] proved that $\mathcal{GI}_C\text{-id}_R M = \mathrm{Gid}_{R\ltimes C} M$ for any R-module M. So $\mathrm{CMid}_R M \leqslant \mathrm{Gid}_R M$. However, the inequality may be strict (see Example 3.6 below).

1.4. Definition. The *Auslander class* $\mathcal{A}_C(R)$ with respect to a semidualizing module C consists of all R-modules M satisfying

(A1) $\mathrm{Tor}^R_{\geqslant 1}(C,M)=0$,

(A2) $\mathrm{Ext}_R^{\geqslant 1}(C, C\otimes_R M)=0$, and

(A3) The natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_R(C, C\otimes_R M)$ is an isomorphism.

1.5. Lemma. Let U be an C-injective R-module. Then $\Gamma_{\mathfrak{a}}(U)$ is C-injective and $\mathrm{H}^i_{\mathfrak{a}}(U) = 0$ for all $i>0$.

Proof. Since U is C-injective, $U=\mathrm{Hom}_R(C,E)$ for some injective R-module E by definition. Then we have

$$\begin{aligned}
\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(C,E)) &= \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n, \mathrm{Hom}_R(C,E))\\
&\cong \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n \otimes_R C, E)\\
&\cong \varinjlim_n \mathrm{Hom}_R(C, \mathrm{Hom}_R(R/\mathfrak{a}^n, E))\\
&\cong \mathrm{Hom}_R(C, \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n, E))\\
&\cong \mathrm{Hom}_R(C, \Gamma_{\mathfrak{a}}(E)).
\end{aligned}$$

Note that the module $\Gamma_{\mathfrak{a}}(E)$ remains injective by [3, Proposition 2.1.4]. So $\Gamma_{\mathfrak{a}}(U)$ is C-injective. The second claim is evident from [1, Lemma 5.9]. □

The key to the proof of Theorem 2.7 below is given in the following proposition.

1.6. Proposition. Let M be a finitely generated R-module in $\mathcal{A}_C(R)$ and U a C-injective R-module. Then $\mathrm{H}^i_{\mathfrak{a}}(M,U)=0$ for all $i>0$.

Proof. Let $0 \to U \to E^0 \to E^1 \to \cdots$ be an injective resolution of U. Applying the functor $\Gamma_{\mathfrak{a}}(-)$ to this exact sequence, we get from [3, 1.2.2] and Lemma 2.5 the exact sequence $0 \to \Gamma_{\mathfrak{a}}(U) \to \Gamma_{\mathfrak{a}}(E^0) \to \Gamma_{\mathfrak{a}}(E^1) \to \cdots$. Also, it is an injective resolution of $\Gamma_{\mathfrak{a}}(U)$. Thus, applying the functor $\mathrm{Hom}_R(M,-)$ to this exact sequence gives $\mathrm{H}^i_{\mathfrak{a}}(M,U) \cong \mathrm{Ext}^i_R(M,\Gamma_{\mathfrak{a}}(U))$. Since M is in $\mathcal{A}_C(R)$, $\mathrm{Tor}^R_{\geqslant 1}(C,M)=0$ by definition. Therefore, the result follows from Lemma 2.5. □

We are ready to present the main result of this section.

1.7. Theorem. *Let M be a finitely generated R-module with finite projective dimension. Let N be an R-module with* $\mathrm{CMid}_R N$ *finite. Then* $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$ *for every* $i > \mathrm{CMid}_R N$.

Proof. We proceed in a similar way as in the proof of [30, Lemma 1.1]. Suppose that $\mathrm{CMid}_R N = n$. Then, by Remark 2.3, $\mathcal{GI}_C\text{-id}_R N = n$ for some semidualizing R-module C. We prove by induction on n. First assume that $n=0$. Then N is C-Gorenstein injective. By the definition of C-Gorenstein injective modules, there are short exact sequences $0 \to N_1 \to U_0 \to N \to 0$ and $0 \to N_i \to U_{i-1} \to N_{i-1} \to 0$ for all $i>1$, where U_i is C-injective and N_i is C-Gorenstein injective. By Proposition 2.6 and [18, Corollary 6.2 and Proposition 3.1], $\mathrm{H}^i_{\mathfrak{a}}(M,U)=0$ for any C-injective R-module U and any $i>0$. Hence [23, Theorem 6.26] implies that $\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M,N_1) \cong \mathrm{H}^{i+2}_{\mathfrak{a}}(M,N_2) \cong \cdots$ for all $i>0$. But by [29, Theorem 2.5] we know that $\mathrm{H}^i_{\mathfrak{a}}(X,Y)=0$ for all $i > ara(\mathfrak{a}) + \mathrm{pd}_R X$, where the arithmetic rank $ara(\mathfrak{a})$ of the ideal $\mathfrak{a}$ is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$. So $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$ for all $i>0$.

Now assume that $n>0$. By definition, one gets a short exact sequence $0 \to N \to G \to N' \to 0$ where G is C-Gorenstein injective and $\mathcal{GI}_C\text{-id}_R N' = n-1$. The inductive hypothesis gives that $\mathrm{H}^i_{\mathfrak{a}}(M,N')=0$ for $i>n-1$. Then we see from the exact sequence $\mathrm{H}^{i-1}_{\mathfrak{a}}(M,N') \to \mathrm{H}^i_{\mathfrak{a}}(M,N) \to \mathrm{H}^i_{\mathfrak{a}}(M,G)$ that $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$ for $i > \mathrm{CMid}_R N$, completing the proof. □

We then have the following immediate corollaries.

1.8. Corollary. Let M be a finitely generated R-module and G a C-Gorenstein injective R-module. Then $\mathrm{H}^i_{\mathfrak{a}}(M,G) \cong \mathrm{Ext}^i_R(M,\Gamma_{\mathfrak{a}}(G))$ for all i.

Proof. By virtue of Theorem 2.7, we have $\mathrm{H}^i_{\mathfrak{a}}(G)=0$ for any $i>0$ and any C-Gorenstein injective R-module G. Therefore, a similar proof of Proposition 2.6 gives the assertion. □

1.9. Corollary. Let M be a finitely generated R-module with finite projective dimension. If N is any R-module, then for each i, $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ can be computed by applying the functor $\mathrm{H}^0_{\mathfrak{a}}(M,-)$ to any $\mathcal{GI}_C(R)$-coresolution of N.

Proof. Observe that $\mathrm{H}^0_{\mathfrak{a}}(M,-) \cong \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(-))$ and $\mathrm{H}^i_{\mathfrak{a}}(M,G)=0$ for any $i>0$ and any C-Gorenstein injective R-module G. So the assertion follows from [12, Chapter III, Proposition 1.2A]. □

Now, we will study the effect of the 0-th generalized local cohomology $\mathrm{H}^0_{\mathfrak{a}}(-,-)$ on C-Gorenstein injective modules. In the proof of Proposition 2.11 below, we will use the following lemma.

1.10. Lemma. Assume that R admits a dualizing R-module ω. Then $\mathcal{GI}_\omega\text{-id}_R M \leqslant \text{id}_R \omega$ for all R-modules M.

Proof. This is a consequence of [8, Theorem 12.3.1], [10, Theorem 4.32] and Remark 2.3. □

1.11. Proposition. Assume that R admits a dualizing R-module ω. Let N be a ω-Gorenstein injective R-module. Then $\text{H}^0_{\mathfrak{a}}(P, N)$ is ω-Gorenstein injective for any finitely generated projective R-module P.

Proof. Suppose that $\text{id}_R\omega = n$. Since N is ω-Gorenstein injective, one has an exact sequence

$$\cdots \to U_{i+1} \to U_i \to \cdots \to U_1 \to U_0 \to N \to 0,$$

where U_i is ω-injective and $K_i = \text{Coker}(U_{i+1} \to U_i)$ is ω-Gorenstein injective for $i \geqslant 1$. Let P be a finitely generated projective R-module. By Lemma 2.5 there is an injective R-module E_i such that $\Gamma_{\mathfrak{a}}(U_i) \cong \text{Hom}_R(\omega, E_i)$, and so $\text{H}^0_{\mathfrak{a}}(P, U_i) \cong \text{Hom}_R(P, \Gamma_{\mathfrak{a}}(U_i)) \cong \text{Hom}_R(P, \text{Hom}_R(\omega, E_i)) \cong \text{Hom}_R(\omega, \text{Hom}_R(P, E_i))$. Hence, $\text{H}^0_{\mathfrak{a}}(P, U_i)$ is also ω-injective for all $i \geqslant 0$. Consider the short exact sequences $0 \to K_1 \to U_0 \to N \to 0$ and $0 \to K_i \to U_{i-1} \to K_{i-1} \to 0$ for all $i > 1$. Now applying the functor $\text{H}^0_{\mathfrak{a}}(P, -)$ to these short exact sequences and using Theorem 2.7, we have the following short exact sequences

$$0 \to \text{H}^0_{\mathfrak{a}}(P, K_1) \to \text{H}^0_{\mathfrak{a}}(P, U_0) \to \text{H}^0_{\mathfrak{a}}(P, N) \to 0,$$
$$0 \to \text{H}^0_{\mathfrak{a}}(P, K_i) \to \text{H}^0_{\mathfrak{a}}(P, U_{i-1}) \to \text{H}^0_{\mathfrak{a}}(P, K_{i-1}) \to 0.$$

Pasting these short exact sequences together, we have the following exact sequence

$$\cdots \to \text{H}^0_{\mathfrak{a}}(P, U_{i+1}) \to \text{H}^0_{\mathfrak{a}}(P, U_i) \to \cdots \to \text{H}^0_{\mathfrak{a}}(P, U_1) \to \text{H}^0_{\mathfrak{a}}(P, U_0) \to \text{H}^0_{\mathfrak{a}}(P, N) \to 0,$$

where $\text{H}^i_{\mathfrak{a}}(P, U_i)$ is ω-injective for all $i \geqslant 0$. Thus it is easily seen from the exact sequence that $\text{H}^0_{\mathfrak{a}}(P, N)$ is the n-th ω-Gorenstein injective cosyzygy of $\text{Ker}(\text{H}^0_{\alpha}(P, U_{n-1}) \to \text{H}^0_{\mathfrak{a}}(P, U_{n-2}))$. So the dual form of [28, Proposition 2.12] and Lemma 2.10 imply that $\text{H}^0_{\mathfrak{a}}(P, N)$ is ω-Gorenstein injective. □

It should be pointed that the special case where $\omega = P = R$ is appeared in [25, Theorem 3.2].

2. Applications

Let $(R, \mathfrak{m}, k)$ be a local ring. We say that R is *Cohen-Macaulay* if $\text{depth}R = \text{dim}R$. R is *Gorenstein* if it has finite self-injective dimension. R is *regular* if it has finite global dimension. In fact, every regular ring is Gorenstein, and every Gorenstein ring is Cohen-Macaulay (see [4, Proposition 3.1.20]). Based on Sazeedeh's idea in [26], we first bring a new characterization of regular local rings among Cohen-Macaulay rings with dualizing modules. The theorem below establishes a relationship between the vanishing properties of generalized local cohomology and the regularity of a local ring.

2.1. Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring with a dualizing module ω. Then the following statements are equivalent.*

1. *For every finitely generated R-module M and every R-module N with* $\mathrm{CMid}_R N$ *finite,* $\mathrm{H}^i_{\mathfrak{a}}(M, N) = 0$ *for all* $i > \mathrm{CMid}_R N$.
2. *For every ideal I and every R-module N with* $\mathrm{CMid}_R N$ *finite,* $\mathrm{H}^i_{\mathfrak{a}}(R/I, N) = 0$ *for all* $i > \mathrm{CMid}_R N$.
3. *For every ideal I and every ω-Gorenstein injective R-module G,* $\mathrm{H}^i_{\mathfrak{a}}(R/I, G) = 0$ *for all* $i > 0$.
4. *Any $\mathfrak{a}$-torsion R-module has finite injective dimension.*
5. *R is regular.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (4). Given an $\mathfrak{a}$-torsion R-module N. Since ω is dualizing, we may by Lemma 2.10 assume that $\mathcal{GI}_{\omega}\text{-id}_R N = n < \infty$. Due to [3, Corollary 2.1.6], we know that there exists an injective resolution $0 \to N \to E^0 \to E^1 \to \cdots$ of N such that E^i is $\mathfrak{a}$-torsion. Moreover, the n-th cosyzygy $\Omega^n(N)$ of N is $\mathfrak{a}$-torsion and ω-Gorenstein injective by the dual result of [28, Proposition 2.12]. Thus, by Corollary 2.8, it follows that $\mathrm{Ext}^i_R(R/I, \Omega^n(N)) \cong \mathrm{Ext}^i_R(R/I, \Gamma_{\mathfrak{a}}(\Omega^n(N)) \cong \mathrm{H}^i_{\mathfrak{a}}(R/I, \Omega^n(N)) = 0$ for any ideal I and any $i > 0$. This means that $\Omega^n(N)$ is injective. Therefore $\mathrm{id}_R N < \infty$.

(4) $\Rightarrow$ (5) follows from [20, Theorem 5.82], as the residue field k is $\mathfrak{m}$-torsion.

(5) $\Rightarrow$ (1) can be derived directly from Theorem 2.7, since any finitely generated R-module has finite projective dimension over a regular ring. □

Next, we turn to the second main result of this section. E. E. Enochs and O. M. G. Jenda in [6, Theorem 4.1] gave a characterization of Gorenstein rings by the finiteness of copure injective dimension of all modules. Sazeedeh in [26, Theorem 3.8] proved that the Gorensteiness of a local ring just depends on the finiteness of copure injective dimension of all $\mathfrak{m}$-torsion modules. By using the results obtained in Section 2, we present an even more simple criterion for a local ring to be Gorenstein.

2.2. Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following statements are equivalent.*

1. $\mathcal{I}_C(R)^{\perp}\text{-id}_R N \leqslant \dim R$ *for any $\mathfrak{m}$-torsion R-module N.*
2. $\mathcal{I}_C(R)^{\perp}\text{-id}_R k \leqslant \dim R$.
3. *For any $\mathfrak{m}$-torsion R-module N, there exists a coproper* $\mathcal{GI}_C(R)$-coresolution $0 \to N \to G^0 \to \cdots \to G^n \to 0$ with $n = \dim R$ and G^i $\mathfrak{m}$-torsion.
4. $\mathrm{id}_R C \leqslant \dim R$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (4). Suppose that $\dim R = n$, and let $0 \to k \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0$ be an $\mathcal{I}_C(R)^{\perp}$-coresolution of k, then $\mathrm{Ext}^{n+i}_R(U, k) \cong \mathrm{Ext}^i_R(U, I^n) = 0$ for all $i \geqslant 1$ and all $U \in \mathcal{I}_C(R)$, as $I^i \in \mathcal{I}_C(R)^{\perp}$. In particular, we have $\mathrm{Ext}^{n+i}_R(\mathrm{Hom}_R(C, E(k)), k) = 0$ for $i \geqslant 1$. But by [8, Corollary 3.4.4], $\mathrm{Hom}_R(C, E(k))$ is Artinian. Thus we conclude that $\mathrm{fd}_R \mathrm{Hom}_R(C, E(k)) < \infty$ by a similar argument as in [2, Corollary 5.1.2]. Since $E(k)$ is an injective cogenerator, $\mathrm{id}_R C$ is finite and hence $\mathrm{id}_R C \leqslant \dim R$ by [8, Corollary 9.2.17].

(4) ⇒ (3). Since C is dualizing, it follows from [17, Theorem B] that $\mathcal{GI}_C(R)$ is preenveloping. Let N be an $\mathfrak{m}$-torsion R-module. Then N has a monic $\mathcal{GI}_C(R)$-preenvelope $f : N \to G$. Applying the left exact functor $\Gamma_{\mathfrak{m}}(-)$ yields a monomorphism $\Gamma_{\mathfrak{m}}(f) : N \to \Gamma_{\mathfrak{m}}(G)$. Because $\Gamma_{\mathfrak{m}}(G)$ is C-Gorenstein injective by Proposition 2.11, it is straightforward to check that $\Gamma_{\mathfrak{m}}(f) : N \to \Gamma_{\mathfrak{m}}(G)$ is also a $\mathcal{GI}_C(R)$-preenvelope of N. Doing continuously in the same way, we can construct an exact sequence $0 \to N \to G^0 \to \cdots \to G^n \to 0$ with desired properties by Lemma 2.10.

(3) ⇒ (1) is trivial, as $\mathrm{Ext}_R^{i\geqslant 1}(U, G) = 0$ for all $U \in \mathcal{I}_C(R)$ and $G \in \mathcal{GI}_C(R)$. □

When $C = R$, for an R-module N, $\mathcal{I}_R(R)^{\perp}\text{-id}_R N$ is exactly the copure injective dimension of N, which is usually denoted by $\mathrm{cid}_R N$. Therefore we have the following corollary.

2.3. Corollary. Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following statements are equivalent.

(1) $\mathrm{cid}_R N \leqslant \mathrm{dim} R$ for any $\mathfrak{m}$-torsion R-module N.
(2) $\mathrm{cid}_R k \leqslant \mathrm{dim} R$.
(3) For any $\mathfrak{m}$-torsion R-module N, there exists a coproper Gorenstein injective coresolution $0 \to N \to G^0 \to \cdots \to G^n \to 0$ with $n = \mathrm{dim} R$ and G^i $\mathfrak{m}$-torsion.
(4) R is Gorenstein.

As a final application of Theorem 2.7, we provide a sufficient condition for testing the Cohen-Macaulayness of a local ring. This result in fact extends a theorem of L.Khatami et al. in [19, Theorem 2.7]. We adopt some of their ideas in the proof of Theorem 3.5 below. Following [13], an R-module M is *cofinite* if there exists an ideal $\mathfrak{a}$ of R such that M is $\mathfrak{a}$-*cofinite* (i.e., $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and all $\mathrm{Ext}_R^i(R/\mathfrak{a}, M)$ are finitely generated). If M is finitely generated, then it is easy to see that M is $\mathrm{Ann}(M)$-cofinite.

2.4. Proposition. Let $(R, \mathfrak{m}, k)$ be a local ring. If N is a non-zero cofinite R-module with $\mathrm{CMid}_R N$ finite, then $\mathrm{dim}_R N \leqslant \mathrm{CMid}_R N$. In particular, if N is finitely generated and $\mathrm{CMid}_R N = 0$, then N is of finite length.

Proof. By Theorem 2.7, $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$ for all $i > \mathrm{CMid}_R N$. On the other hand, we have $\mathrm{H}^{\mathrm{dimN}}_{\mathfrak{m}}(N) \neq 0$ by [21, Theorem 2.9]. Hence $\mathrm{dim}_R N \leqslant \mathrm{CMid}_R N$ follows. Now, assume that N is finitely generated. If $\mathrm{CMid}_R N = 0$, then $\mathrm{dim}_R N = 0$. So N has finite length by [22, Theorem 13.4]. □

With the aid of Proposition 3.4, we are now able to prove the following theorem, which partially answers a question of R. Takahashi in [27]: Is a local ring Cohen-Macaulay if it admits a non-zero finitely generated module of finite Gorenstein injective dimension?

2.5. Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring. If R admits a non-zero cofinite R-module N with $\mathrm{CMid}_R N$ finite and $\mathrm{dim}_R N = \mathrm{dim} R$, then R is Cohen-Macaulay.*

Proof. Since $\mathrm{CMid}_R N$ is finite, it follows from Remark 2.3 that $\mathrm{CMid}_R N = \mathrm{Gid}_{R\ltimes C} N$ is finite for some semidualizing R-module C. By [19, Theorem 2.3], there exists a prime ideal $\mathfrak{q}$ of $R \ltimes C$ in $\mathrm{Supp}_{R\ltimes C} N$ with $\mathrm{Gid}_{R\ltimes C} N \leqslant \mathrm{depth}_{(R\ltimes C)_{\mathfrak{q}}}\ (R \ltimes C)_{\mathfrak{q}}$. Note the fact that depth$R\ltimes C=$ depth$R$ and dim$R\ltimes C=$ dim$R$ by [15]. Then Proposition 3.4 implies that

$$\dim R\ltimes C=\dim N\leqslant \mathrm{Gid}_{R\ltimes C}N\leqslant \mathrm{depth}_{(R\ltimes C)_{\mathfrak{q}}}(R\ltimes C)_{\mathfrak{q}}\leqslant \dim(R\ltimes C)_{\mathfrak{q}}=\mathrm{ht}\mathfrak{q}.$$

Hence $\mathfrak{q}$ must be the maximal ideal of $R\ltimes C$. Therefore we have

$$\dim R\leqslant \mathrm{depth}R\ltimes C=\mathrm{depth}R.$$

This means that R is Cohen-Macaulay. □

Since $\dim_R C=\dim R$ and $\mathrm{CMid}_R C\leq \mathcal{I}_C\text{-id}_R(C)$, a special case of Theorem 3.5 has been given by S. Sather-Wagstaff and S. Yassemi in [24, Lemma 2.11]. Finally, we end this paper with an example showing that the Cohen-Macaulay injective dimension is strictly less than the Gorenstein injective dimension.

2.6. Example. Let k be a field and $R=k[[x^3,x^4,x^5]]$. From [9, Example 3.3], R is a non-Gorenstein Cohen-Macaulay ring with a dualizing module. By [15, Theorem 5.1], $\mathrm{CMid}_R k<\infty$. But R is not Gorenstein, we know from [11, Theorem A] that $\mathrm{Gid}_R k=\infty$. So $\mathrm{CMid}_R k<\mathrm{Gid}_R k$.

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References

[1] Asadollahi, J. and Salarian, Sh. *Cohomology theories based on Gorenstein injective modules*, Trans. Amer. Math. Soc. **358**(5), 2183-2203, 2006.

[2] Asadollahi, J., Jahanshahi, F. and Salarian, Sh. *Complete cohomology and Gorensteinness of schemes*, J. Algebra **319**(6), 2626-2651, 2008.

[3] Brodmann, M. and Sharp, R. Y. *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics **60** (Cambridge University Press, 1998).

[4] Bruns, W. and Herzog, J. *Cohen-Macaulay Rings*, Studies in Advanced Mathematics **39** (Cambridge University Press, 1998).

[5] Divaani-Aazar, K. and Hajikarimi, A. *Generalized local cohomology modules and homological Gorenstein dimensions*, Comm. Algebra, **39**(6), 2051-2067, 2011.

[6] Enochs, E. E. and Jenda, O. M. G. *Copure injective resolutions, flat resolvents and dimensions*, Comment. Math. Univ. Carolin. **34**(2), 203-211, 1993.

[7] Enochs, E. E. and Jenda, O. M. G. *Gorenstein injective and projective modules*, Math. Z. **220**(4), 611-633, 1995.

[8] Enochs, E. E. and Jenda, O. M. G. *Relative Homological Algebra* (Walter de Gruyter, Berlin, 2000).

[9] Enochs, E. E. and Yassemi, S. *Foxby equivalence and cotorsion theories relative to semi-dualizing modules*, Math. Scand. **95**(1), 33-43, 2004.

[10] Fossum, R. M., Griffith, P. A. and Reiten, I. *Trivial Extensions of Abelian Categories*, Lecture Notes in Math. **456** (Springer-Verlag, Berlin, 1975).

[11] Foxby, H.-B. and Frankild, A. J. *Cyclic modules of finite Gorenstein injective dimension and Gorenstein rings*, Illinois J. Math. **51** (1), 67-82, 2007.

[12] Hartshorne, R. *Algebraic Geometry*, Graduate Texts in Mathematics **52** (Springer-Verlag, New York-Heidelberg, 1977).

[13] Hellus, M. *A note on the injective dimension of local cohomology modules*, Proc. Amer. Math. Soc. **136** (7), 2313-2321, 2008.

[14] Herzog, J. *Komplex Auflösungen und Dualität in der lokalen Algebra*, Habilitationsschrift, Universität Regensburg, 1974.

[15] Holm, H. and Jrgensen, P. *Cohen-Macaulay homological dimensions*, Rend. Semin. Mat. Univ. Padova **117**, 87-112, 2007.

[16] Holm, H. and Jrgensen, P. *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2), 423-445, 2006.

[17] Holm, H. and Jrgensen, P. *Cotorsion pairs induced by duality pairs*, J. Commutative Algebra **1** (4), 621-633, 2009.

[18] Holm, H. and White, D. *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (4), 781-808, 2007.

[19] Khatami, L., Tousi, M. and Yassemi, S. *Finiteness of Gorenstein injective dimension of modules*, Proc. Amer. Math. Soc. **137** (7), 2201-2207, 2009.

[20] Lam, T. Y. *Lectures on Modules and Rings* (Springer, Berlin, 1999).

[21] Mafi, A. *Some results on local cohomology modules*, Arch. Math. **87** (3), 211-216, 2006.

[22] Matsumura, H. *Commutative Ring Theory* (Cambridge University Press, 1986).

[23] Rotman, J. J. *An Introduction to Homological Algebra* (New York: Academic Press, 1979).

[24] Sather-Wagstaff, S. and Yassemi, S. *Modules of finite homological dimension with respect to a semidualizing module*, Arch. Math. **93** (2), 111-121, 2009.

[25] Sazeedeh, R. *Gorenstein injective modules and local cohomology*, Proc. Amer. Math. Soc. **132** (10), 2885-2891, 2004.

[26] Sazeedeh, R. *Gorenstein injective, Gorenstein flat modules and the section functor*, J. Pure Appl. Algebra **211** (3), 773-783, 2007.

[27] Takahashi, R. *The existence of finitely generated modules of finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **134** (11), 3115-3121, 2006.

[28] White, D. *Gorenstein projective dimension with respect to a semidualizing module*, J. Commutative algebra **2** (1), 111-137, 2010.

[29] Yassemi, S. *Generalized section functors*, J. Pure Appl. Algebra **95** (1), 103-119, 1994.

[30] Yassemi, S. *A generalization of a theorem of Bass*, Comm. Algebra **35** (1), 249-251, 2007.