A Unified Family of Generalized $q$-Hermite Apostol Type Polynomials and its Applications

Subuhi Khan$^1$ and Tabinda Nahid$^1$

Abstract
The intended objective of this paper is to introduce a new class of generalized $q$-Hermite based Apostol type polynomials by combining the $q$-Hermite polynomials and a unified family of $q$-Apostol-type polynomials. The generating function, series definition and several explicit representations for these polynomials are established. The $q$-Hermite-Apostol Bernoulli, $q$-Hermite-Apostol Euler and $q$-Hermite-Apostol Genocchi polynomials are studied as special members of this family and corresponding relations for these polynomials are obtained.

Keywords: $q$-Hermite polynomials, Generalized $q$-Apostol type polynomials, Generalized $q$-Hermite Apostol type polynomials, Explicit representation

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1. Introduction and preliminaries

The $q$-calculus has been extensively studied for a long time by many mathematicians, physicists and engineers. The $q$-calculus is a generalization of many subjects, like the hypergeometric series, complex analysis and particle physics. The $q$-analogues of many orthogonal polynomials and functions assume a very pleasant form reminding directly of their classical counterparts. The $q$-calculus is mostly being used by physicists at a high level. In short, $q$-calculus is quite a popular subject today.

Throughout the present paper, $\mathbb{C}$ indicates the set of complex numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0$ indicates the set of non-negative integers. Further, the variable $q \in \mathbb{C}$ such that $|q| < 1$. The following $q$-standard notations and definitions are taken from [1].

The $q$-analogue of the shifted factorial $(a)_n$ is defined by

$$(a; q)_0 = 1, (a; q)_n = \prod_{m=0}^{n-1} (1 - q^m a), \ n \in \mathbb{N}.$$  

The $q$-analogues of a complex number $a$ and of the factorial function are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \ q \in \mathbb{C} - \{1\}; \ a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n} [m]_q = \frac{(q; q)_n}{(1 - q)_n}, \ q \neq 1; \ n \in \mathbb{N}, \ [0]_q! = 1, \ q \in \mathbb{C}.$$
The $q$-binomial coefficient $\left[\begin{array}{c} n \\ k \end{array}\right]_q$ is defined by
\[
\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad k = 0, 1, \ldots, n.
\]

The $q$-exponential function is defined as:
\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-q)xq}, \quad |x| < |1-q|^{-1}.
\]

The $q$-Hermite polynomials are special or limiting cases of the orthogonal polynomials as they contain no parameter other than $q$ and appears to be at the bottom of a hierarchy of the classical $q$-orthogonal polynomials [2]. The $q$-Hermite polynomials constitute a 1-parameter family of orthogonal polynomials, which for $q = 1$ reduce to the well known Hermite polynomials. We recall that the $q$-Hermite polynomials $H_{n,q}(x)$ is defined by the following generating function [3]:
\[
F_q(x,t) := F_q(t)e_q(x) = \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!},
\]
\[
F_q(t) := \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{2^n}{[2n]_q!!}, \quad [2n]_q!! = [2n]_q[2n-2]_q \ldots [2]_q.
\]
\[
D_{q,\lambda}H_{n,q}(x) = [n]_q H_{n-1,q}(x).
\]

Recently, many mathematicians studied the unification of the Bernoulli and Euler polynomials. Luo and Srivastava [4, 5] introduced the generalized Apostol-Bernoulli polynomials of order $\alpha$ and the generalized Apostol-Euler polynomials $E^{(\alpha)}_{n,q}(x)$ of order $\alpha$ are investigated by Luo [6, 7]. Thereafter, in 2014 Ernst [8] defined the $q$-analogues of the generalized Apostol type polynomials.

The generalized $q$-Apostol-Bernoulli polynomials $B^{(\alpha)}_{n,q,\lambda}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:
\[
\left(\frac{1}{\lambda e_q(t) - 1}\right)^\alpha e_q(x) = \sum_{n=0}^{\infty} B^{(\alpha)}_{n,q,\lambda}(x) \frac{t^n}{[n]_q!}.
\]

The generalized $q$-Apostol-Euler polynomials $E^{(\alpha)}_{n,q,\lambda}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:
\[
\left(\frac{2}{\lambda e_q(t) + 1}\right)^\alpha e_q(x) = \sum_{n=0}^{\infty} E^{(\alpha)}_{n,q,\lambda}(x) \frac{t^n}{[n]_q!}.
\]

The generalized $q$-Apostol-Genocchi polynomials $G^{(\alpha)}_{n,q,\lambda}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function [8]:
\[
\left(\frac{2t}{\lambda e_q(t) + 1}\right)^\alpha e_q(x) = \sum_{n=0}^{\infty} G^{(\alpha)}_{n,q,\lambda}(x) \frac{t^n}{[n]_q!}.
\]

In view of equations (1.3)-(1.5), the generalized $q$-Apostol type polynomials $\varphi^{(\alpha)}_{n,q,\lambda}(x;k,a,b)$ ($\alpha \in \mathbb{N}_0, \lambda, a, b \in \mathbb{C}$) of order $\alpha$ are defined by the following generating function:
\[
\left(\frac{2^{1-k}k^k}{\beta^\alpha e_q(t) - a^\beta}\right)^\alpha e_q(x) = \sum_{n=0}^{\infty} \varphi^{(\alpha)}_{n,q,\lambda}(x;k,a,b) \frac{t^n}{[n]_q!},
\]
where $\varphi^{(\alpha)}_{n,q,\lambda}(k,a,b) = \varphi^{(\alpha)}_{n,q,\lambda}(0;k,a,b)$ are the $q$-Apostol type numbers of order $\alpha$.

If we take the limit $q \to 1$, the generalized $q$-Apostol type polynomials defined by equation (1.6) reduces to the unified Apostol type polynomials [9]. In fact, the following special cases hold:
\[
\lim_{q \to 1} \varphi^{(\alpha)}_{n,q,\lambda}(x;1,1,1) = B^{(\alpha)}_{n,\lambda}(x).
\]
\[
\lim_{q \to 1} \mathcal{P}_{n,q,\lambda}^{(\alpha)}(x;0,-1,1) = E_{n,\lambda}^{(\alpha)}(x),
\]
\[
\lim_{q \to 1} \mathcal{P}_{n,q,\frac{1}{2}}^{(\alpha)}(x;1,-1/2,1) = G_{n,\lambda}^{(\alpha)}(x),
\]
where \(B_{n,\lambda}^{(\alpha)}(x), E_{n,\lambda}^{(\alpha)}(x)\) and \(G_{n,\lambda}^{(\alpha)}(x)\) are the generalized forms of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

In the current article, the \(q\)-Hermite-Apostol type polynomials are introduced and their explicit relations are proved. The corresponding results for the \(q\)-Hermite-Apostol Bernoulli, \(q\)-Hermite-Apostol Euler and \(q\)-Hermite-Apostol Genocchi polynomials are established.

### 2. Generalized \(q\)-Hermite Apostol type polynomials

In this section, a new hybrid class of the generalized \(q\)-Hermite-Apostol type polynomials \(\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b)\) is introduced by convoluting the \(q\)-Hermite polynomials and generalized \(q\)-Apostol type polynomials. In order to establish the generating function for the these polynomials, the following result is proved:

**Theorem 2.1.** The following generating function for the generalized \(q\)-Hermite based Apostol type polynomials \(\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b)(\alpha \in \mathbb{N}_0, \alpha, a, b \in \mathbb{C})\) holds true:

\[
\left( \frac{2^{1-k}k}{\beta^k e_q(t) - at^k} \right)^\alpha F_q(t) e_q(x t) = \sum_{n=0}^{\infty} \mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b) \frac{t^n}{[n]_q!},
\]

(2.1)

**Proof.** Expanding the exponential function \(e_q(x t)\) and then replacing the powers of \(x, i.e., x^0, x^1, x^2, \ldots, x^n\) by the correlating \(q\)-Hermite polynomials \(H_0(q;x), H_1(q;x), \ldots, H_n(q;x)\) in the l.h.s. of equation (1.6) and after summing up the terms of the resultant equation and denoting the resultant \(q\)-HATyP in the r.h.s. by \(\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b)\), assertion (2.1) is proved.

Taking \(x = 0\) in equation (2.1), we get

\[
\mathcal{H}_{n,q,\alpha,\beta}(k,a,b) = \mathcal{H}_{n,q,\alpha,\beta}(0;k,a,b),
\]

where \(\mathcal{H}_{n,q,\alpha,\beta}(k,a,b)\) are the \(q\)-Hermite Apostol type numbers of order \(\alpha\).

Next, the series expansions for the \(q\)-HATyP \(\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b)\) is obtained by proving the following result:

**Theorem 2.2.** The following series expansions for the generalized \(q\)-Hermite based Apostol type polynomials \(\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b)\) hold true:

\[
\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{H}_{r,q,\alpha,\beta}(k,a,b) H_{n-r,q}(x),
\]

(2.2)

\[
\mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{H}_{r,q,\alpha,\beta}(k,a,b) x^{n-r}.
\]

(2.3)

**Proof.** Utilizing equations (1.2) and (1.6) in the l.h.s. of generating function (2.1) and then using Cauchy-product rule in the l.h.s. of the resultant equation, it follows that

\[
\sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \mathcal{H}_{r,q,\alpha,\beta}(k,a,b) H_{n-r,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b) \frac{t^n}{[n]_q!}.
\]

(2.4)

Equating the coefficients of identical powers of \(t\) in both sides of equation (2.4), assertion (2.2) follows. Utilizing equation (1.1) in the l.h.s. of generating function (2.1), it follows that

\[
\sum_{n=0}^{\infty} \mathcal{H}_{n,q,\alpha,\beta}(x;k,a,b) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \sum_{r=0}^{\infty} \mathcal{H}_{r,q,\alpha,\beta}(k,a,b) \frac{t^r}{[r]_q!},
\]

which on applying the Cauchy product rule in the r.h.s. and then comparing the coefficients of same powers of \(t\) in both sides of resultant equation yields assertion (2.3).
Different members of the generalized $q$-Hermite-Apostol family can be obtained by making suitable selections of the parameters $k, a, b$ and $\beta$ in generating relation (2.1). Some of these members are listed in Table 1.

**Proposition 2.3.** The following relations for the generalized $q$-Hermite based Apostol type polynomials $H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ holds true:

$$D_{q,t} e_q(xt) = x e_q(xt),$$

$$D_{q,u} \left( H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) \right) = [n]_q H_\mathcal{P}^{(\alpha)}_{n-1,q,\beta}(x;k,a,b).$$  

**Theorem 2.4.** For each $n \in \mathbb{N}$ and for the $q$-commuting variables $x$ and $u$ such that $xu = qux$, the generalized $q$-Hermite based Apostol type polynomials $H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ satisfy the following relations:

$$H_\mathcal{P}^{(\alpha+\gamma)}_{n,q,\beta}(x;k,a,b) = \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right]_q H_\mathcal{P}^{(\alpha)}_{r,q,\beta}(x;k,a,b) H_\mathcal{P}^{(\gamma)}_{n-r,q,\beta}(k,a,b).$$  

$$H_\mathcal{P}^{(\alpha+\gamma)}_{n,q,\beta}(x+u;k,a,b) = \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right]_q H_\mathcal{P}^{(\alpha)}_{r,q,\beta}(x;k,a,b) H_\mathcal{P}^{(\gamma)}_{n-r,q,\beta}(u;k,a,b).$$  

**Proof.** Replacing $\alpha$ by $\alpha + \gamma$ in definition (2.1), we have

$$\sum_{n=0}^{\infty} H_\mathcal{P}^{(\alpha+\gamma)}_{n,q,\beta}(x;k,a,b) \frac{t^n}{[n]_q!} = \left( \frac{2^{1-k} t}{\beta^q e_q(t) - a^q} \right)^{\alpha+\gamma} F_q(t) e_q(xt)$$

$$= \left( \sum_{n=0}^{\infty} H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} H_\mathcal{P}^{(\gamma)}_{n,q,\beta}(k,a,b) \frac{t^n}{[n]_q!} \right).$$

Using Cauchy-product rule in the r.h.s. of above equation, it follows that

$$\sum_{n=0}^{\infty} H_\mathcal{P}^{(\alpha+\gamma)}_{n,q,\beta}(x;k,a,b) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right]_q H_\mathcal{P}^{(\alpha)}_{r,q,\beta}(x;k,a,b) H_\mathcal{P}^{(\gamma)}_{n-r,q,\beta}(k,a,b) \frac{t^n}{[n]_q!}.$$  

Equating the coefficients of identical powers of $t$ in both sides of equation (2.7), assertion (2.5) follows. Further, replacing $\alpha$ by $\alpha + \gamma$ and $x$ by $x + u$ in Definition 2.1 and proceeding on the same lines of proof as above, assertion (2.6) follows.  

**Theorem 2.5.** For each $n \in \mathbb{N}$ and for the $q$-commuting variables $x$ and $u$ such that $xu = qux$, the generalized $q$-Hermite based Apostol type polynomials $H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b)$ satisfy the following relation:

$$\beta^b H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x+1;k,a,b) - \alpha^b H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x;k,a,b) = \frac{2^{1-k} [n]_q!}{[n-k]_q!} H_\mathcal{P}^{(\alpha-1)}_{n-k,q,\beta}(x;k,a,b).$$

<table>
<thead>
<tr>
<th>S. No.</th>
<th>$k; a; b; \beta$</th>
<th>Generating function</th>
<th>Name of the polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>$k = 1; a = 1; b = 1; \beta = \lambda$</td>
<td>$\left( \frac{t}{x e_q(t)+t} \right)^{(\alpha)} F_q(t) e_q(t) = \sum_{n=0}^{\infty} H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x) \frac{t^n}{[n]_q!}$</td>
<td>The generalized $q$-Hermite-Apostol Bernoulli polynomials (GqHABP)</td>
</tr>
<tr>
<td>II.</td>
<td>$k = 0; a = -1; b = 1; \beta = \lambda$</td>
<td>$\left( \frac{2}{x e_q(t)+t} \right)^{(\alpha)} F_q(t) e_q(t) = \sum_{n=0}^{\infty} H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x) \frac{t^n}{[n]_q!}$</td>
<td>The generalized $q$-Hermite-Apostol Euler polynomials (GqHAEP)</td>
</tr>
<tr>
<td>III.</td>
<td>$k = 1; a = -1/2; b = 1; \beta = \lambda/2$</td>
<td>$\left( \frac{2}{x e_q(t)+t} \right)^{(\alpha)} F_q(t) e_q(t) = \sum_{n=0}^{\infty} H_\mathcal{P}^{(\alpha)}_{n,q,\beta}(x) \frac{t^n}{[n]_q!}$</td>
<td>The generalized $q$-Hermite-Apostol Genocchi polynomials (GqHAGP)</td>
</tr>
</tbody>
</table>
Proof. From generating relation (2.1), we have

and are given in Table 2.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Special polynomials</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. GqHABP</td>
<td>$\mathcal{H}^{(\alpha)}<em>{n,q,\lambda} (x) = \sum</em>{r=0}^{n} \left[ \begin{array}{l} n \ r \end{array} \right] q_r \mathcal{H}_{n-r,q,\lambda} H_n^{(\alpha)} (x)$</td>
<td>$\beta^b H^{(\alpha)}<em>{n,q,\lambda} (x+1) - \delta^b H^{(\alpha)}</em>{n,q,\lambda} (x) = \frac{2^{1-k}</td>
</tr>
<tr>
<td>II. GqHAEP</td>
<td>$\mathcal{H}^{(\alpha)}<em>{n,q,\lambda} (x) = \sum</em>{r=0}^{n} \left[ \begin{array}{l} n \ r \end{array} \right] q_r \mathcal{H}_{n-r,q,\lambda} x^{n-r}$</td>
<td>$\beta^b H^{(\alpha)}<em>{n,q,\lambda} (x+1) - \delta^b H^{(\alpha)}</em>{n,q,\lambda} (x) = \frac{2^{1-k}</td>
</tr>
<tr>
<td>II. GqHAGP</td>
<td>$\mathcal{H}^{(\alpha)}<em>{n,q,\lambda} (x) = \sum</em>{r=0}^{n} \left[ \begin{array}{l} n \ r \end{array} \right] q_r \mathcal{H}_{n-r,q,\lambda} x^{n-r}$</td>
<td>$\beta^b H^{(\alpha)}<em>{n,q,\lambda} (x+1) - \delta^b H^{(\alpha)}</em>{n,q,\lambda} (x) = \frac{2^{1-k}</td>
</tr>
</tbody>
</table>

| Table 2. Certain results for the GqHABP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$, GqHAEP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$ and GqHAGP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$ |

Equating the coefficients of same powers of $t$ in both sides of the above equation, assertion (2.8) follows.

In view of Table 1, certain results for the GqHABP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$, GqHAEP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$ and GqHAGP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$ are established and are given in Table 2.

In the next section, certain explicit representations for the GqHAEP $\mathcal{H}^{(\alpha)}_{n,q,\lambda} (x)$ are established.
3. Explicit representations

In order to derive the explicit representations for the \(GqHATyP_{H_n,q,\beta}(x,k,a,b)\), we recall the following definition:

**Definition 3.1.** The generalized \(q\)-Stirling numbers \(S_q(n,\nu,a,b,\beta)\) of the second kind of order \(\nu\) is defined as [10]:
\[
\sum_{n=0}^{\infty} S_q(n,\nu,a,b,\beta) \frac{t^n}{[n]_q!} = (\beta^{b}e_q(t) - a^b)^\nu.
\]

**Theorem 3.2.** The following explicit formula for the generalized \(q\)-Hermite based Apostol type polynomials \(H_{n,q,\beta}(x,k,a,b)\) in terms of the generalized \(q\)-Stirling numbers of the second kind \(S_q(n,\nu,a,b,\beta)\) holds true:
\[
H_{n-vk,q,\beta}(x,k,a,b) = 2^{v(k-1)} \left\{ \sum_{l=0}^{\infty} \frac{[n]_{[l]}!}{[l]_q!} \left( \sum_{i=0}^{\infty} \frac{t^i}{[i]_q!} \right) \right\} H_{l,q,\beta}(x,k,a,b) S_q(n-l,\nu,a,b,\beta).
\]

**Proof.** From generating relation (2.1), we have
\[
\sum_{n=0}^{\infty} H_{n,q,\beta}(x,k,a,b) \frac{t^n}{[n]_q!} = (\beta^{b}e_q(t) - a^b)^\nu.
\]
Applying the Cauchy product rule on the r.h.s. of the above equation, it follows that
\[
\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \left\{ \sum_{l=0}^{\infty} \frac{[n]_{[l]}!}{[l]_q!} \left( \sum_{i=0}^{\infty} \frac{t^i}{[i]_q!} \right) \right\} H_{l,q,\beta}(x,k,a,b) S_q(n-l,\nu,a,b,\beta).
\]
Equating the coefficients of identical powers of \(t\) in both sides of equation (3.2) yields assertion (3.1).

**Theorem 3.3.** The following explicit relation for the generalized \(q\)-Hermite based Apostol type polynomials \(H_{n,q,\beta}(x,k,a,b)\) in terms of the generalized \(q\)-Apostol Bernoulli polynomials \(B_{n,q,\lambda}(x)\) holds true:
\[
H_{n,q,\beta}(x,k,a,b) = \frac{1}{[n+1]_q} \left\{ \sum_{r=0}^{n+1} \frac{[n+1]_r}{[r]_q} \sum_{m=0}^{r} \frac{[m]_q}{[r]_q} \right\} B_{n+1-r,q,\lambda}(x)
\]

**Proof.** Consider generating function (2.1) in the following form:
\[
\left( \frac{2^{1-k}k}{\beta^{b}e_q(t) - a^b} \right)^\alpha F_q(t)e_q(xt) = \left( \frac{2^{1-k}k}{\beta^{b}e_q(t) - a^b} \right)^\alpha F_q(t) \left( \frac{t}{\beta^{b}e_q(t) - a^b} - 1 \right) \frac{\lambda e_q(t) - 1}{t} e_q(xt),
\]
which on simplifying and rearranging the terms becomes
\[
\left( \frac{2^{1-k}k}{\beta^{b}e_q(t) - a^b} \right)^\alpha F_q(t)e_q(xt) = \left( \frac{2^{1-k}k}{\beta^{b}e_q(t) - a^b} \right)^\alpha F_q(t) \left( \frac{t}{\beta^{b}e_q(t) - a^b} - 1 \right) e_q(xt) \left( \frac{\lambda}{t} e_q(t) \right) - \frac{1}{t} \left( \frac{2^{1-k}k}{\beta^{b}e_q(t) - a^b} \right)^\alpha F_q(t) \left( \frac{t}{\beta^{b}e_q(t) - a^b} - 1 \right) e_q(xt).
\]
Using equations (1.3) and (2.1) in equation (3.4), we have
\[
\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \left\{ \sum_{r=0}^{n+1} \frac{[n+1]_r}{[r]_q} \sum_{m=0}^{r} \frac{[m]_q}{[r]_q} \right\} B_{n+1-r,q,\lambda}(x)
\]
\[
\sum_{m=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} \sum_{m=0}^{r} \frac{[m]_q}{[r]_q} \right\} B_{n+1-r,q,\lambda}(x) \frac{t^n}{[n]_q!}.
\]
Corollary 3.5. in terms of the generalized q-Apostol Euler polynomials $E_{n+1-m,q,\lambda}(x)$.

Corollary 3.4. The following explicit relation for the generalized q-Hermite based Apostol type polynomials $H_{n,q,\beta}(x;k,a,b)$ in terms of the the generalized q-Apostol Euler polynomials $E_{n,q,\lambda}(x)$ holds true:

$$H_{n,q,\beta}(x;k,a,b) = \frac{1}{2} \sum_{m=0}^{n} \sum_{r=0}^{n} \left[ \sum_{m=0}^{n+1} \binom{n+1}{m} \sum_{r=0}^{n} \binom{n}{r} E_{n-r,q,\lambda}(x) + E_{n-m,q,\lambda}(x) \right] H_{m,q,\beta}(k,a,b).$$

Corollary 3.5. The following explicit relation for the generalized q-Hermite based Apostol type polynomials $H_{n,q,\beta}(x;k,a,b)$ in terms of the generalized q-Apostol Genocchi polynomials $G_{n,q,\lambda}(x)$ holds true:

$$H_{n,q,\beta}(x;k,a,b) = \frac{1}{2(n+1)} \sum_{r=0}^{n+1} \sum_{m=0}^{n} \binom{n+1}{m} \sum_{r=0}^{n} \binom{n}{r} G_{n+1-r,q,\lambda}(x)$$

$$- \sum_{m=0}^{n+1} \binom{n+1}{m} \sum_{r=0}^{n} \binom{n}{r} G_{n+1-m,q,\lambda}(x) \right] H_{m,q,\beta}(k,a,b).$$

In view of Table 1, certain explicit representations for the GqHABP $H_{n,q,\beta}(x)$, GqHAEP $E_{n,q,\lambda}(x)$ and GqHAGP $G_{n,q,\lambda}(x)$ are established and are given in Table 3.
Note. It is to be observed that for $\lambda = 1$, the results derived above for the generalized $q$-Hermite-Apostol Bernoulli polynomials $H_{n,q,\lambda}(x)$, the generalized $q$-Hermite-Apostol Euler polynomials $E_{n,q,\lambda}^{(\alpha)}(x)$ and generalized $q$-Hermite-Apostol Genocchi polynomials $G_{n,q,\lambda}^{(\alpha)}(x)$ gives the analogous results for the generalized $q$-Hermite Bernoulli polynomials $H_{n,q}(x)$, the generalized $q$-Hermite Euler polynomials $E_{n,q}(x)$ and generalized $q$-Hermite Genocchi polynomials $G_{n,q}(x)$.

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