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A New Theorem on The Existence of Positive Solutions of Singular Initial-Value Problem for Second Order Differential Equations

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Abstract
We proved a new theorem on the existence of positive solutions to initial-value problems for second-order nonlinear singular differential equations. The existence of solutions is proven under considerably weaker than previously known conditions.

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1. Introduction

We consider the problem

$$\begin{aligned}(py')' + pqg(y) &= 0, \;\; t \in [0,T] \\ y(0) &= a > 0, \\ \lim_{t\to 0+} p(t)y'(t) &= 0\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}(py')' + pqg(y) &= 0, \;\; t \in [0,T] \\ y(0) &= a > 0, \\ y'(0) &= 0,\end{aligned} \tag{1.2}$$

where $0 < T < \infty$, $p \geq 0$, $q \geq 0$ and $g : [0,\infty) \to [0,\infty)$.

Agarwal and O'Regan [1] established the existence theorems for the positive solution of the problem (1.1) and (1.2):

Theorem 1.1. *[1] Suppose the following conditions are satisfied*

$$p \in C[0,T) \cap C^1(0,T) \;\; \textit{with } p > 0 \textit{ on } (0,T) \tag{1.3}$$

$$q \in L^1_p[0,t^*] \text{ for any } t^* \in (0,T) \text{ with } q > 0 \text{ on } (0,T), \tag{1.4}$$

where $L^1_r[0,a]$ *is the space of functions* $u(t)$ *with* $\int_0^a |u(t)|\, r(t)dt < \infty$,

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \text{ for any } t^* \in (0,T) \tag{1.5}$$

and

$$g : [0,\infty) \to [0,\infty) \text{ is continuous, nondecreasing on } [0,\infty) \text{ and } g(u) > 0 \text{ for } u > 0. \tag{1.6}$$

Let

$$H(z) = \int_z^a \frac{dx}{g(x)} \text{ for } 0 < z \le a$$

and assume

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)dxds < a \text{ for any } t^* \in (0,T), \tag{1.7}$$

here

$$\tau(x) = g\left(H^{-1}\left(\int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)dzdw\right)\right).$$

Then (1.1) *has a solution* $y \in C[0,T)$ *with* $py' \in C[0,T)$, $(py')' \in L^1_{pq}(0,T)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T)$. *In addition if either*

$$p(0) \neq 0$$

or

$$p(0) = 0 \text{ and } \lim_{t\to 0+} \frac{p(t)q(t)}{p'(t)} = 0$$

holds, then y is a solution of (1.2)*.*

The condition (1.7) in connection with the definition of the function $\tau(x)$, makes this theorem difficult for an application. In [2] we proved more easy and applicable theorem:

Theorem 1.2. *Suppose* (1.3)-(1.5) *hold. In addition, we assume*

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dxds < a$$

for any $t^* \in (0,T_0)$. *Then*

a) (1.1) *has a solution* $y \in C[0,T_0)$ *with* $py' \in C[0,T_0)$, $(py')' \in L^1_{pq}(0,T_0)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T_0)$.

b) If $\int_{T_0}^{T_1} \frac{1}{p(s)} \int_0^s p(x)q(x)g(T_0)ds < y(T_0)$, *and conditions* (1.3)-(1.6) *satisfied then solution can be extended into the interval* $[0,T_1)$.

In this paper we generalized the Theorem 1.2.

2. Main result

Theorem 2.1. *Suppose the following conditions are satisfied*

$$\begin{aligned} p &\in C[0,T)\cap C^1(0,T) \text{ with } p>0 \text{ on } (0,T], \\ q &\geq 0, \end{aligned}$$

$$\int_0^{t^*}\frac{1}{p(s)}\int_0^s p(x)q(x)dxds<\infty \text{ for any } t^*\in(0,T],$$

$$g:[0,\infty)\to[0,\infty) \text{ is continuous, nondecreasing on } [0,\infty),$$

and assume

$$\begin{aligned} \int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(a-\varphi(x))dxds &\leq a-\varphi(t), \\ \int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(\varphi(x))dxds &\geq \varphi(t) \end{aligned}$$

for some $\varphi(t)\in C[0,T]$, with $0\leq\varphi(t)\leq a$. Then (1.1) has a solution $y\in C[0,T]$ with $py'\in C[0,T]$, $(py')'\in L^1_{pq}(0,T)$ and $0<y(t)\leq a$ for $t\in[0,T]$. In addition if either

$$p(0)\neq 0$$

or

$$p(0)=0 \text{ and } \lim_{t\to 0+}\frac{p(t)q(t)}{p'(t)}=0$$

holds, then y is a solution of (1.2).

Remark 2.2. *The case $\varphi(t)\equiv 0$, corresponds to the case of inequality*

$$\int_0^{t^*}\frac{1}{p(s)}\int_0^s p(x)q(x)g(a)dxds<a \text{ for any } t^*\in(0,T].$$

Proof of Theorem 2.1. Consider the sequence $\{y_n(t)\}, n=0,1,2,...$ with $y_0(t)\equiv a-\varphi(t)$,

$$y_n(t)=a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_{n-1}(x))dxds,\ n=1,2,...,\ t\leq T.$$

We have

$$\begin{aligned} y_0(t) &\equiv a-\varphi(t), \\ y_1(t) &= a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(a-\varphi(x))dxds\geq\varphi(t), \end{aligned}$$

$$\begin{aligned} y_2(t) &= a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(y_1(x))dxds \\ &\leq a-\int_0^t\frac{1}{p(s)}\int_0^s p(x)q(x)g(\varphi(x))dxds \\ &\leq a-\varphi(t), \end{aligned}$$

$$\begin{aligned}
y_3(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_2(x))dxds \geq \varphi(t) \\
y_4(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_3(x))dxds \leq a - \varphi(t), \ldots \\
y_{2n-1}(t) &\geq \varphi(t), \\
y_{2n}(t) &\leq a - \varphi(t), \ldots
\end{aligned}$$

The sequences $\{y_{2n}(t)\}$ and $\{y_{2n+1}(t)\}$ are equicontinuous. Indeed we have

$$|y_n(t) - y_n(r)| = \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{n-1}(x))dxds \leq M \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds, \tag{2.1}$$

where

$$M = \max\{g(u) : 0 \leq u \leq a\}$$

and the right hand side of (2.1) can be taken $< \varepsilon$ for $|t-r| < \delta$, regardless of the choice of t and r: the function $\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds$ is (uniformly) continuous on $[0,T]$. It follows from Ascoli Arzela Theorem that the sequence $\{y_{2n}(t)\}$ has the (uniformly) convergent subsequence, $y_{2n_k}(t) \to u(t)$. The Lebesgue dominated theorem guarantees that

$$\begin{aligned}
y_{2n_k+1}(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{2n_k}(x))dxds \to v(t), \\
v(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds \\
\text{and } u(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(v(x))dxds.
\end{aligned}$$

If $u(t) = v(t)$ we have that the function $u(t)$ is the solution of the problem (1.1), indeed it follows from

$$u(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds$$

that

$$\begin{aligned}
u'(t) &= -\frac{1}{p(t)} \int_0^t p(x)q(x)g(u(x))dx, \\
pu' &= -\int_0^t p(x)q(x)g(u(x))dx, \\
\left(pu'\right)' &= -pqg(u).
\end{aligned}$$

So we suppose $u(t) \neq v(t)$. We have $u(0) = v(0) = a$ and if for example, $u(t) > v(t)$ on the interval $(0,b)$, then we obtain

$$u(b) - v(b) = \int_0^b \frac{1}{p(s)} \int_0^s p(x)q(x)\left[g(u(x)) - g(v(x)\right]dxds > 0$$

and therefore $u(t) > v(t)$ on the whole interval $(0,T]$. The same holds for all points of intersections $t_0 : u(t_0) = v(t_0)$. That is if $u(t_0) = v(t_0)$, then for any $\varepsilon > 0$ there are infinitely many points $t_n \in [t_0, t_0 + \varepsilon)$ such that $u(t_n) = v(t_n)$. Therefore, $u(t) > v(t)$ (or $<$) on $(t_0, T]$. Without loss of generality let us suppose $u(t) > v(t)$ on $(0,T]$

and consider the operator $N : C[0,T] \to C[0,T]$ defined by

$$Ny(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y(x))dxds.$$

Next let

$$K = \{y \in C[0,T] : v(t) \le y(t) \le u(t) \text{ for } t \in [0,T]\}.$$

Clearly K is closed, convex, bounded subset of $C[0,T]$ and $N : K \to K$. Let us show that $N : K \to K$ is continuous and compact operator. Continuity follows from Lebesgue dominated convergence theorem: if $y_n(t) \to y(t)$, then $Ny_n(t) \to Ny(t)$. To show that N is completely continuous let $y(t) \in K$, then

$$|Ny(t) - Ny(r)| \le M \left| \int_r^t \frac{1}{p(x)} \int_0^x p(z)q(z)dzds \right| \text{ for } t, r \in [0,T],$$

that is N completely continuous on $[0,T]$.

The Schauder-Tychonoff theorem guarantees that N has a fixed point $w \in K$, i.e. w is a solution of (1.1). It follows from

$$w'(t) = -\frac{1}{p(t)} \int_0^t p(x)q(x)g(w(x))dx,$$

that if $p(0) \neq 0$ then $w'(0) = 0$. Now if $p(0) = 0$ but $\lim_{t\to 0+} \frac{p(t)q(t)}{p'(t)} = 0$ we have from (**??**) that

$$w'(0+) = -\lim_{t\to 0+} \int_0^t \frac{p(x)q(x)}{p(t)} g(w(x))dx = 0,$$

that is w is a solution of (1.2).

The proof is completed.

Conflict of interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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