On Signomial Constrained Optimal Control Problems

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Abstract
In this paper, using the notions of variational differential system, adjoint differential system and modified Legendrian duality, we formulate and prove necessary optimality conditions in signomial constrained optimal control problems.

Keywords: Optimal control, Maximum principle, Variational differential system, Adjoint differential system, Modified Legendrian duality.

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1. Introduction and problem formulation

Optimal control theory (see Lee and Markus [1], Pontriaguine et al. [2], Evans [3]), due to important applications in various branches of pure and applied science, has attracted many researchers over the years. Wagner [4] established a Pontryagin-type maximum principle associated with some Dieudonnée-Rashevsky type problems governed by Lipschitz functions. Later, Udrişte [5], using the multi-time concept, formulated and proved, under the simplified hypothesis, a maximum principle based on multiple/curvilinear integral cost functional and m-flow type PDE constraints. Treanţa and Vârsan [6] derived that solutions associated with an extended affine control system can be obtained as a limit process using solutions for a parameterized affine control system and weak small controls.

In this paper, taking into account Treanţa and Udrişte [7] and Treanţă [8], we introduce necessary conditions of optimality for a new class of optimal control problems involving signomial type constraints. For other different but connected points of view regarding this subject, the reader is directed to Mititelu and Treanţa [9] and Treanţă [10, 11].

In the following, for $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ we shall write $x > 0$ if $x^i > 0$, $i = 1, n$, and $x \geq 0$ if $x^i \geq 0$, $i = 1, n$. The set $\mathbb{R}^+_n = \{ x \in \mathbb{R}^n : x \geq 0 \}$ is said to be the positive orthant and, most of the times, we shall use the open positive orthant $\mathbb{P}^n = \{ x \in \mathbb{R}^n : x > 0 \}$. On the set $\mathbb{P}^n$, we consider the distinct monomials of the form $v^k = v^k(x) = (x^1)^{\alpha_{1k}} \cdots (x^n)^{\alpha_{nk}}$, $k = 1, m$, where $\alpha_{ik}$ are real numbers. If $\alpha^i_k$ are real numbers, then the functions $a^i_k v^k$, with summation upon $k$, are called signomials. The controlled signomial dynamical systems are defined as follows:

$$x^i(t) = a^i_k v^k(x(t), u(t)), \quad i = 1, n,$$

where $v^k(x, u) := (x^i)^{\alpha_{ik}} \cdots (x^n)^{\alpha_{nk}}(u^1)^{\gamma_{1k}} \cdots (u^r)^{\gamma_{rk}}$, $\alpha_{ik}, \gamma_{lk} \in R$, $k = 1, m$, $i = 1, n$, $\beta = 1, r$, $t \in I \subseteq R$, and $\mathbb{P}^r \ni u = (u^j)$, $\beta = 1, r$, is a control.
Further, let us consider an optimal control problem based on a simple integral cost functional, constrained by a controlled signomial dynamical system:

$$\max_{u(\cdot), x_0} I(u(\cdot)) = \int_0^{t_0} X(x(t), u(t)) \, dt \quad (1.1)$$

subject to

$$\dot{x}(t) = a_i^k v^k(x(t), u(t)), \quad i = 1, n, \; k = 1, m$$  \hspace{1cm} (1.2)

$$u(t) \in U, \; \forall t \in [0, t_0]; \; x(0) = x_0, \; x(t_0) = x_{00}.$$  \hspace{1cm} (1.3)

In the aforementioned optimal control problem we used the following terminology and notations: $t \in [0, t_0]$ is parameter of evolution, or the time; $[0, t_0] \subset R_+$ is the time interval; $x : [0, t_0] \to P^n$, $x(t) = (x^i(t))$, $i = 1, n$, is a $C^2$-class function, called state vector; $u : [0, t_0] \to P^r$, $u(t) = (u^a(t))$, $a = 1, r$, is a continuous control vector; $U$ is the set of all admissible controls; the running cost $X(x(t), u(t))$ is a $C^1$-class function, called autonomous Lagrangian.

Through this work, the summation over the repeated indices is assumed. Further, we introduce the Lagrange multiplier $p(t) = (p_i(t))$, also called co-state variable (vector), and a new Lagrange function

$$L(x(t), u(t), p(t)) = X(x(t), u(t)) + p_i(t) \left[ a_i^k v^k(x(t), u(t)) - \dot{x}(t) \right].$$

In this way, we change the initial optimal control problem into a free optimization problem

$$\max_{u(\cdot), x_0} \int_0^{t_0} L(x(t), u(t), p(t)) \, dt$$

subject to

$$u(t) \in U, \; p(t) \in P, \; \forall t \in [0, t_0]$$

$$x(0) = x_0, \; x(t_0) = x_{00},$$

where $P$ is the set of co-state variables, which will be defined later. The control Hamiltonian

$$H(x(t), u(t), p(t)) = X(x(t), u(t)) + a_i^k p_i(t) v^k(x(t), u(t)),$$

or, equivalently,

$$H = L + p_i \dot{x}^i \quad \text{(modified Legendrian duality)}$$

permits us to rewrite the previous optimal control problem as follows

$$\max_{u(\cdot), x_0} \int_0^{t_0} \left[ H(x(t), u(t), p(t)) - p_i(t) \dot{x}(t) \right] \, dt$$

subject to

$$u(t) \in U, \; p(t) \in P, \; \forall t \in [0, t_0]$$

$$x(0) = x_0, \; x(t_0) = x_{00}.$$
1.1 Variational and adjoint differential systems

Let us suppose that (1.2) is satisfied. Fix the control $u(t)$ and a corresponding solution $x(t)$ of (1.2). Let $x(t, \varepsilon)$ be a differentiable variation of the state variable $x(t)$, fulfilling

$$\dot{x}(t, \varepsilon) = d^k_i v^k_i(x(t, \varepsilon), u(t))$$

$$x(0, \varepsilon) = x(t), \quad i = 1, n.$$  

Denote by $y_i(t) := \dot{x}(t, 0)$. Taking the partial derivative with respect to $\varepsilon$, evaluating at $\varepsilon = 0$, we obtain the following system

$$\dot{y}(t) = d^k_i v^k_i(x(t, u(t)) \cdot y^i(t),$$

called variational differential system. The differential system

$$\dot{p}_i(t) = -d^k_i p_i(t) v^k_i(x(t), u(t))$$

is called the adjoint differential system of the previous variational differential system since the scalar product $p_i(t) \cdot y^i(t)$ is a first integral of the two systems. Indeed, we have

$$\frac{d}{dt} [p_i(t) \cdot y^i(t)] = 0.$$

2. Main result

Let $\hat{u}(t) = (\hat{u}^\beta(t))$, $\beta = 1, r$, be a continuous control vector defined on the closed interval $[0, t_0]$, with $\hat{u}(t) \in \text{Int} U$, which is an optimal point for the aforementioned control problem. Consider $u(t, \varepsilon) = \hat{u}(t) + \varepsilon h(t)$ a variation of the optimal control vector $\hat{u}(t)$, where $h$ is an arbitrary continuous vector function. We have $\hat{u}(t) \in \text{Int} U$ and, since a continuous function on a compact interval $[0, t_0]$ is bounded, there exists $\varepsilon_h > 0$ such that $u(t, \varepsilon) = \hat{u}(t) + \varepsilon h(t) \in \text{Int} U$, $\forall |\varepsilon| < \varepsilon_h$. This $\varepsilon$ is a "small" parameter used in our variational arguments.

Define $x(t, \varepsilon)$ as the state variable corresponding to the control variable $u(t, \varepsilon)$, i.e.,

$$\dot{x}(t, \varepsilon) = d^k_i v^k_i(x(t, \varepsilon), u(t, \varepsilon)),$$  

and $x(0, \varepsilon) = x_0$. As well, consider (for $|\varepsilon| < \varepsilon_h$) the function (integral with parameter)

$$I(\varepsilon) := \int_0^{t_0} X(x(t, \varepsilon), u(t, \varepsilon)) dt.$$

Since $\hat{u}(t)$ is an optimal control variable we get $I(0) \geq I(\varepsilon), \forall |\varepsilon| < \varepsilon_h$. Also, for any continuous vector function $p(t) = (p_i(t)) : [0, t_0] \rightarrow \mathbb{R}^n$, we have

$$\int_0^{t_0} p_i(t) \left[ d^k_i v^k_i(x(t, \varepsilon), u(t, \varepsilon)) - \dot{x}(t, \varepsilon) \right] dt = 0.$$

The variations involve

$$L(x(t, \varepsilon), u(t, \varepsilon), p(t)) = X(x(t, \varepsilon), u(t, \varepsilon))$$

$$+ p_i(t) \left[ d^k_i v^k_i(x(t, \varepsilon), u(t, \varepsilon)) - \dot{x}(t, \varepsilon) \right]$$

and the associated function (integral with parameter)

$$I(\varepsilon) = \int_0^{t_0} L(x(t, \varepsilon), u(t, \varepsilon), p(t)) dt.$$

Now, assume that the co-state variable $p(t) = (p_i(t))$ is of $C^1$-class. The control Hamiltonian with variations

$$H(x(t, \varepsilon), u(t, \varepsilon), p(t)) = X(x(t, \varepsilon), u(t, \varepsilon)) + d^k_i p_i(t) v^k_i(x(t, \varepsilon), u(t, \varepsilon))$$

changes the above integral with parameter as follows

$$I(\varepsilon) = \int_0^{t_0} [H(x(t, \varepsilon), u(t, \varepsilon), p(t)) - p_i(t) \dot{x}(t, \varepsilon)] dt.$$
Differentiating with respect to $\varepsilon$, evaluating at $\varepsilon = 0$, and using the formula of integration by parts, it follows

$$
I'(0) = \int_0^\varepsilon [H(x(t), \dot{u}(t), p(t)) + \dot{p}_j(t)] \cdot x^i(t, 0) dt
$$

$$
+ \int_0^\varepsilon H_{\theta}(x(t), \dot{u}(t), p(t)) \cdot h^\beta(t) dt - \left( p_j(t) \cdot x^i(t, 0) \right) |^0\varepsilon.
$$

where $x(t)$ is the state variable corresponding to the optimal control variable $\dot{u}(t)$. We must have $I'(0) = 0$, for any continuous vector function $h(t) = h^\beta(t)$, $\beta = 1, r$. Also, the functions $x^i(t, 0)$ solve the following Cauchy problem

$$
\nabla x^i(t, 0) = a^k d^k (x(t, 0), u(t)) \cdot x^i(t, 0) + d^k k (x(t, 0), u(t)) \cdot h(t)
$$

$t \in [0, t_0]$, $x^i(0, 0) = 0$.

Consequently, we obtain

$$
\frac{\partial H}{\partial u^\beta} (x(t), \dot{u}(t), p(t)) = 0, \quad \forall t \in [0, t_0]. \tag{2.1}
$$

Using the adjoint differential system introduced in Sect. 1.1, we define $P$ as the set of solutions for the following problem

$$
\dot{p}_j(t) = -\frac{\partial H}{\partial x^i_j} (x(t), \dot{u}(t), p(t)), \quad p_j(t_0) = 0, \quad \forall t \in [0, t_0]. \tag{2.2}
$$

Moreover, we get

$$
x^i(t) = \frac{\partial H}{\partial p_j} (x(t), \dot{u}(t), p(t)), \quad x(0) = x_0, \quad \forall t \in [0, t_0]. \tag{2.3}
$$

**Remark 2.1.** The algebraic system (2.1) describes the critical points of the control Hamiltonian $H$ with respect to the control vector $u = (u^\beta)$.

Now, taking into account the previous computations, we are able to formulate the main result of this paper.

**Theorem 2.2. (Simplified maximum principle)** Let assume that the problem of maximizing the functional (1.1), subject to the signomial constraints (1.2) and to the conditions (1.3), with $X$, $v^k$ of $C^1$-class, has an interior solution $\dot{u}(t) \in IntU$ determining the optimal state variable $x(t) = (x^i(t))$. Then there exists the $C^1$-class co-state variable $p = (p_i)$, defined on the closed interval $[0, t_0]$, such that the relations (1.2), (2.1), (2.2) and (2.3) hold.

Further, by using the new Lagrange function $L$ and the above mentioned theorem, the following result is obvious.

**Corollary 2.3.** Consider the problem of maximizing the functional (1.1), subject to the signomial constraints (1.2) and to the conditions (1.3), with $X$, $v^k$ of $C^1$-class, has an interior solution $\dot{u}(t) \in IntU$ determining the optimal state variable $x(t) = (x^i(t))$. Then there exists the $C^1$-class co-state variable $p = (p_i)$, defined on the closed interval $[0, t_0]$, such that

$$
x^i(t) = a^k d^k (x(t), u(t)), \quad i = 1, n, \quad k = 1, m
$$

and the following Euler-Lagrange ODEs associated with the Lagrangian $L$

$$
\frac{\partial L}{\partial u^\beta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{u}^\beta} = 0, \quad \beta = 1, r
$$

$$
\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = 0, \quad \frac{\partial L}{\partial p_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{p}_i} = 0, \quad i = 1, n
$$

are satisfied.
3. Conclusion and further development

In this paper, using the concepts of variational differential system, adjoint differential system and modified Legendrian duality, we have formulated and proved a simplified maximum principle associated with a signomial constrained optimal control problem. An immediate perspective of the present paper is to obtain the Euler-Lagrange and Hamilton ODEs, with many applications in Optimization Theory and Mechanics.

References