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On Signomial Constrained Optimal Control Problems

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Abstract
In this paper, using the notions of *variational differential system*, *adjoint differential system* and *modified Legendrian duality*, we formulate and prove necessary optimality conditions in signomial constrained optimal control problems.

Keywords: Optimal control, Maximum principle, Variational differential system, Adjoint differential system, Modified Legendrian duality.
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1. Introduction and problem formulation

Optimal control theory (see Lee and Markus [1], Pontriaguine *et al.* [2], Evans [3]), due to important applications in various branches of pure and applied science, has attracted many researchers over the years. Wagner [4] established a Pontryagin-type maximum principle associated with some Dieudonné-Rashevsky type problems governed by Lipcshitz functions. Later, Udrişte [5], using the *multi-time* concept, formulated and proved, under the simplified hypothesis, a maximum principle based on multiple/curvilinear integral cost functional and m-flow type PDE constraints. Treanţă and Vârsan [6] derived that solutions associated with an extended affine control system can be obtained as a limit process using solutions for a parameterized affine control system and weak small controls.

In this paper, taking into account Treanţă and Udrişte [7] and Treanţă [8], we introduce necessary conditions of optimality for a new class of optimal control problems involving signomial type constraints. For other different but connected points of view regarding this subject, the reader is directed to Mititelu and Treanţă [9] and Treanţă [10, 11].

In the following, for $x = (x^1,...,x^n) \in R^n$ we shall write $x > 0$ if $x^i > 0,\ i = \overline{1,n}$, and $x \geq 0$ if $x^i \geq 0,\ i = \overline{1,n}$. The set $R^n_+ = \{x \in R^n : x \geq 0\}$ is said to be the *positive orthant* and, most of the times, we shall use the *open positive orthant* $\mathbf{P}^n = \{x \in R^n : x > 0\}$. On the set $\mathbf{P}^n$, we consider the distinct monomials of the form $v^k = v^k(x) = (x^1)^{\alpha_{1k}} \cdots (x^n)^{\alpha_{nk}},\ k = \overline{1,m}$, where α_{ik} are real numbers. If a^i_k are real numbers, then the functions $a^i_k v^k$, with summation upon k, are called *signomials*. The *controlled signomial dynamical systems* are defined as follows:

$$\dot{x}^i(t) = a^i_k v^k\left(x(t),u(t)\right), \quad i = \overline{1,n},$$

where $v^k(x,u) := (x^1)^{\alpha_{1k}} \cdots (x^n)^{\alpha_{nk}} (u^1)^{\gamma_{1k}} \cdots (u^r)^{\gamma_{rk}},\ \alpha_{ik}, \gamma_{\beta k} \in R,\ k = \overline{1,m},\ i = \overline{1,n},\ \beta = \overline{1,r},\ t \in I \subseteq R$, and $\mathbf{P}^r \ni u = (u^\beta),\ \beta = \overline{1,r}$, is a *control*.

Further, let us consider an optimal control problem based on a simple integral cost functional, constrained by a controlled signomial dynamical system:

$$\max_{u(\cdot),x_{t_0}} I(u(\cdot)) = \int_0^{t_0} X(x(t),u(t))\,dt \tag{1.1}$$

subject to

$$\dot{x}^i(t) = a^i_k v^k(x(t),u(t)), \quad i=\overline{1,n},\ k=\overline{1,m} \tag{1.2}$$

$$u(t)\in U,\ \forall t\in[0,t_0]; \quad x(0)=x_0,\ x(t_0)=x_{t_0}. \tag{1.3}$$

In the aforementioned optimal control problem we used the following terminology and notations: $t\in[0,t_0]$ is *parameter of evolution*, or the *time*; $[0,t_0]\subset R_+$ is the *time interval*; $x:[0,t_0]\to\mathbf{P}^n$, $x(t)=(x^i(t)),\ i=\overline{1,n}$, is a C^2-class function, called *state vector*; $u:[0,t_0]\to\mathbf{P}^r$, $u(t)=(u^\beta(t)),\ \beta=\overline{1,r}$, is a continuous *control vector*; U is the set of all admissible controls; the *running cost* $X(x(t),u(t))$ is a C^1-class function, called *autonomous Lagrangian*.

Through this work, the summation over the repeated indices is assumed. Further, we introduce the *Lagrange multiplier* $p(t)=(p_i(t))$, also called *co-state variable (vector)*, and a new Lagrange function

$$L(x(t),u(t),p(t)) = X(x(t),u(t)) + p_i(t)\left[a^i_k v^k(x(t),u(t)) - \dot{x}^i(t)\right].$$

In this way, we change the initial optimal control problem into a free optimization problem

$$\max_{u(\cdot),x_{t_0}} \int_0^{t_0} L(x(t),u(t),p(t))\,dt$$

subject to

$$u(t)\in U,\ p(t)\in P,\ \forall t\in[0,t_0]$$

$$x(0)=x_0,\ x(t_0)=x_{t_0},$$

where P is the set of co-state variables, which will be defined later. The *control Hamiltonian*

$$H(x(t),u(t),p(t)) = X(x(t),u(t)) + a^i_k p_i(t) v^k(x(t),u(t)),$$

or, equivalently,

$$H = L + p_i\dot{x}^i \quad (modified\ Legendrian\ duality)$$

permits us to rewrite the previous optimal control problem as follows

$$\max_{u(\cdot),x_{t_0}} \int_0^{t_0} \left[H(x(t),u(t),p(t)) - p_i(t)\dot{x}^i(t)\right] dt$$

subject to

$$u(t)\in U,\ p(t)\in P,\ \forall t\in[0,t_0]$$

$$x(0)=x_0,\ x(t_0)=x_{t_0}.$$

1.1 Variational and adjoint differential systems

Let us suppose that (1.2) is satisfied. Fix the control $u(t)$ and a corresponding solution $x(t)$ of (1.2). Let $x(t,\varepsilon)$ be a differentiable variation of the state variable $x(t)$, fulfilling

$$\dot{x}^i(t,\varepsilon) = a^i_k v^k(x(t,\varepsilon),u(t))$$

$$x(t,0) = x(t), \quad i = \overline{1,n}.$$

Denote by $y^i(t) := x^i_\varepsilon(t,0)$. Taking the partial derivative with respect to ε, evaluating at $\varepsilon = 0$, we obtain the following system

$$\dot{y}^i(t) = a^i_k v^k_{x^j}(x(t),u(t)) \cdot y^j(t),$$

called *variational differential system*. The differential system

$$\dot{p}_j(t) = -a^i_k p_i(t) v^k_{x^j}(x(t),u(t))$$

is called the *adjoint differential system* of the previous variational differential system since the scalar product $p_i(t)\cdot y^i(t)$ is a first integral of the two systems. Indeed, we have

$$\frac{d}{dt}\left[p_i(t)\cdot y^i(t)\right] = 0.$$

2. Main result

Let $\hat{u}(t) = \left(\hat{u}^\beta(t)\right)$, $\beta = \overline{1,r}$, be a continuous control vector defined on the closed interval $[0,t_0]$, with $\hat{u}(t) \in Int\,U$, which is an optimal point for the aforementioned control problem. Consider $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t)$ a variation of the optimal control vector $\hat{u}(t)$, where h is an arbitrary continuous vector function. We have $\hat{u}(t) \in Int\,U$ and, since a continuous function on a compact interval $[0,t_0]$ is bounded, there exists $\varepsilon_h > 0$ such that $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t) \in Int\,U$, $\forall\, |\varepsilon| < \varepsilon_h$. This ε is a "small" parameter used in our variational arguments.

Define $x(t,\varepsilon)$ as the state variable corresponding to the control variable $u(t,\varepsilon)$, i.e.,

$$\dot{x}^i(t,\varepsilon) = a^i_k v^k(x(t,\varepsilon),u(t,\varepsilon)), \quad i = \overline{1,n},\ \forall t \in [0,t_0]$$

and $x(0,\varepsilon) = x_0$. As well, consider (for $|\varepsilon| < \varepsilon_h$) the function (integral with parameter)

$$I(\varepsilon) := \int_0^{t_0} X(x(t,\varepsilon),u(t,\varepsilon))\,dt.$$

Since $\hat{u}(t)$ is an optimal control variable we get $I(0) \geq I(\varepsilon)$, $\forall\, |\varepsilon| < \varepsilon_h$. Also, for any continuous vector function $p(t) = (p_i)(t) : [0,t_0] \to R^n$, we have

$$\int_0^{t_0} p_i(t)\left[a^i_k v^k(x(t,\varepsilon),u(t,\varepsilon)) - \dot{x}^i(t,\varepsilon)\right] dt = 0.$$

The variations involve

$$L(x(t,\varepsilon),u(t,\varepsilon),p(t)) = X(x(t,\varepsilon),u(t,\varepsilon))$$

$$+p_i(t)\left[a^i_k v^k(x(t,\varepsilon),u(t,\varepsilon)) - \dot{x}^i(t,\varepsilon)\right]$$

and the associated function (integral with parameter)

$$I(\varepsilon) = \int_0^{t_0} L(x(t,\varepsilon),u(t,\varepsilon),p(t))\,dt.$$

Now, assume that the co-state variable $p(t) = (p_i(t))$ is of C^1-class. The control Hamiltonian with variations

$$H(x(t,\varepsilon),u(t,\varepsilon),p(t)) = X(x(t,\varepsilon),u(t,\varepsilon)) + a^i_k p_i(t) v^k(x(t,\varepsilon),u(t,\varepsilon))$$

changes the above integral with parameter as follows

$$I(\varepsilon) = \int_0^{t_0} \left[H(x(t,\varepsilon),u(t,\varepsilon),p(t)) - p_i(t)\dot{x}^i(t,\varepsilon)\right] dt.$$

Differentiating with respect to ε, evaluating at $\varepsilon = 0$, and using the formula of integration by parts, it follows

$$I'(0) = \int_0^{t_0} \left[H_{x^j}\left(x(t), \hat{u}(t), p(t)\right) + \dot{p}_j(t)\right] \cdot x^j_\varepsilon(t,0)dt$$

$$+ \int_0^{t_0} H_{u^\beta}\left(x(t), \hat{u}(t), p(t)\right) \cdot h^\beta(t)dt - \left(p_i(t) \cdot x^i_\varepsilon(t,0)\right)|_0^{t_0},$$

where $x(t)$ is the state variable corresponding to the optimal control variable $\hat{u}(t)$. We must have $I'(0) = 0$, for any continuous vector function $h(t) = \left(h^\beta(t)\right)$, $\beta = \overline{1,r}$. Also, the functions $x^i_\varepsilon(t,0)$ solve the following Cauchy problem

$$\nabla_t x^i_\varepsilon(t,0) = a^i_k v^k_x\left(x(t,0), u(t)\right) \cdot x_\varepsilon(t,0) + a^i_k v^k_u\left(x(t,0), u(t)\right) \cdot h(t)$$

$$t \in [0,t_0], \quad x_\varepsilon(0,0) = 0.$$

Consequently, we obtain

$$\frac{\partial H}{\partial u^\beta}\left(x(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in [0,t_0]. \tag{2.1}$$

Using the adjoint differential system introduced in *Sect.* 1.1, we define P as the set of solutions for the following problem

$$\dot{p}_j(t) = -\frac{\partial H}{\partial x_j}\left(x(t), \hat{u}(t), p(t)\right), \quad p_j(t_0) = 0, \ \forall t \in [0,t_0]. \tag{2.2}$$

Moreover, we get

$$\dot{x}^j(t) = \frac{\partial H}{\partial p_j}\left(x(t), \hat{u}(t), p(t)\right), \quad x(0) = x_0, \ \forall t \in [0,t_0]. \tag{2.3}$$

Remark 2.1. *The algebraic system (2.1) describes the critical points of the control Hamiltonian H with respect to the control vector $u = (u^\beta)$.*

Now, taking into account the previous computations, we are able to formulate the main result of this paper.

Theorem 2.2. *(**Simplified maximum principle**) Let assume that the problem of maximizing the functional (1.1), subject to the signomial constraints (1.2) and to the conditions (1.3), with X, v^k of C^1-class, has an interior solution $\hat{u}(t) \in Int U$ determining the optimal state variable $x(t) = \left(x^i(t)\right)$. Then there exists the C^1-class co-state variable $p = (p_i)$, defined on the closed interval $[0,t_0]$, such that the relations (1.2), (2.1), (2.2) and (2.3) hold.*

Further, by using the new Lagrange function L and the above mentioned theorem, the following result is obvious.

Corollary 2.3. *Consider the problem of maximizing the functional (1.1), subject to the signomial constraints (1.2) and to the conditions (1.3), with X, v^k of C^1-class, has an interior solution $\hat{u}(t) \in Int U$ determining the optimal state variable $x(t) = \left(x^i(t)\right)$. Then there exists the C^1-class co-state variable $p = (p_i)$, defined on the closed interval $[0,t_0]$, such that*

$$\dot{x}^i(t) = a^i_k v^k\left(x(t), u(t)\right), \quad i = \overline{1,n}, \ k = \overline{1,m}$$

and the following Euler-Lagrange ODEs associated with the Lagrangian L

$$\frac{\partial L}{\partial u^\beta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}^\beta} = 0, \quad \beta = \overline{1,r}$$

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0, \quad \frac{\partial L}{\partial p_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i} = 0, \quad i = \overline{1,n}$$

are satisfied.

3. Conclusion and further development

In this paper, using the concepts of *variational differential system*, *adjoint differential system* and *modified Legendrian duality*, we have formulated and proved a simplified maximum principle associated with a signomial constrained optimal control problem. An immediate perspective of the present paper is to obtain the Euler-Lagrange and Hamilton ODEs, with many applications in Optimization Theory and Mechanics.

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