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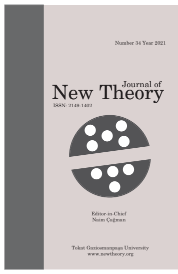


# On Ostrowski-Type Inequalities via Strong $s$-Godunova-Levin Functions

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**Abstract** — In this paper, we first introduce a new class of convex functions called strong $s$-Godunova-Levin functions, which encompass the strong Godunova-Levin, $s$-Godunova-Levin, and Godunova-Levin function classes. By relying on the identity given by Cerone *et al.* [Demonstratio Math., 37 (2004))] and by some simple technical methods, we derive some new Ostrowski-type inequalities for functions whose derivatives in absolute value at a certain power $q \geq 1$ lies in the above cited new class of functions. Some special cases are discussed. The results obtained can be considered a generalization of certain known results.

***Keywords*** — *Ostrowski inequality, Hölder inequality, power mean inequality, strong s-Godunova-Levin functions*

***Mathematics Subject Classification (2020)*** — 26D10, 26D15

## 1. Introduction

Let $H$ be an interval in $\mathbb{R}$. The following concepts are known in the literature.

**Definition 1.1.** [1] A function $\eta : H \to \mathbb{R}$ is said to be convex if

$$\eta\,(tx + (1-t)\,y) \leq t\eta\,(x) + (1-t)\,\eta\,(y)$$

holds for all $x, y \in H$ and $t \in [0,1]$.

**Definition 1.2.** [2] A function $\eta : H \to \mathbb{R}$ is called strongly convex with modulus $c$ if

$$\eta\,(tx + (1-t)\,y) \leq t\eta\,(x) + (1-t)\,\eta\,(y) - ct(1-t)\,|x-y|^2$$

holds for all $x, y \in H$ and $t \in (0,1)$.

**Definition 1.3.** [3] A nonnegative function $\eta : H \to \mathbb{R}$ is said to be $p$-convex if

$$\eta\,(tx + (1-t)\,y) \leq \eta\,(x) + \eta(y)$$

holds for all $x, y \in H$ and all $t \in [0,1]$.

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**Definition 1.4.** [4] A nonnegative function $\eta : H \to \mathbb{R}$ is said to be strongly $p$-convex if

$$\eta\left(tx+\left(1-t\right)y\right) \leq \eta\left(x\right)+\eta(y)-ct(1-t)\left|x-y\right|^{2}$$

holds for all $x, y \in H$ and all $t \in [0,1]$.

**Definition 1.5.** [5] A function $\eta : H \to [0,+\infty)$ is said to be Godunova-Levin function if

$$\eta\left(tx+\left(1-t\right)y\right) \leq \frac{\eta(x)}{t}+\frac{\eta(y)}{1-t}$$

holds for all $x, y \in H$ and $t \in (0,1)$.

**Definition 1.6.** [4] A function $\eta : H \to [0,+\infty)$ is said to be strong Godunova-Levin function if

$$\eta\left(tx+\left(1-t\right)y\right) \leq \frac{\eta(x)}{t}+\frac{\eta(y)}{1-t}-ct(1-t)\left|x-y\right|^{2}$$

holds for all $x, y \in H$ and all $t \in (0,1)$.

**Definition 1.7.** [6] A function $\eta : H \to [0,+\infty)$ is said to be $s$-Godunova-Levin function, where $s \in [0,1]$, if

$$\eta\left(tx+\left(1-t\right)y\right) \leq \frac{\eta(x)}{t^{s}}+\frac{\eta(y)}{(1-t)^{s}}$$

holds for all $x, y \in H$ and all $t \in (0,1)$.

The most important inequality to study the error estimation for different numerical quadrature rules is undoubtedly that known as the Ostrowski inequality which can be stated as follows:

**Theorem 1.8.** [7] Let $\eta : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$ is an interval, be a mapping, and $e, g \in U^{\circ}$, with $e < g$. If $|\eta'| \leq \mathbb{M}$ for all $x \in [e,g]$, then

$$\left|\eta(x)-\frac{1}{g-e}\int_{e}^{g}\eta\left(t\right)dt\right| \leq \mathbb{M}\left(g-a\right)\left[\frac{1}{4}+\frac{\left(x-\frac{e+g}{2}\right)^{2}}{(g-e)^{2}}\right] \tag{1}$$

In recent decades, the inequality (1) has generated much interest from researchers, several papers dealing with its generalizations and extensions has appeared, see [8–19], and references have been cited therein.

In [20], Cerone and Dragomir have shown the following identity:

**Lemma 1.9.** [20] Let $\eta : H \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $H^{\circ}$, where $e, g \in H$ with $e < g$. If $\eta' \in L[e,g]$, then

$$\eta\left(\upsilon\right)-\frac{1}{g-e}\int_{e}^{g}\eta\left(u\right)du=\frac{(x-e)^{2}}{g-e}\int_{0}^{1}t\eta'\left(t\upsilon+\left(1-t\right)a\right)dt-\frac{(g-x)^{2}}{g-e}\int_{0}^{1}t\eta'\left(t\upsilon+\left(1-t\right)g\right)dt$$

for each $\upsilon \in [e,g]$.

Based on the above lemma, they have established some Ostrowski-type inequalities via different types of convexity. We cited the results therein.

**Theorem 1.10.** Let $\eta : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. If $|\eta'|$ is convex on $[a,x]$ and $[x,b]$, then

$$\left|\eta\left(x\right)-\frac{1}{b-a}\int_{a}^{b}\eta\left(u\right)du\right| \leq \frac{b-a}{6}\left[\left(\frac{x-a}{b-a}\right)^{2}\left|\eta'\left(a\right)\right|+\left(\frac{b-x}{b-a}\right)^{2}\left|\eta'\left(b\right)\right| + \left(1+2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2}\right)\left|\eta'\left(x\right)\right|\right]$$

In [21], Noor et al. have established the following Ostrowski-type inequalities for differential $s$-Godunova-Levin functions.

**Theorem 1.11.** Let $\eta : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$ and $\eta' \in L([a, b])$ for all $x \in [a, b]$. If $|\eta'|$ is $s$-Godunova-Levin function of the second kind and $|\eta'| \leq M$, then the following inequality holds,

$$\left| \eta(x) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{M\left((b-x)^2 + (x-a)^2\right)}{(b-a)(1-s)}$$

**Theorem 1.12.** Let $\eta : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$ and $\eta' \in L([a, b])$ for all $x \in [a, b]$. If $|\eta'|^q$ is $s$-Godunova-Levin function of the second kind where $q \geq 1$ and $|\eta'| \leq M$, then the following inequality holds,

$$\left| \eta(x) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{M\left((b-x)^2 + (x-a)^2\right)}{(b-a)(1-s)^{\frac{1}{q}}\, 2^{1-q}}$$

**Theorem 1.13.** Let $\eta : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$ and $\eta' \in L([a, b])$ for all $x \in [a, b]$. If $|\eta'|^q$ is $s$-Godunova-Levin function of the second kind where $q, p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $|\eta'| \leq M$, then the following inequality holds,

$$\left| \eta(x) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{M\left((b-x)^2 + (x-a)^2\right)}{(b-a)(1-s)^{\frac{1}{q}}\, (p+1)^{\frac{1}{p}}}$$

The last result is based on the Hölder inequality, which can be stated as follows:

**Theorem 1.14.** [22, Hölder Inequality for Integrals] Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f$ and $g$ are real functions defined on $[e, g]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[e, g]$, then

$$\int_e^g |f(u)\, g(u)|\, du \leq \left( \int_e^g |f(u)|^p\, du \right)^{\frac{1}{p}} \left( \int_e^g |g(u)|^q\, du \right)^{\frac{1}{q}}$$

with equality if and only if $\alpha\, |f(u)|^p = \beta\, |g(u)|^q$ almost everywhere for some constants $\alpha$ and $\beta$.

Motivated by the above and some other existing results, we establish some new Ostrowski-type inequalities for functions whose derivatives in absolute values lie in a new class of convex functions called strong $s$-Godunova-Levin functions.

## 2. Main Results

**Definition 2.1.** A function $\eta : H \to [0, +\infty)$ is said to be strong $s$-Godunova-Levin functions with modulus $c > 0$, where $s \in [0, 1]$, if

$$\eta(tx + (1-t)y) \leq \frac{\eta(x)}{t^s} + \frac{\eta(y)}{(1-t)^s} - ct(1-t)|x-y|^2$$

holds for all $x, y \in H$ and all $t \in (0, 1)$.

**Remark 2.2.** Clearly all strong $s$-Godunova-Levin functions is $s$-Godunova-Levin functions. Moreover, we note that Definition 2.1 recaptures all definitions cited above by fixing the value of $s$ or by tending $c$ towards 0, with exception of Definitions 1 and 2.

**Theorem 2.3.** Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$, where $a<b$, and $\eta' \in L[a,b]$. If $|\eta'|$ is strong $s$-Godunova-Levin functions with modulus $c>0$, where $s \in [0,1)$, then the following inequality

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(a)|}{(1-s)(2-s)}\right) + \frac{(b-x)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(b)|}{(1-s)(2-s)}\right) - \frac{c}{12}\left(\frac{(b-x)^4+(x-a)^4}{b-a}\right)$$

holds for all $x \in [a,b]$.

Proof. From Lemma 1.9, modulus, and strong $s$-Godunova-Levin convexity of $|\eta'|$, we obtain

$$\begin{aligned}
\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| &\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)b)\right|dt \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left(\frac{|\eta'(x)|}{t^s} + \frac{|\eta'(a)|}{(1-t)^s} - ct(1-t)(x-a)^2\right)dt \\
&\quad + \frac{(b-x)^2}{b-a}\int_0^1 t\left(\frac{|\eta'(x)|}{t^s} + \frac{|\eta'(b)|}{(1-t)^s} - ct(1-t)(b-x)^2\right)dt \\
&= \frac{(x-a)^2}{b-a}\left(\left|\eta'(x)\right|\int_0^1 t^{1-s}dt + \left|\eta'(a)\right|\int_0^1 t(1-t)^{-s}dt \right.\\
&\quad \left. - c(x-a)^2\int_0^1 t^2(1-t)\,dt\right) \\
&\quad + \frac{(b-x)^2}{b-a}\left(\left|f'(x)\right|\int_0^1 t^{1-s}dt + \left|f'(b)\right|\int_0^1 t(1-t)^{-s}dt \right.\\
&\quad \left. - c(b-x)^2\int_0^1 t^2(1-t)\,dt\right) \\
&= \frac{(x-a)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(a)|}{(1-s)(2-s)}\right) + \frac{(b-x)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(b)|}{(1-s)(2-s)}\right) \\
&\quad - \frac{c}{12}\left(\frac{(b-x)^4+(x-a)^4}{b-a}\right)
\end{aligned}$$

The proof is completed. □

**Corollary 2.4.** In Theorem 2.3, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{4(1-s)(2-s)}\left(\left|\eta'(a)\right| + 2(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right| + \left|\eta'(b)\right|\right) - \frac{c(b-a)^3}{96}$$

**Corollary 2.5.** In Theorem 2.3, if we tend $c$ to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(a)|}{(1-s)(2-s)}\right) + \frac{(b-x)^2}{b-a}\left(\frac{(1-s)|\eta'(x)|+|\eta'(b)|}{(1-s)(2-s)}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{4(1-s)(2-s)}\left(\left|f'(a)\right| + 2(1-s)\left|f'\left(\frac{a+b}{2}\right)\right| + \left|f'(b)\right|\right)$$

**Remark 2.6.** In Corollary 2.5, if we assume that $|\eta'(u)| \leq M$, then the first inequality recaptures Corollary 3.1 from [21].

**Corollary 2.7.** In Theorem 2.3, if we take $s = 0$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\left(\frac{|\eta'(x)|+|\eta'(a)|}{2}\right) + \frac{(b-x)^2}{b-a}\left(\frac{|\eta'(x)|+|\eta'(b)|}{2}\right) - \frac{c}{12}\left(\frac{(b-x)^4+(x-a)^4}{b-a}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{8}\left(\left|\eta'(a)\right| + 2\left|\eta'\left(\frac{a+b}{2}\right)\right| + \left|\eta'(b)\right| - \frac{c(b-a)^2}{12}\right)$$

**Corollary 2.8.** In Corollary 2.7, if we tend $c$ to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\left(\frac{|\eta'(x)|+|\eta'(a)|}{2}\right) + \frac{(b-x)^2}{b-a}\left(\frac{|\eta'(x)|+|\eta'(b)|}{2}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{8}\left(\left|\eta'(a)\right| + 2\left|\eta'\left(\frac{a+b}{2}\right)\right| + \left|\eta'(b)\right|\right)$$

**Theorem 2.9.** Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$, and $\eta' \in L[a,b]$. If $|\eta'|^q$ is strong $s$-Godunova-Levin functions with modulus $c > 0$, where $s \in [0,1)$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{b-a}\right)^2\left(\frac{|\eta'(x)|^q+|\eta'(a)|^q}{1-s} - \frac{c}{6}(x-a)^2\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{b-x}{b-a}\right)^2\left(\frac{|\eta'(x)|^q+|\eta'(b)|^q}{1-s} - \frac{c}{6}(b-x)^2\right)^{\frac{1}{q}}\right)$$

holds for all $x \in [a,b]$.

Proof. From Lemma 1.9, properties of modulus, and Hölder inequality, we have

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|\eta'(tx+(1-t)b)\right|dt$$
$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\eta'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\eta'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leq \frac{1}{(p+1)^{\frac{1}{p}}} \left( \frac{(x-a)^2}{b-a} \left( \int_0^1 \left| \eta' (tx + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \frac{(b-x)^2}{b-a} \left( \int_0^1 \left| \eta' (tx + (1-t)b) \right|^q dt \right)^{\frac{1}{q}} \right) \\
&\leq \frac{1}{(p+1)^{\frac{1}{p}}} \left( \frac{(x-a)^2}{b-a} \left( \int_0^1 \left( \frac{|\eta'(x)|^q}{t^s} + \frac{|\eta'(a)|^q}{(1-t)^s} - ct(1-t)(x-a)^2 \right) dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \frac{(b-x)^2}{b-a} \left( \int_0^1 \left( \frac{|\eta'(x)|^q}{t^s} + \frac{|\eta'(b)|^q}{(1-t)^s} - ct(1-t)(b-x)^2 \right) dt \right)^{\frac{1}{q}} \right) \\
&= \frac{1}{(p+1)^{\frac{1}{p}}} \left( \frac{(x-a)^2}{b-a} \left( \left| \eta'(x) \right|^q \int_0^1 t^{-s} dt + \left| \eta'(a) \right|^q \int_0^1 (1-t)^{-s} dt \right. \right. \\
&\quad \left. - c(x-a)^2 \int_0^1 t(1-t)\, dt \right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^2}{b-a} \left( \left| \eta'(x) \right|^q \int_0^1 t^{-s} dt + \left| \eta'(b) \right|^q \int_0^1 (1-t)^{-s} dt \right. \\
&\quad \left. \left. - c(b-x)^2 \int_0^1 t(1-t)\, dt \right)^{\frac{1}{q}} \right) \\
&= \frac{b-a}{(p+1)^{\frac{1}{p}}} \left( \left( \frac{x-a}{b-a} \right)^2 \left( \frac{|\eta'(x)|^q + |\eta'(a)|^q}{1-s} - \frac{c}{6}(x-a)^2 \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left( \frac{b-x}{b-a} \right)^2 \left( \frac{|\eta'(x)|^q + |\eta'(b)|^q}{1-s} - \frac{c}{6}(b-x)^2 \right)^{\frac{1}{q}} \right)
\end{aligned}
$$

The proof is completed. □

**Corollary 2.10.** In Theorem 2.9, if we choose $x = \frac{a+b}{2}$, we obtain

$$
\begin{aligned}
\left| \eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq & \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left( \left( \frac{|\eta'(a)|^q + \left|\eta'\left(\frac{a+b}{2}\right)\right|^q}{1-s} - \frac{c(b-a)^2}{24} \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \frac{\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q}{1-s} - \frac{c(b-a)^2}{24} \right)^{\frac{1}{q}} \right)
\end{aligned}
$$

**Corollary 2.11.** In Theorem 2.9, if we tend $c$ to 0, i.e. $c \to 0^+$, we obtain

$$
\left| \eta(x) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}} \left( \left( \frac{x-a}{b-a} \right)^2 \left( \frac{|\eta'(x)|^q + |\eta'(a)|^q}{1-s} \right)^{\frac{1}{q}} + \left( \frac{b-x}{b-a} \right)^2 \left( \frac{|\eta'(x)|^q + |\eta'(b)|^q}{1-s} \right)^{\frac{1}{q}} \right)
$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$
\left| \eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \eta(u)\, du \right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left( \left( \frac{\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(a)|^q}{1-s} \right)^{\frac{1}{q}} + \left( \frac{\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q}{1-s} \right)^{\frac{1}{q}} \right)
$$

**Remark 2.12.** In Corollary 2.11, if we assume that $|\eta'(u)| \leq M$, then the first inequality gives

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}}\left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right)M,$$

which is the correct result of Corollary 3.2 from [21].

**Corollary 2.13.** In Theorem 2.9, if we take $s = 0$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{b-a}\right)^2\left(\left|\eta'(x)\right|^q + \left|\eta'(a)\right|^q - \frac{c}{6}(x-a)^2\right)^{\frac{1}{q}} + \left(\frac{b-x}{b-a}\right)^2\left(\left|\eta'(x)\right|^q + \left|\eta'(b)\right|^q - \frac{c}{6}(b-x)^2\right)^{\frac{1}{q}}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(a)\right|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}} + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(b)\right|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right)$$

**Corollary 2.14.** In Corollary 2.13, if we tend $c$ to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{b-a}\right)^2\left(\left|\eta'(x)\right|^q + \left|\eta'(a)\right|^q\right)^{\frac{1}{q}} + \left(\frac{b-x}{b-a}\right)^2\left(\left|\eta'(x)\right|^q + \left|\eta'(b)\right|^q\right)^{\frac{1}{q}}\right)$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(a)\right|^q\right)^{\frac{1}{q}} + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(b)\right|^q\right)^{\frac{1}{q}}\right)$$

**Theorem 2.15.** Let $\eta : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ where $a < b$, and $\eta' \in L[a,b]$. If $|\eta'|^q$ strong $s$-Godunova-Levin functions with modulus $c > 0$, where $s \in [0,1)$ and $q > 1$, then we have

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(a)|^q}{(1-s)(2-s)} - \frac{c(x-a)^2}{12}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(b)|^q}{(1-s)(2-s)} - \frac{c(b-x)^2}{12}\right)^{\frac{1}{q}}$$

for all $x \in [a,b]$.

Proof. From Lemma 1.9, properties of modulus, and power mean inequality, we get

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|\eta'(tx + (1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|\eta'(tx + (1-t)b)\right|dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\eta'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\eta'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^2}{(b-a)2^{1-\frac{1}{q}}}\left(\int_0^1 t\left|\eta'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{(b-a)2^{1-\frac{1}{q}}}\left(\int_0^1 t\left|\eta'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\int_0^1 t\left(\frac{|\eta'(x)|^q}{t^s}+\frac{|\eta'(a)|^q}{(1-t)^s}-ct(1-t)(x-a)^2\right)dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\int_0^1 t\left(\frac{|\eta'(x)|^q}{t^s}+\frac{|\eta'(b)|^q}{(1-t)^s}-ct(1-t)(b-x)^2\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^2}{(b-a)2^{1-\frac{1}{q}}}\left(\left|\eta'(x)\right|^q\int_0^1 t^{1-s}dt+\left|\eta'(a)\right|^q\int_0^1 t(1-t)^{-s}dt\right.$$
$$\left.-c(x-a)^2\int_0^1 t^2(1-t)dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\left|\eta'(x)\right|^q\int_0^1 t^{1-s}dt+\left|\eta'(b)\right|^q\int_0^1 t(1-t)^{-s}dt\right.$$
$$\left.-c(b-x)^2\int_0^1 t^2(1-t)dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q+|\eta'(a)|^q}{(1-s)(2-s)}-\frac{c(x-a)^2}{12}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q+|\eta'(b)|^q}{(1-s)(2-s)}-\frac{c(b-x)^2}{12}\right)^{\frac{1}{q}}$$

The proof is completed. □

**Corollary 2.16.** In Theorem 2.15, if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b \eta(u)du\right| \leq \frac{b-a}{2^{3-\frac{1}{q}}}\left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q+|\eta'(a)|^q}{(1-s)(2-s)}-\frac{c(b-a)^2}{48}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2^{3-\frac{1}{q}}}\left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q+|\eta'(b)|^q}{(1-s)(2-s)}-\frac{c(b-a)^2}{48}\right)^{\frac{1}{q}}$$

**Corollary 2.17.** In Theorem 2.15, if we tend $c$ to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(a)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2^{1-\frac{1}{q}}(b-a)}\left(\frac{(1-s)|\eta'(x)|^q + |\eta'(b)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{2^{3-\frac{1}{q}}}\left(\left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(b)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}} + \left(\frac{(1-s)\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + |\eta'(a)|^q}{(1-s)(2-s)}\right)^{\frac{1}{q}}\right)$$

**Remark 2.18.** In Corollary 2.17, if we assume that $|\eta'(u)| \leq M$, then the first inequality recaptures Corollary 3.3 from [21].

**Corollary 2.19.** In Theorem 2.15, if we take $s = 0$, we obtain

$$\begin{aligned}\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq & \frac{(x-a)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(a)\right|^q - \frac{c(x-a)^2}{6}\right)^{\frac{1}{q}} \\ & + \frac{(b-x)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(b)\right|^q - \frac{c(b-x)^2}{6}\right)^{\frac{1}{q}}\end{aligned}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\begin{aligned}\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq & \frac{b-a}{8}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(a)\right|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(b)\right|^q - \frac{c(b-a)^2}{24}\right)^{\frac{1}{q}}\right)\end{aligned}$$

**Corollary 2.20.** In Corollary 2.19, if we tend $c$ to 0, i.e. $c \to 0^+$, we obtain

$$\left|\eta(x) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{(x-a)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(a)\right|^q\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2(b-a)}\left(\left|\eta'(x)\right|^q + \left|\eta'(b)\right|^q\right)^{\frac{1}{q}}$$

Moreover, if we choose $x = \frac{a+b}{2}$, we obtain

$$\left|\eta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \eta(u)\,du\right| \leq \frac{b-a}{8}\left(\left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(a)\right|^q\right)^{\frac{1}{q}} + \left(\left|\eta'\left(\frac{a+b}{2}\right)\right|^q + \left|\eta'(b)\right|^q\right)^{\frac{1}{q}}\right)$$

## 3. Conclusion

Ostrowski-type inequalities are of great importance when studying the error estimation for different numerical quadrature rules. It suffices to take for example $x = (a+b)/2$, and we obtain the rule of midpoint or fix some values of $x$ and use the triangular inequality to estimate the error of the Simpson rule and the Trapezoidal rule. In this study, we introduce the concept of strong $s$-Godunova-Levin functions and established new Ostrowski-type inequalities for this new class of functions and their associated corollaries. The results obtained generalize those of [22].

## Conflicts of Interest

The authors declare no conflict of interest.

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