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On the Hyperharmonic Function

Ayhan DIL

Akdeniz University, Faculty of Science, Department of Mathematics, 07050, Antalya, Turkey
(ORCID: https://orcid.org/0000-0003-1273-6704)

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Abstract: In this paper we investigate some properties of the Hyperharmonic function defined by

$$H_z^{(w)} = \frac{(z)_w}{z\Gamma(w)} (\Psi(z+w) - \Psi(w)), \quad w, z+w \in \mathbb{C} \setminus \left(\mathbb{Z}^- \cup \{0\}\right).$$

Using this definition we introduce harmonic numbers with complex index and we give some series of these numbers. Also formulas for the calculation of harmonic numbers with rational index are obtained. For the simplicity of differentiation we reorganized representation of $H_z^{(w)}$. With the help of this new form we get higher order derivatives of the Hyperharmonic function more easily. Besides these, owing to the fact that the Hyperharmonic function is composed of some important functions, we interested in properties and connections of it. We get connections between the Hyperharmonic function and trigonometric functions. Infinite product representation, integral representation and differentiation identities of this function are also obtained.

Hiperharmonik Fonksiyon Üzerine

Anahtar Kelimeler
Harmonik sayılar,
Hiperharmonik sayılar,
Gamma fonksiyonu,
Digamma fonksiyonu,
Beta fonksiyonu

Özet: Bu çalışmada

$$H_z^{(w)} = \frac{(z)_w}{z\Gamma(w)} (\Psi(z+w) - \Psi(w)), \quad w, z+w \in \mathbb{C} \setminus \left(\mathbb{Z}^- \cup \{0\}\right)$$

eşitliği ile tanımlanan Hiperharmonik fonksiyonun bazı özellikleri araştırılmıştır. Bu tanımdan faydalanarak karmaşık indeksli harmonik sayılar tanıtılmış ve bu sayıların bazı serileri verilmiştir. Ayrıca rasyonel indeksli harmonik sayıların hesaplanması için formüller elde edilmiştir. $H_z^{(w)}$ fonksiyonunun türevlerinin daha kolay hesaplanabilmesi için, mevcut gösterim yeniden düzenlenmiştir. Bu yeni gösterim yardımıyla Hiperharmonik fonksiyonun yüksek mertebeli türevleri daha kolay hesaplanabilmektedir. Bunların yanı sıra, Hiperharmonik fonksiyonun özel bazı fonksiyonların birleşimi biçiminde ifade edilebildiği gerçeğinden hareketle, bazı özellikleri ve bağlantıları çalışılmıştır. Hiperharmonik fonksiyonun trigonometrik fonksiyonlarla ilişkileri elde edilmiş, sonsuz çarpım gösterimi, integral gösterimi ve bazı türevsel özdeşlikleri verilmiştir.

1. Introduction

The n-th harmonic number H_n is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

J.H. Conway and R.K. Guy have defined the notion of hyperharmonic numbers ([4]). $H_n^{(0)} := \frac{1}{n}$, and for all $r \in \mathbb{Z}^+$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \tag{1}$$

be the n-th hyperharmonic number of order r. These numbers can be expressed by the binomial coefficients and the ordinary harmonic numbers ([4, 11]):

$$H_n^{(r)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1}). \tag{2}$$

Hyperharmonic numbers have been studied in a variety of contexts, including in Euler sums (see [5–7, 11, 14, 15, 17–19]).

Corresponding author: adil@akdeniz.edu.tr

In [11] Mező and Dil generalized (2) as:

$$\binom{k+r-1}{k} H_n^{(k+r)} = \binom{n+k}{n} H_{n+k}^{(r)} - \binom{n+k+r-1}{n} H_k^{(r)}. \tag{3}$$

The representation in (2) is quite interesting because the right-hand side can be written in terms of continuous function instead of discrete variables n and r. Starting with this point, we define Hyperharmonic function $H_z^{(w)}$ where $w, z+w \in \mathbb{C}\backslash(\mathbb{Z}^- \cup \{0\})$. In this paper we investigate properties of this function and introduce relations in which it plays particular roles.

2. Material and Method

The polygamma function of order m is a meromorphic function on $\mathbb{C}$ and defined as the $(m+1)-th$ derivative of the logarithm of the usual gamma function $\Gamma(z)$ as:

$$\Psi^{(m)}(z) := \frac{d^m}{dz^m}\Psi(z) = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z).$$

Here

$$\Psi^{(0)}(z) = \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

where $\Psi(z)$ is the digamma function. For all $m \geq 0$ the function $\Psi^{(m)}(z)$ is holomorphic on $\mathbb{C}\backslash(\mathbb{Z}^- \cup \{0\})$ ([13]). With the help of $\Gamma(z)$ and $\Psi(z)$ we can state the hyperharmonic number of order r as:

$$H_n^{(r)} = \frac{(n)_r}{n\Gamma(r)}(\Psi(n+r) - \Psi(r)) \tag{4}$$

where $(x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$ is the Pochhammer symbol.
Considering (4), Mező [10] defined the Hyperharmonic function as:

$$H_z^{(w)} = \frac{(z)_w}{z\Gamma(w)}(\Psi(z+w) - \Psi(w)), \tag{5}$$

where $w, z+w \in \mathbb{C}\backslash(\mathbb{Z}^- \cup \{0\})$. Using this definition, Mező computed the first derivatives of $H_z^{(w)}$ respect to variables z and w. For the simplicity of differentiation we reorganized representation of $H_z^{(w)}$. Using this new form we get the higher derivatives of $H_z^{(w)}$ more easily. We also consider the special case of $H_z^{(w)}$ as $H_z^{(1)} := H_z$, and we call H_z as a harmonic numbers with complex index. Some representations of H_z in terms of infinite series are given. Also formulas for the calculation of harmonic numbers with rational index are obtained. Besides these, owing to the fact that the Hyperharmonic function is a compose of some important functions, we interested in investigating properties and connections of it.

3. Results

3.1. Harmonic numbers with complex index

In this section we consider two special cases of the Hyperharmonic function $H_z^{(w)}$; these are $z = 1$ and $w = 1$. The case $z = 1$ is not interesting because we get

$$H_1^{(w)} := H^{(w)} = \frac{(1)_w}{\Gamma(w)}(\Psi(1+w) - \Psi(w)).$$

By considering the well-known identity ([16])

$$\Psi(1+w) = \Psi(w) + \frac{1}{w} \tag{6}$$

it turns out that

$$H^{(w)} = \frac{(1)_w}{w\Gamma(w)} = 1 \tag{7}$$

for any $w \in \mathbb{C}\backslash(\mathbb{Z}^- \cup \{0\})$.
On the other hand setting $w = 1$ in (5) we have

$$H_z^{(1)} := H_z = \frac{(z)_1}{z\Gamma(1)}(\Psi(z+1) - \Psi(1))$$

from which it follows that

$$H_z = \Psi(z+1) + \gamma, \tag{8}$$

where γ is the Euler-Mascheroni constant. So we extended the definition of harmonic numbers; more precisely we have "complex indexed harmonic numbers".
With the help of (8) and the following fractional values of Ψ function (see [1])

$$\Psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2$$

and

$$\Psi\left(\frac{1}{2} \pm n\right) = -\gamma - 2\ln 2 + 2\sum_{k=0}^{n-1}\frac{1}{2k+1}$$

we get "fractional indexed harmonic numbers":

$$H_{-\frac{1}{2}} = -2\ln 2$$

and for $n \geq 1$

$$H_{-\frac{1}{2}\pm n} = -2\ln 2 + 2\sum_{k=0}^{n-1}\frac{1}{2k+1}.$$

For instance:

$$H_{\frac{1}{2}} = H_{-\frac{3}{2}} = -2\ln 2 + 2,$$

$$H_{\frac{3}{2}} = H_{-\frac{5}{2}} = -2\ln 2 + \frac{8}{3}.$$

Actually we can give a more general result than above. Let p and q be positive integers such that $0 < p < q$, Gauss proved the following equation,

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \frac{\pi}{2}\cot\frac{\pi p}{q} - \ln q + 2\sum_{n=1}^{\lfloor \frac{q}{2} \rfloor}\cos\frac{2\pi np}{q}\ln\left(2\sin\frac{\pi n}{q}\right). \tag{9}$$

Here $\lfloor \frac{q}{2} \rfloor$ denotes the greatest integer in $\frac{q}{2}$. In the light of this equation one can calculate harmonic numbers with negative rational index as:

$$H_{\frac{p}{q}-1} = -\frac{\pi}{2}\cot\frac{\pi p}{q} - \ln q + 2\sum_{n=1}^{\lfloor \frac{q}{2} \rfloor}\cos\frac{2\pi np}{q}\ln\left(2\sin\frac{\pi n}{q}\right),$$

where $-1 < \frac{p}{q} - 1 < 0$. For instance

$$H_{-\frac{1}{3}} = -\frac{\pi\sqrt{3}}{6} - \frac{3}{2}\ln 3.$$

3.1.1. Some series expansions of H_z

Equation (8) is important because using series expansions of the digamma function $\Psi(z)$ we get series expansions for H_z.
For the function $\Psi(z+1)$ let us consider the following well-known series expansion,

$$\Psi(z+1) = -\gamma + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}, \qquad z \in \mathbb{C}\backslash\mathbb{Z}^-,$$

(see [1]) and Taylor series

$$\Psi(z+1) = -\gamma - \sum_{k=1}^{\infty} \zeta(k+1)(-z)^k, \qquad |z| < 1,$$

([2]) and Newton series

$$\Psi(s+1) = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\binom{s}{k}.$$

Then immediately we have

$$H_z = \sum_{k=1}^{\infty} \frac{z}{k(k+z)}, \qquad z \in \mathbb{C}\backslash\mathbb{Z}^-, \tag{10}$$

$$H_z = -\sum_{k=1}^{\infty} \zeta(k+1)(-z)^k, \qquad |z| < 1,$$

and

$$H_s = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\binom{s}{k}$$

respectively.

3.2. Differentiation identities of the Hyperharmonic function

In [10] Mező gave the first derivative of $H_z^{(w)}$ with respect to the variables w and z. In this section we consider the higher order derivatives of $H_z^{(w)}$.

Derivatives with respect to the variable w To obtain higher order derivatives we write another representation of $H_z^{(w)}$. Since

$$(z)_w = \frac{\Gamma(z+w)}{\Gamma(z)} \tag{11}$$

equation (5) yields

$$H_z^{(w)} = \frac{\Gamma(z+w)}{z\Gamma(z)\Gamma(w)}\left(\Psi(z+w) - \Psi(w)\right) \tag{12}$$

which can equally well be written

$$H_z^{(w)} = \frac{(w)_z}{\Gamma(z+1)}\left(\Psi(z+w) - \Psi(w)\right). \tag{13}$$

Equation (13) is more suitable form of $H_z^{(w)}$ to obtain derivatives with respect to w.

Let us note that

$$\frac{d}{dw}(w)_z = (w)_z\left(\Psi(z+w) - \Psi(w)\right). \tag{14}$$

In the light of (14) we have

$$\frac{d^{n+1}}{dw^{n+1}}(w)_z = \Gamma(z+1)\frac{d^n}{dw^n}H_z^{(w)} \tag{15}$$

Therefore to obtain the higher order derivatives of $H_z^{(w)}$ we can consider the higher order derivatives of $(w)_z$. For the sake of simplicity let us denote

$$\Phi(z,w) = \Psi(z+w) - \Psi(w).$$

As an example here we give the first few higher order derivatives of $(w)_z$ using the abbreviations $\Phi(z,w) = \Phi_w$;

$$\begin{aligned}
\frac{d}{dw}(w)_z &= (w)_z\Phi_w \\
\frac{d^2}{dw^2}(w)_z &= (w)_z\left\{\Phi_w^2 + \Phi_w^{(\prime)}\right\} \\
\frac{d^3}{dw^3}(w)_z &= (w)_z\left\{\Phi_w^3 + 3\Phi_w^{(\prime)}\Phi_w + \Phi_w^{(\prime\prime)}\right\} \\
\frac{d^4}{dw^4}(w)_z &= (w)_z\left\{\Phi_w^4 + 6\Phi_w^2\Phi_w^{(\prime)} + 4\Phi_w\Phi_w^{(\prime\prime)}\right. \\
&\qquad \left. + \Phi_w^{(\prime\prime\prime)} + 3\left(\Phi_w^{(\prime)}\right)^2\right\} \\
\frac{d^5}{dw^5}(w)_z &= (w)_z\left\{\Phi_w^5 + 10\Phi_w^3\Phi_w^{(\prime)} + 10\Phi_w^2\Phi_w^{(\prime\prime)}\right. \\
&\qquad \left. + 5\Phi_w\Phi_w^{(\prime\prime\prime)} + \Phi_w^{(iv)} + 10\Phi_w'\Phi_w'' + 15\Phi_w\left(\Phi_w'\right)^2\right\} \\
\frac{d^6}{dw^6}(w)_z &= (w)_z\left\{\Phi_w^6 + 15\Phi_w^4\Phi_w^{(\prime)} + 20\Phi_w^3\Phi_w^{(\prime\prime)}\right. \\
&\qquad + 15\Phi_w^2\Phi_w^{(\prime\prime\prime)} + 6\Phi_w\Phi_w^{(iv)} + \Phi_w^{(v)} \\
&\qquad + 45\Phi_w^2\left(\Phi_w^{(\prime)}\right)^2 + 60\Phi_w\Phi_w^{(\prime)}\Phi_w^{(\prime\prime)} + 15\Phi_w^{(\prime)}\Phi_w^{(\prime\prime\prime)} \\
&\qquad \left. + 15\left(\Phi_w'\right)^3 + 10\left(\Phi_w^{(\prime\prime)}\right)^2\right\} \\
&\ \ \vdots
\end{aligned}$$

Using these information, for instance we have

$$\begin{aligned}
\frac{d}{dw}H_z^{(w)} &= H_z^{(w)}\left(\Psi(z+w) - \Psi(w)\right) \\
&\quad + \frac{(z)_w}{z\Gamma(w)}\left(\Psi'(z+w) - \Psi'(w)\right).
\end{aligned} \tag{16}$$

Remark 3.1. $\frac{d}{dz}H_z^{(w)}$ is also given in [10] but there is a misprint.

Derivatives with respect to the variable z Let us make a preparation to get the higher order derivatives of $H_z^{(w)}$ with respect to variable z.
Since

$$\frac{(z)_w}{z} = (z+1)_{w-1}$$

we have

$$H_z^{(w)} = \frac{(z+1)_{w-1}}{\Gamma(w)}\left(\Psi(z+w) - \Psi(w)\right). \tag{17}$$

Now we are ready to obtain the derivatives of $H_z^{(w)}$ with respect to z. For instance we have

$$\frac{d}{dz}H_z^{(w)} = \frac{1}{\Gamma(w)}\frac{d}{dz}\left[(z+1)_{w-1}\left(\Psi(z+w)-\Psi(w)\right)\right]. \tag{18}$$

Also we know that

$$\frac{d}{dw}(w)_z = (w)_z\Phi_w$$

so we get

$$\begin{aligned}\frac{d}{dz}H_z^{(w)} &= H_z^{(w)}\left(\Psi(z+w)-\Psi(z+1)\right)\\ &\quad+\frac{(z)_w}{z\Gamma(w)}\Psi'(z+w). \end{aligned}\tag{19}$$

Remark 3.2. Equation (19) also given in [10]. Considering (17) together with the derivatives of $(w)_z$ given before we obtain the higher order derivatives of $H_z^{(w)}$ more easily.

3.3. Relationship with some important special functions

Having regard to importance of the trigonometric functions we firstly investigate connections between the trigonometric functions and the Hyperharmonic function. To this aim we need to remind some well-known facts about the Gamma and the Digamma functions. One of facts is the Legendre's duplication formula for the gamma function ([16])

$$\Gamma(2z) = \pi^{-\frac{1}{2}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{20}$$

Also the Digamma function satisfies the following reflection formula ([2]),

$$\Psi(z)-\Psi(1-z) = -\pi\cot\pi z. \tag{21}$$

The following lemma shows a connection with the trigonometric functions.

Proposition 3.3. *For the non-integer values of z we have,*

$$H_{2z-1}^{(1-z)} = \frac{-\pi^{\frac{3}{2}}\cot\pi z}{2^{2z-1}\Gamma\left(z+\frac{1}{2}\right)\Gamma(1-z)}. \tag{22}$$

Proof. Replacing in equation (12) w by $1-z$ and z by $2z-1$, we obtain

$$H_{2z-1}^{(1-z)} = \frac{\Gamma(z)}{\Gamma(2z)\Gamma(1-z)}\left(\Psi(z)-\Psi(1-z)\right)$$

which combines with the functional relation (21) to give

$$H_{2z-1}^{(1-z)} = \frac{\Gamma(z)}{\Gamma(2z)\Gamma(1-z)}\left(-\pi\cot\pi z\right).$$

Using the Legendre's duplication formula (20) for $\Gamma(2z)$ we obtain desired equation. □

Remark 3.4. As a result of Proposition 3.3, it can be seen easily that $H_{2n}^{(\frac{1}{2}-n)} = 0$ for all positive integer n.

We have one more relation with the trigonometric functions. For this we remind the following reflection formula for the gamma function ([2]):

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}. \tag{23}$$

Proposition 3.5. *For the non-integer values of z we have,*

$$H_z^{(1-z)} = -\frac{\sin\pi z}{\pi z}\left(\Psi(z)+\pi\cot\pi z+\gamma\right).$$

Proof. In (12) replacing w with $1-z$ we have

$$H_z^{(1-z)} = \frac{1}{z\Gamma(z)\Gamma(1-z)}\left(-\gamma-\Psi(1-z)\right). \tag{24}$$

Here using equation (21) and (23) we complete proof. □

Now as a result we give an infinite product representation of the Hyperharmonic function, for this recall that :

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \tag{25}$$

(see [3]).

Corollary 3.6. *We have the infinite product*

$$H_z^{(1-z)} = -\left(\Psi(z)+\pi\cot\pi z+\gamma\right)\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right). \tag{26}$$

As an application of this formula let us consider the case when z approaches to an $m\in\mathbb{Z}^+$. Then we write

$$\begin{aligned}H_m^{(1-m)} &= \prod_{\substack{n=1\\ n\neq m}}^{\infty}\left(1-\frac{m^2}{n^2}\right)\\ &\quad\times\lim_{z\to m} -\left(\Psi(z)+\pi\cot\pi z+\gamma\right)\left(1-\frac{z^2}{m^2}\right).\\ &= \frac{2}{m}\prod_{\substack{n=1\\ n\neq m}}^{\infty}\left(1-\frac{m^2}{n^2}\right).\end{aligned}$$

For example we have

$$H_1^{(0)} = 1 = 2\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$$

and

$$H_2^{(-1)} = -3\prod_{n=3}^{\infty}\left(1-\frac{4}{n^2}\right).$$

Proposition 3.7. *The following relation holds:*

$$H_{\frac{1}{q}}^{\left(\frac{p-1}{q}\right)} = \frac{q^2\Gamma\left(\frac{p}{q}\right)}{\Gamma\left(\frac{1}{q}\right)\Gamma\left(\frac{p-1}{q}\right)}\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+kq)^n-1}$$

where $p,q\notin\mathbb{Z}$ and $\frac{p}{q}\notin\mathbb{Z}^-$.

Proof. Replacing in equation (12) z by $\frac{1}{q}$ and w by $\frac{p-1}{q}$, we obtain

$$H_{\frac{1}{q}}^{\left(\frac{p-1}{q}\right)} = \frac{\Gamma\left(\frac{p}{q}\right)}{\frac{1}{q}\Gamma\left(\frac{1}{q}\right)\Gamma\left(\frac{p-1}{q}\right)}\left(\Psi\left(\frac{p}{q}\right)-\Psi\left(\frac{p-1}{q}\right)\right).$$

Also we remind the following known equation for the Digamma function ([8])

$$\Psi\left(\frac{p}{q}\right)-\Psi\left(\frac{p-1}{q}\right)=q\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+kq)^{n}-1}$$

In view of these two equations we get the desired result. □

We continue this section to obtaining relations of $H_z^{(w)}$ with two other important functions; $B(z,w)$ and $\beta(z)$. The function $B(z,w)$ is usual Beta function which has the following connections with the Gamma function:

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{27}$$

And the function $\beta(z)$ is defined as ([8]):

$$\beta(z)=\Psi\left(\frac{z+1}{2}\right)-\Psi\left(\frac{z}{2}\right). \tag{28}$$

It is time to present the Hypeharmonic function in terms of β and Γ functions.

Proposition 3.8. *We have*

$$H_{\frac{1}{2}}^{(z)}=2^{2-2z}\frac{\Gamma(2z)}{\Gamma^{2}(z)}\beta(2z).$$

Proof. Replacing in equation (12) z by $\frac{1}{2}$ and w by $\frac{z}{2}$, and considering (28) we obtain

$$H_{\frac{1}{2}}^{\left(\frac{z}{2}\right)}=\frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{z}{2}\right)}2\beta(z). \tag{29}$$

Besides, from the Legendre's duplication formula (20) we have

$$\Gamma\left(\frac{z+1}{2}\right)=2^{1-z}\sqrt{\pi}\frac{\Gamma(z)}{\Gamma\left(\frac{z}{2}\right)}$$

which combines with (29) to give desired result. □

The following proposition shows the relation between the Hyperharmonic function with the digamma and the Beta function.

Proposition 3.9. *We have*

$$H_z^{(w)}=\frac{(\Psi(z+w)-\Psi(w))}{zB(z,w)}. \tag{30}$$

Proof. Combining (27) and (12) we have (30). □

Equation (30) connects the Hyperharmonic function with the well-known the Beta function. This equation enables us to investigate some properties of the Hyperharmonic function considering properties of the Digamma and the Beta function. Also equation (30) is convenient to obtain special values of the Hyperharmonic function using information about $\Psi(z)$ and $B(z,w)$ functions. For example,

$$H_{\frac{1}{2}}^{\left(\frac{1}{2}\right)}=\frac{4\ln 2}{\pi},$$

$$H_{\frac{2}{3}}^{\left(\frac{1}{3}\right)}=\frac{27\ln 3}{8\pi\sqrt{3}}+\frac{3}{8},$$

and

$$H_{\frac{3}{4}}^{\left(\frac{1}{4}\right)}=\frac{4\ln 2}{\pi\sqrt{2}}+\frac{2}{3\sqrt{2}}.$$

Using equation (30) we obtain more general relation than these special values:

Corollary 3.10. *We have*

$$H_{1-z}^{(z)}=\frac{\sin\pi z}{\pi(z-1)}H_{z-1}.$$

Proof. Replacing z by $1-z$ and w by z in (30), we obtain

$$H_{1-z}^{(z)}=\frac{\Psi(z)+\gamma}{(z-1)B(z,1-z)}.$$

Remembering, $B(z,1-z)=\frac{\pi}{\sin\pi z}$ ([1]) and also with the help of equation (8) we get desired result. □

Yet another benefits of the Proposition 3.9 is obtaining some relations about the Hyperharmonic function considering properties of $B(z,w)$ function. The following proposition shows a symmetry between $H_z^{(w)}$ and $H_w^{(z)}$.

Corollary 3.11. *The Hyperharmonic function has the following symmetric relation:*

$$\frac{H_z^{(w)}}{H_w^{(z)}}=\frac{w}{z}\left(\frac{\Psi(z+w)-\Psi(w)}{\Psi(z+w)-\Psi(z)}\right). \tag{31}$$

Proof. (30) can be written as

$$B(z,w)=\frac{(\Psi(z+w)-\Psi(w))}{zH_z^{(w)}}. \tag{32}$$

Also we know from (27) that $B(z,w)$ is a symmetric function i.e.

$$B(z,w)=B(w,z). \tag{33}$$

In view of equation (32), equation (33) shows validity of (31). □

In the paper [11] authors gave a recurrence relation for the Hyperharmonic function. Here we obtain more general recurrence than (3).

Corollary 3.12. *The following recurrence holds for $H_z^{(w)}$*

$$H_z^{(w+1)}=H_z^{(w)}\frac{(z+w)(\Psi(z+w+1)-\Psi(w+1))}{w(\Psi(z+w)-\Psi(w))}. \tag{34}$$

Proof. Under the condition $\mathrm{Re}(z)>0$ and $\mathrm{Re}(w)>0$ the Beta function has the following recurrence:

$$B(z,w+1)=\frac{w}{z+w}B(z,w)$$

which combines with (32) to give the desired result. □

At the end of this section we give a relation between $H_z^{(w)}$ and the higher order Bernoulli polynomials given by the identity:

$$\frac{t^n e^{xt}}{(e^t-1)^n}=\sum_{v=0}^{\infty}B_v^{(n)}(x)\frac{t^v}{v!}.$$

Note that higher order Bernoulli polynomials satisfy the following relation (see [12]):

$$B_v^{(m+1)}(x)=\frac{v!}{m!}\frac{d^{m-v}}{dx^{m-v}}\left[(x-1)(x-2)\ldots(x-m)\right]. \tag{35}$$

Proposition 3.13. *Let m and v be positive integers such that $m>v+1$, then we have*

$$B_v^{(m+1)}(1-x)=(-1)^{m+1}v!\frac{d^{m-v-1}}{dx^{m-v-1}}H_m^{(x)}.$$

Proof. Considering (35) we see

$$B_v^{(m+1)}(1-x) = (-1)^{m+1}\frac{v!}{m!}\frac{d^{m-v}}{dx^{m-v}}(x)_m.$$

In the light of (15) we get the result. □

3.4. Integral representations of the Hyperharmonic function

For the real variable x, considering the series representation (10), we can state γ by a definite integral of H_x as

$$\int_0^1 H_x dx = \gamma$$

and also more generally

$$\int_0^n H_x dx = n\gamma + \ln(n!).$$

Besides, in the light of (8) and the following well-known equation (see [8])

$$\int_0^1 \frac{1-x^a}{1-x}x^{b-1}dx = \Psi(a+b) - \Psi(b) \text{ where } a,b\in\mathbb{R}$$

we get the following integral representation of $H_a^{(b)}$ as

$$H_a^{(b)} = \frac{(a)_b}{a\Gamma(b)}\int_0^1 \frac{1-x^a}{1-x}x^{b-1}dx.$$

As a result of this we have

$$H_a = \int_0^1 \frac{1-x^a}{1-x}dx.$$

In [9] authors proved that

$$\int_0^1 x^{b-1}(1-x)^{a-1}\ln x dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(\Psi(b) - \Psi(a+b))$$

where $a,b\in\mathbb{R}^+$. Using this equation with (12) we have

$$H_a^{(b)} = -\frac{1}{a}\left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right)^2\int_0^1 x^{b-1}(1-x)^{a-1}\ln x dx.$$

Setting $b=1$ we get

$$H_a = -a\int_0^1 (1-x)^{a-1}\ln x dx.$$

Now we give another integral relation for $H_n^{(y)}$:

Proposition 3.14. *Let $n\in\mathbb{Z}^+$ and $y\in\mathbb{R}\backslash(\mathbb{Z}^-\cup\{0\})$. Then*

$$\int H_n^{(y)}dy = \frac{(y)_n}{n!}$$

Proof. Considering (6) we write

$$\Psi(y+n) - \Psi(y) = \sum_{k=0}^{n-1}\frac{1}{y+k}$$

which combines with (13) to give

$$H_n^{(y)} = \frac{(y)_n}{n!}\sum_{k=0}^{n-1}\frac{1}{y+k}.$$

The above equation can be written as

$$H_n^{(y)} = \frac{d}{dy}\frac{(y)_n}{n!}$$

Hence the formula has been established. □

The Digamma function has the following integral representation ([1]):

$$\Psi(z) = -\gamma + \int_0^\infty \frac{e^{-t}-e^{-zt}}{1-e^{-t}}dt \tag{36}$$

Owing to this representation of the Ψ we have the following result.

Proposition 3.15. *We have*

$$H_z^{(w)} = \frac{(w)_z}{\Gamma(z+1)}\int_0^\infty \frac{e^{-wt}(1-e^{-zt})}{1-e^{-t}}dt. \tag{37}$$

Proof. Let us write the integral representations of $\Psi(z+w)$ and $\Psi(w)$ via equation (36). Hence we get

$$\Psi(z+w) - \Psi(w) = \int_0^\infty \frac{e^{-wt}(1-e^{-zt})}{1-e^{-t}}dt$$

which implies (37) after multiplying both sides of this equation by $\frac{(w)_z}{\Gamma(z+1)}$. □

4. Discussion and Conclusion

So far we present some formulas to calculate special values of $H_z^{(w)}$. Actually more general results can be obtained with the help of (9). In the light of (9) and (13) one can calculate all rational upper and lower indexed hyperharmonic numbers as:

$$H_{\frac{p_1}{q_1}}^{\left(\frac{p_2}{q_2}\right)} = \frac{\left(\frac{p_2}{q_2}\right)_z}{\Gamma\left(\frac{p_1}{q_1}+1\right)}\left(\Psi\left(\frac{p_1}{q_1}+\frac{p_2}{q_2}\right) - \Psi\left(\frac{p_2}{q_2}\right)\right).$$

In our work we present a way to obtain higher derivatives of the Hyperharmonic function with respect to the variables z and w. Closed formulas for these derivatives are open problems; they might be quite complicated.

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References

[1] Abramowitz, M., Stegun, I. 1972. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. Dover, New York, 1046s.

[2] Andrews, G. E., Askey, R., Roy, R. 2000. Special Functions, Cambridge University Press, 682s.

[3] Bak, J., Newman, D. J. 1997. Complex Analysis, Springer, 328s.

[4] Conway, J. H., Guy, R. K. 1996. The Book of Numbers, New York, Springer-Verlag, 310s.

[5] Dil, A., Mező, I., Cenkci, M. 2017. Evaluation of Euler-like sums via Hurwitz zeta values. Turk. J. Math., 41(6), 1640-1655.

[6] Dil A, Boyadzhiev KN. 2015. Euler sums of hyperharmonic numbers. J. Number Theory, 147: 490-498.

[7] Gaboury S. 2014. Further Expansion and Summation Formulas Involving the Hyperharmonic Function. Commun. Korean Math. Soc., 29 (2): 269-83.

[8] Gradshteyn, I. S., Ryzhik, I. M. 2007. Table of Integrals, Series, and Products, Elsevier Academic Press, USA, 1163s.

[9] Medina, L. A., Moll, V. H. 2009. The Integrals in Gradshteyn and Ryzhik. Part 10: The Digamma Function. SCIENTIA, Series A: Mathematical Sciences, Vol. 17, 45–66.

[10] Mező, I. 2009. Analytic Extension of Hyperharmonic Numbers. Online Journal of Analytic Combinatorics, Issue 4 (2009) .

[11] Mező, I., Dil, A. 2010. Hyperharmonic Series Involving Hurwitz Zeta Function. J. Number Theory, 130, 2 : 360-369.

[12] Milne-Thomson, L. M. 1965. The Calculus of Finite Differences. MacMillan & Co., 558s.

[13] Rainville, E. D. 1960. Special Functions. MacMillan, New York, 365s.

[14] Sofo, A., Srivastava, H. M. 2015. A Family of Shifted Harmonic Sums. Ramanujan J. 37, 89-108.

[15] Sofo, A. 2014. Shifted Harmonic Sums of Order Two. Commun. Korean Math. Soc. 29 (2), 239-255.

[16] Wang, Z. X., Guo, D. R. 1989. Special Functions, World Scientific, 720s.

[17] Xu, C. 2018. Euler Sums of Generalized Hyperharmonic Numbers. J. Korean Math. Soc. 55, No. 5, 1207-1220.

[18] Xu, C. 2018. Computation and Theory of Euler Sums of Generalized Hyperharmonic Numbers. C. R. Acad. Sci. Paris, Ser. I 356, 243-252.

[19] Xu, C. 2017. Identities for the Shifted Harmonic Numbers and Binomial Coeffcients. Filomat 31:19, 6071-6086.