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Degenerate Saccheri Quadrilaterals, Möbius Transformations and Conjugate Möbius Transformations

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Abstract

In this paper, we define a new geometric concept that we will call "degenerate Saccheri quadrilateral" and use it to give a new characterization of Möbius transformations. Our proofs are based on a geometric approach.

Keywords: *Saccheri quadrilateral, Möbius transformation*

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1. Introduction

Möbius transformations are rational functions of the form $f(z) = \frac{az+b}{cz+d}$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$. They are the automorphisms of extended complex plane $\overline{\mathbb{C}}$, that is, the meromorphic bijections $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Möbius transformations are also directly conformal homeomorphisms of $\overline{\mathbb{C}}$ onto itself and they have beautiful properties. For example, a map is Möbius if, and only if it preserves cross ratios. As for geometric aspect, circle-preserving is another important characterization of Möbius transformations. There are well-known elementary proofs that if f is a continuous injective map of $\overline{\mathbb{C}}$ that maps circles into circles, then f is Möbius. In addition to this the following result is well known and fundamental in complex analysis.

Theorem 1.1. *[1] If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a circle preserving map, then f is a Möbius transformation if and only if f is a bijection.*

The transformations $f(z) = \frac{a\overline{z}+b}{c\overline{z}+d}$ with $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$ are known as conjugate Möbius transformations of $\overline{\mathbb{C}}$. It is easy to see that each conjugate Möbius transformation f is the composition of complex conjugation with a Möbius transformation, since both of these are homeomorphisms of $\overline{\mathbb{C}}$ onto itself (complex conjugation being given by reflection in the plane through $\mathbb{R} \cup \{\infty\}$), so is f. Notice that the composition of a conjugate Möbius transformation with a Möbius transformation is a conjugate Möbius transformation and composition of two conjugate Möbius transformations is a Möbius transformation. There is a topological distinction between Möbius transformations and conjugate Möbius transformations in that Möbius transformations preserve the orientation of $\overline{\mathbb{C}}$ while conjugate Möbius transformations reverse it. To see more details about conjugate Möbius transformations, we refer [11].

C. Carathéodory [4] proved that every arbitrary one to one correspondence between the points of a circular disc C and a bounded point set C' by which circles lying completely in C are transformed into circles lying in C' must always be either a Möbius transformation $f(z)$ or $f(\overline{z})$. R. Höfer generalized the Carathéodory's theorem to arbitrary dimensions in [9]. R. Höfer proved that for a domain D of $\mathbb{R}^n$, if any injective mapping $f : D \to \mathbb{R}^n$ which takes hyperspheres whose interior is contained in D to hyperspheres in $\mathbb{R}^n$, then f is the restriction of a Möbius transformation. For more details about sphere preserving maps, see [2].

Since Möbius transformations play a major role in complex analysis and hyperbolic geometry, some authors tried to present new characterizations of Möbius transformations by using various Euclidean polygons and hyperbolic polygons. For example, in [8], H. Haruki and T.M. Rassias proved the following result by using Apollonius quadrilaterals as follows:

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Theorem 1.2. *[8] Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a analytic and univalent transformation in a non-empty domain R on the $z-$plane. Then f is a Möbius transformation if and only if f preserves Apollonius quadrilaterals in R.*

S. Yang and A. Fang presented a new characterization of Möbius transformations by using Lambert quadrilaterals and Saccheri quadrilaterals in the hyperbolic plane $B^2 = \{z : |z| < 1\}$ as follows:

Theorem 1.3. *[14], [15] Let $f : B^2 \to B^2$ be a continuous bijection. Then f is Möbius if and only if f preserves Lambert quadrilaterals in B^2.*

Theorem 1.4. *[14], [15] Let $f : B^2 \to B^2$ be a continuous bijection. Then f is Möbius if and only if f preserves Saccheri quadrilaterals in B^2.*

Definition 1.1. [3] The Lambert quadrilateral is a hyperbolic quadrilateral with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ and θ, where $0 < \theta < \frac{\pi}{2}$.

Definition 1.2. [3] The Saccheri quadrilateral is a hyperbolic quadrilateral with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, θ and θ, where $0 < \theta < \frac{\pi}{2}$.

To see other characterizations of Möbius transformations with the help of hyperbolic polygons, we refer [10], [5], [6] and [7].

Distorting of the non-adjacent right angles of a Lambert quadrilateral, degenerate Lambert quadrilateral concept is defined as follows:

Definition 1.3. [7] A degenerate Lambert quadrilateral is a hyperbolic convex quadrilateral with ordered angles $\frac{\pi}{2} + \epsilon$, $\frac{\pi}{2}$, $\frac{\pi}{2} - \epsilon$, θ where $0 < \theta < \frac{\pi}{2}$ and $0 < \epsilon < \frac{\pi}{2} - \frac{\theta}{2}$.

In [7], O. Demirel proved the following result:

Theorem 1.5. *[7] Let $f : B^2 \to B^2$ be a surjective transformation. Then f is a Möbius transformation or a conjugate Möbius transformation if and only if f preserves all $\epsilon-$Lambert quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$.*

Distorting of the right angles of a Saccheri quadrilateral, degenerate Saccheri quadrilateral concept is defined as follows:

Definition 1.4. A degenerate Saccheri quadrilateral is a hyperbolic convex quadrilateral with ordered angles $\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon, \theta, \theta$ where $0 < \theta < \frac{\pi}{2}$ and $0 < \epsilon < \frac{\pi}{2} - \frac{\theta}{2}$.

Notice that, for a degenerate Saccheri quadrilateral, the sum of the measures of mutual distorted angles are $\frac{\pi}{2} + \epsilon + \theta$ and $\frac{\pi}{2} - \epsilon + \theta$.

In this paper we call the degenerate Saccheri quadrilaterals having ordered angles $\frac{\pi}{2} - \epsilon$, $\frac{\pi}{2} + \epsilon$, θ, θ briefly as $\epsilon-$Saccheri quadrilaterals. We consider the hyperbolic plane $B^2 = \{z : |z| < 1\}$ with length differential $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$.

Throughout of the paper we denote by X' the image of X under f , by $[P, Q]$ the geodesic segment between points P and Q, by PQ the geodesic through points P and Q, by PQR the hyperbolic triangle with three ordered vertices P, Q and R, by $PQRS$ the hyperbolic quadrilateral with four ordered vertices P, Q, R and S, and by $\angle PQR$ the angle between $[P, Q]$ and $[P, R]$.

2. A Characterization of Möbius Transformations by use of Degenerate Saccheri Quadrilaterals

In this section, by the $\epsilon-$Saccheri quadrilaterals preserving property of functions, we meant that if $ABCD$ is a $\epsilon-$Saccheri quadrilateral having ordered angles $\frac{\pi}{2} - \epsilon$, $\frac{\pi}{2} + \epsilon$, θ and θ, then $A'B'C'D'$ is a $\epsilon-$Saccheri quadrilateral having ordered angles $\frac{\pi}{2} - \epsilon$, $\frac{\pi}{2} + \epsilon$, θ' and θ'.

Lemma 2.1. *Let $f : B^2 \to B^2$ be a mapping which preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$. Then f is injective.*

Proof. Let us take two different points P and Q in B^2. Then by constructing a $\epsilon-$Saccheri quadrilateral $PQRS$, one can easily get that $P'Q'R'S'$ is also a $\epsilon-$Saccheri quadrilateral by the property of f. Therefore, the points P' and Q' must be different which implies that f is injective. □

Lemma 2.2. *Let $f : B^2 \to B^2$ be a mapping which preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the collinearity and betweenness properties of the points.*

Proof. Let P and Q be two different points in B^2 and assume that S is an interior point of $[P, Q]$. Then S must be lie on all $\epsilon-$Saccheri quadrilaterals whose vertices are P and Q. By the property of f, the images of all $\epsilon-$Saccheri quadrilaterals with vertices P and Q are $\epsilon-$Saccheri quadrilaterals with vertices P' and Q' and must contain S'. Since f is injective by *Lemma 2.1*, we get $P' \neq S' \neq Q'$. Therefore, S' must be an interior point of $[P', Q']$ which implies that f preserves the collinearity and betweenness properties of the points. □

Lemma 2.3. *Let $f : B^2 \to B^2$ be a mapping which preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the angles $\frac{\pi}{2} + \epsilon$ and $\frac{\pi}{2} - \epsilon$.*

Proof. Let $ABCD$ be a $\epsilon-$Saccheri quadrilateral with $\angle ABC = \frac{\pi}{2} - \epsilon$, $\angle BCD = \frac{\pi}{2} + \epsilon$, $\angle CDA = \angle DAB = \theta$ and denote the midpoint of $[B, C]$ by M. Now, for a fixed $\alpha \in \mathbb{R}$ satisfying $0 < \alpha < \epsilon < \frac{\pi}{2}$, pick a point on DC, say E, satisfying $\angle MEC = \frac{\pi}{2} - \alpha$ and $C \in [E, D]$. Let F be the common point of the geodesics ME and AB. Notice that, if E is close enough to C, then F is close enough to B. Because of the fact that $\angle ECM = \angle MBF = \frac{\pi}{2} - \epsilon$, $\angle CME = \angle BMF$ and $d_H = (B, M) = d_H(M, C)$ hold true, where d_H is the hyperbolic distance function, by *hyperbolic angle-side-angle theorem* [12], the triangles FBM and ECM are congruent. Hence $AFED$ must be a $\alpha-$ Saccheri quadrilateral with $\angle AFE = \frac{\pi}{2} + \alpha$, $\angle FED = \frac{\pi}{2} - \alpha$, $\angle CDA = \angle DAB = \theta$. Since f preserves all degenerate Saccheri quadrilaterals, then the hyperbolic quadrilaterals $A'B'C'D'$ and $A'F'E'D'$ are $\epsilon-$Saccheri quadrilateral and $\alpha-$ Saccheri quadrilateral, respectively. Let us denote the measures of the angles of $A'B'C'D'$ and $A'F'E'D'$ by $\frac{\pi}{2} - \epsilon$, $\frac{\pi}{2} + \epsilon$, θ', θ' and $\frac{\pi}{2} - \alpha$, $\frac{\pi}{2} + \alpha$, θ', θ', respectively. Notice that since $\angle EDA = \angle CDA = \angle BAD = \angle FAD = \theta$, then we have $\angle E'D'A' = \angle C'D'A' = \angle B'A'D' = \angle F'A'D' = \theta'$. Because of the fact that f preserves the collinearity and betweenness of the points by *Lemma 2.2*, one can easily see that the points F' and C' must be lie on $[A', B']$ and $[D', E']$, respectively. Therefore, we have $\angle A'B'C' = \frac{\pi}{2} - \epsilon$ or $\angle A'B'C' = \frac{\pi}{2} + \epsilon$. Now, assume that $\angle A'B'C' = \frac{\pi}{2} + \epsilon$. Thus we have $\angle B'C'D' = \frac{\pi}{2} - \epsilon$. Obviously, $\angle A'F'M' = \frac{\pi}{2} + \alpha$ must be hold, otherwise, if $\angle A'F'M' = \frac{\pi}{2} - \alpha$ holds which implies $\angle M'F'B' = \frac{\pi}{2} + \alpha$, then the sum of the measures of interior angles of the triangle $F'B'M'$ is greater than π which is not possible in hyperbolic geometry. Thus we get $\angle A'F'M' = \frac{\pi}{2} + \alpha$ and $\angle M'F'B' = \frac{\pi}{2} - \alpha$. Let P and Q be the common points of the unit disc B^2 and the hyperbolic disc $D'E'$. Assume that E' lies on $[C', Q]$. If X is a point moving from Q to P on PQ, then $\angle M'XP$ must be increase from 0 to π. Hence, we get $\frac{\pi}{2} - \alpha < \frac{\pi}{2} - \epsilon$ which implies $\epsilon < \alpha$. This is a contradiction since $\alpha < \epsilon$. Therefore, we get $\angle A'B'C' = \frac{\pi}{2} - \epsilon$ which implies $\angle B'C'D' = \frac{\pi}{2} + \epsilon$. □

Lemma 2.4. *Let $f : B^2 \to B^2$ be a mapping which preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the measures of equal angles.*

Proof. Let $ABCD$ be a $\epsilon-$Saccheri quadrilateral with $\angle ABC = \frac{\pi}{2} - \epsilon$, $\angle BCD = \frac{\pi}{2} + \epsilon$, $\angle CDA = \angle DAB = \theta$. By *Lemma 2.3*, we have $\angle A'B'C' = \frac{\pi}{2} - \epsilon$, $\angle B'C'D' = \frac{\pi}{2} + \epsilon$. Now we have to prove that $\theta = \angle CDA = \angle DAB = \angle C'D'A' = \angle D'A'B' = \theta'$. Let $PQRS$ be a hyperbolic square with center is O, where O is the origin of B^2 satisfying $\angle PQR = \angle QRS = \angle RSP = \angle SPQ = \frac{\theta}{2}$. Without loss of generality, we may assume that the points P and R lie on the $y-$axis and the points Q and S lie on the $x-$axis. Let X and Y be the midpoints of $[P, Q]$ and $[S, R]$, respectively. Notice that the hyperbolic quadrilateral $PXYS$ is a Saccheri quadrilateral with $\angle PXY = \angle XYS = \frac{\pi}{2}$ by the property of a hyperbolic square. Now, pick a point on $[S, Y]$, say K, such that $\angle OKY = \theta$. The existence of K is clear since $\angle OSY = \frac{\theta}{4}$ and $\angle OYS = \frac{\pi}{2}$. Now pick a point on $[Q, X]$, say L, such that $d_H(L, Q) = d_H(S, K)$. It is not hard to see that the hyperbolic triangles OLQ and OKS are congruent. Hence, we get $\angle OLX = \angle OKY = \theta$. Because of $\theta < \frac{\pi}{2}$, we may represent θ as $\theta := \frac{\pi}{2} - \alpha$. Thus we have $\angle OLQ = \pi - \theta = \pi - (\frac{\pi}{2} - \alpha) = \frac{\pi}{2} + \alpha$ which implies that $KLQR$ is a $\alpha-$Saccheri quadrilateral with $\angle RKL = \frac{\pi}{2} - \alpha$, $\angle KLQ = \frac{\pi}{2} + \alpha$, $\angle LQR = \angle QRK = \frac{\theta}{2}$. The angle $\angle CDA$ of the $\epsilon-$Saccheri quadrilateral $ABCD$ can be moved to the point K by an appropriate hyperbolic isometry g, such that the points $g(A)$ and $g(C)$ lie on the geodesics KL and KR, respectively. Because of the fact that f preserves $\epsilon-$Saccheri quadrilaterals for all $0 < \epsilon < \frac{\pi}{2}$, then $K'L'Q'R'$ is a $\alpha-$Saccheri quadrilateral with $\angle L'K'R' = \frac{\pi}{2} - \alpha = \theta$, $\angle K'L'Q' = \frac{\pi}{2} + \alpha = \pi - \theta$, $\angle L'Q'R' = \angle Q'R'K'$ by *Lemma 2.3*. Thus we get $\angle A'D'C' = \angle g(A)g(D)g(C) = \angle L'K'R' = \theta$ holds true. Similarly, one can easily prove that $\angle D'A'B' = \theta$ holds true. Hence f preserves the measures of equal angles. □

Lemma 2.5. *Let $f : B^2 \to B^2$ be a mapping which preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$. Then f preserves hyperbolic distance.*

Proof. Let A and B be two different points in B^2. These points define two different hyperbolic squares. Choose one of them and denote it by $ABCD$. Assume $\angle ABC = \angle BCD = \angle CDA = \angle DAB := \theta$ and denote it center by M. The geodesics passing through M, must intersect $ABCD$ at two points. Let p be a hyperbolic geodesic passing through M but not passing through the points $A, B, C, D, M_1, M_2, M_3, M_4$ where M_1, M_2, M_3, M_4 are the midpoints of $[A, B], [B, C], [C, D], [D, A]$, respectively. Therefore the hyperbolic square $ABCD$ and hyperbolic geodesic p define two congruent degenerate Saccheri quadrilaterals. Assume that these degenerate Saccheri quadrilaterals be two $\beta-$Saccheri quadrilaterals. By *Lemma 2.3* and *Lemma 2.4*, the images of these two degenerate Saccheri quadrilaterals are $\beta-$Saccheri quadrilaterals with angles $\frac{\pi}{2} - \beta$, $\frac{\pi}{2} + \beta$, θ, θ. By *Lemma 2.4*, we get $\angle A'B'C' = \angle B'C'D' = \angle C'D'A' = \angle D'A'B' := \theta$. Because of the fact that the angles at the vertices of a hyperbolic square define its lengths, we get $d_H(A, B) = d_H(A', B')$, see [13]. □

Theorem 2.1. *Let $f : B^2 \to B^2$ be a surjective transformation. Then f is a Möbius transformation or a conjugate Möbius transformation if and only if f preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$.*

Proof. The "only if " part is obvious because f is an isometry. Conversely, we may assume that f preserves $\epsilon-$Saccheri quadrilaterals for all $0 < \epsilon < \frac{\pi}{2}$ in B^2 and $f(O) = O$ by composing an hyperbolic isometry if necessary. Let us take two different points in B^2 and denote them by x, y. By *Lemma 2.5*, we immediately get $d_H(O, x) = d_H(O, x')$ and $d_H(O, y) = d_H(O, y')$, namely $|x| = |x'|$ and $|y| = |y'|$, where $|\cdot|$ denotes the Euclidean norm. Therefore we get $|x - y| = |x' - y'|$ by since f preserves angular sizes by *Lemma 2.4*. As

$$2\langle x, y\rangle = |x|^2 + |y|^2 - |x - y|^2 = |x'|^2 + |y'|^2 - |x' - y'|^2 = 2\langle x', y'\rangle,$$

f preserves inner-products and then is the restriction on B^2 of an orthogonal transformation, that is, f is a Möbius transformation or a conjugate Möbius transformation by *Carathéodory's theorem*. □

Corollary 2.1. *Let $f : B^2 \to B^2$ be a conformal (angle preserving with sign) surjective transformation. Then f is a Möbius transformation if and only if f preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$.*

Corollary 2.2. *Let $f : B^2 \to B^2$ be a angle reversing surjective transformation. Then f is a conjugate Möbius transformation if and only if f preserves all $\epsilon-$Saccheri quadrilaterals where $0 < \epsilon < \frac{\pi}{2}$.*

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