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Frenet Curves in Euclidean 4-Space

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ABSTRACT

In this paper, we study rectifying curves arising through the dilation of unit speed curves on the unit sphere $\mathbb{S}^3$ and conclude that arcs of great circles on $\mathbb{S}^3$ do not dilate to rectifying curves, which develope previously obtained results for rectifying curves in Eucidean spaces. This fact allows us to prove that there exists an associated rectifying curve for each Frenet curve in the Euclidean space $\mathbb{E}^4$ and a result of the fact rectifying curves in the Euclidean space $\mathbb{E}^4$ are ample , indeed as an appication, we present an ordinary differential equation satisfied by the distance function of a Frenet curve in $\mathbb{E}^4$ which alows us to characterize the spherical curves and rectifying curves in $\mathbb{E}^4$. Furthermore, we study ccr-curves in the Euclidean space $\mathbb{E}^4$ which are generalizations of helices in $\mathbb{E}^3$ and show that the property of a helix that its tangent vector field makes a constant angel with a fixed vector (axis of helix) does not go through for a ccr-curve.

Keywords: *Frenet curves, rectifying curves, Chen curves, ccr-curves, curvatures.*
AMS Subject Classification (2010): *53A04*

1. Introduction

The study of rectifying curves in the Euclidean space $\mathbb{E}^3$ was initiated by Professor B. Y. Chen (cf. [3, 4]) as space curves whose position vector always lies in its rectifying plane, spanned by the tangent and the binormal vector fields T and B of the curve. Accordingly, the position vector with respect to some chosen origin, of a rectifying curve α in $\mathbb{E}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

where $\lambda(s)$ and $\mu(s)$ are arbitrary differentiable functions in arclength parameter $s \in I \subset \mathbb{R}$.

It is well kown that rectifying curves have important role in physics as well as in joint kinematics (cf. [1, 6, 7, 14, 16, 17]). In [2, 9], the authors have extended the notion of rectifying curves in the Euclidean spaces $\mathbb{E}^4$ and to higher dimensional Euclidean spaces $\mathbb{E}^n$ respectively. As a matter of tribute to the enormous contributions of Professor Bang-Yen Chen to geometry and its applications in physics, we shall call a rectifying curve as a *Chen curve*. The main source of examples of Chen curves in the Euclidean space $\mathbb{E}^4$ is through dilation of unit speed curves on the unit sphere $\mathbb{S}^3$ (cf. [9]). In [9], Ilarslan and Nesovic show that, given a unit speed curve $y(t)$ on the unit sphere $\mathbb{S}^3$ and a positive differentiable function $f(t)$, the dilation of $y(t)$ with dilation factor $f(t)$, that is, the curve $\alpha(t) = f(t)y(t)$ is a Chen curve if and only if $f(t) = a\sec(t + t_0)$, where $a > 0$ and t_0 are constants.

In this paper, we observe that the arcs of great circles do not dilate to a Chen curves (cf. Corollary 3.1) and accordingly the Theorem 3.5 in [9] and Theorem 4.7 in [2] need to be refined and thus we restate this result of [9]. Unlike to Chen curves in the Euclidean space $\mathbb{E}^3$, where examples are not too many, it is interesting to see that in the Euclidean space $\mathbb{E}^4$ they are in abundance. Also, we show that for each Frenet curve in $\mathbb{E}^4$ there exists an associated Chen curve (cf. Theorem 3.1) and thus there are as many Chen curves in $\mathbb{E}^4$ as Frenet curves. As in case of Frenet curves in $\mathbb{E}^3$ (cf. [7]), we derive a differential equation satisfied by the distance function of unit speed Frenet curve $\alpha : I \to \mathbb{E}^4$ (cf. Theorem 4.1) and as its applications we have several corollaries, which give characterizations of spherical curves as well as Chen curves in the Euclidean space $\mathbb{E}^4$.

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On the other hand, helices in the Euclidean space $\mathbb{E}^3$ are very important curves as they appear in physical applications as well as in medical sciences (specially in the structure of DNA), and are characterized by torsion curvature ratio is a constant (*Lancret theorem*) [11]. Moreover, an important property of a helix is that its tangent vector field makes a constant angle with a fixed direction (a constant unit vector in $\mathbb{E}^3$) called the axis of helix. As generalization of a helix in $\mathbb{E}^3$ to higher dimensional spaces a curve with constant curvature ratios or a ccr-curve is introduced, and these curves are studied in [13] and [15]. It is well known that, a Frenet curve $\alpha : I \to \mathbb{E}^n$ is said to be *ccr-curve* if the curvature ratios $\frac{\kappa_{i+1}}{\kappa_i}$ are constants, $i = 1, .., n-1$. A natural question arises as to whether the property that the tangent vector field makes a constant angle with a constant vector in $\mathbb{E}^3$ that of a helix holds for ccr-curves? Finally, we study this question and show that this property of helix in $\mathbb{E}^3$ does not go through for ccr-curves in the Euclidean space $\mathbb{E}^4$ (cf. Theorem 5.1).

2. Preliminaries

Let $\alpha : I \to \mathbb{E}^4$ be a unit speed Frenet curve with Frenet-Serret apparatus $\{\kappa_1, \kappa_2, \kappa_3, T, N, B_1, B_2\}$, $\kappa_i \neq 0$ and $\rho_i = \frac{1}{\kappa_i}$, $i = 1, 2, 3$ ([11]). Then Frenet equations are given by

$$T^{'} = \kappa_1 N, \quad N^{'} = -\kappa_1 T + \kappa_2 B_1, \quad B_1^{'} = -\kappa_2 N + \kappa_3 B_2, \quad B_2^{'} = -\kappa_3 B_1. \tag{2.1}$$

A Frenet curve $\alpha(t)$ is said to be a Chen curve if $\langle \alpha(t), N(t) \rangle = 0$, $t \in I$, (cf. [2, 9]). The distance function $f(t) = \|\alpha(t)\|$ of the Chen curve satisfies

$$f(t) = \sqrt{t^2 + c_1 t + c_2}, \tag{2.2}$$

where c_1, c_2 are constants and the converse is also true (cf. [9]).

A major source of examples of Chen curves in the Euclidean space $\mathbb{E}^4$ is the dilation of a unit speed curve on the unit sphere $\mathbb{S}^3$ by a positive dilation factor (cf. [2, 9]). In [9], it is shown that if $y(t)$ is a unit speed curve on the unit sphere $\mathbb{S}^3$ and $f(t)$ is a positive differentiable function, then the curve $\alpha(t) = f(t)y(t)$ is a Chen curve if and only if

$$f(t) = a \sec(t + t_0), \tag{2.3}$$

where $a > 0$ and t_0 are constants.

3. Chen curves through dilation of curves on $\mathbb{S}^3$

The major source of examples of Chen curves is provided by the dilated curves $\alpha(t) = a \sec(t + t_0) y(t)$, where $y(t)$ is a unit speed curve on the unit sphere $\mathbb{S}^3$ and $a > 0$, t_0 are constants.(cf. [9]). However, if we consider the arc of a great circle $y(t)$ on $\mathbb{S}^3$, we can choose its Frenet apparatus as $\left\{1, 0, 0, y^{'}, -y, Y_1, Y_2\right\}$, where binormals Y_1, Y_2 are constant unit vectors. Then we have the dilated curve (cf. Theorem 3.5, [9]),

$$\alpha(t) = a \sec(t + t_0) y(t),$$

and we find the speed v_a and the tangent vector filed T_α of α as

$$v_\alpha = a \sec^2(t + t_0), T_\alpha = \sin(t + t_0) y + \cos(t + t_0) y^{'}.$$

Differentiating T_α, and denoting the first curvature of α as κ_1, we get

$$\kappa_1 a \sec^2(t + t_0) N_\alpha = \cos(t + t_0) y + \sin(t + t_0) y^{'} - \sin(t + t_0) y^{'} + \cos(t + t_0) y^{''} = \overrightarrow{0},$$

where we used first equation (2.1) as $y^{''} = -y$. Hence, the first curvature $\kappa_1 = 0$ and consequently, $\alpha(t)$ is not a Chen curve, as the definition of Chen curve requires that its curvatures are nonzero (cf. [9]). Thus, not all curves which are dilation of unit speed curve $y(t)$ on $\mathbb{S}^3$ of the type $\alpha(t) = a \sec(t + t_0) y(t)$ are Chen curves. To understand this relationship between the Chen curve $\alpha(t)$ and the unit speed curve $y(t)$, first we prove the following Lemma.

Lemma 3.1 *Let $y(t)$ be the unit speed curve on the unit sphere $\mathbb{S}^3$ and $\alpha(t) = a\sec(t+t_0)y(t)$ be the dilation of $y(t)$. Then the first curvatures κ_1^α* and *κ_1 of the curves $\alpha(t)$ and $y(t)$ respectively satisfy*

$$\kappa_1^\alpha = \frac{1}{a}\cos^3(t+t_0)\sqrt{\kappa_1^2-1}.$$

Proof. Let κ_1^α be the first curvature of $\alpha(t)$ and T_α, N_α be the tangent vector field and principal normal vector field of $\alpha(t)$, and κ_1, T, N be those of the unit speed curve $y(t)$. Note that we have

$$T = y', y'' = T' = \kappa_1 N, \ \langle y, y\rangle = \left\langle y', y'\right\rangle = 1, \ \left\langle y, y'\right\rangle = 0. \tag{3.1}$$

Differentiating expression for $\alpha(t)$, we get

$$\alpha'(t) = a\sec(t+t_0)\left(\tan(t+t_0)y + y'\right).$$

Thus, the speed of $\alpha(t)$ is given by $v_\alpha = a\sec^2(t+t_0)$, and consequently, we get

$$T_\alpha = \sin(t+t_0)y + \cos(t+t_0)y'.$$

Differentiating above equation and using equation (2.1), we get

$$\kappa_1^\alpha a\sec^2(t+t_0)N_\alpha = \cos(t+t_0)y + \cos(t+t_0)y''.$$

This equation gives

$$\kappa_1^\alpha = \frac{1}{a}\cos^3(t+t_0)\sqrt{1+\|y''\|^2 + 2\langle y, y''\rangle}. \tag{3.2}$$

Using equation (3.1), we have $\left\|y''\right\|^2 = \kappa_1^2$ and differentiating the equation $\left\langle y, y'\right\rangle = 0$, we get $1 + \left\langle y, y''\right\rangle = 0$. Thus equation (3.2) proves the Lemma. □

Since, by Theorem 3.5 (cf. [9], p-27) $\alpha(t) = a\sec(t+t_0)y(t)$ is a Chen curve, which requires $\kappa_1^\alpha > 0$. Hence, by above Lemma, we have the following:

Corollary 3.1. *Let $y(t)$ be an unit speed curve on the unit sphere $\mathbb{S}^3$ that is not an arc of the great circle, then $\alpha(t) = a\sec(t+t_0)y(t)$ is a Chen curve.*

Consequently, great circles are to be excluded in the Theorem 3.5 of [9] (as well as in Theorem 4.7 of [2]). We restate the result in [9] (similarly Theorem 4.7 in [2] is to be restated with dimensional adjustment) as follows:

Theorem ([9]). *Let $y : I \to S^3$ be a unit speed curve on the unit sphere S^3 centered at the origin $o \in E^4$ and $f(t)$ a positive differentiable function defined on open interval I. Then $\alpha(t) = f(t)y(t)$ is a Chen curve if and only if $f(t) = a\sec(t+t_0)$ for some constants $a > 0$ and t_0 such that there exists no subinterval $J \subset I$ with $y(J) \subset S^3 \cap E^2$ for any 2-plane E^2 containing the origin of E^4.*

In the rest of this section, we prove the following theorem, which asserts that to each unit speed Frenet curve in $\mathbb{E}^4$, there is an associated Chen curve, showing the Chen curves are in abundance, indeed as many as unit speed Frenet curves in the Euclidean space $\mathbb{E}^4$.

Theorem 3.1. *Let $\alpha : I \to E^4$ be a unit Frenet curve of class C^5, with Frenet apparatus $\{\kappa_1, \kappa_2, \kappa_3, T, N, B_1, B_2\}$. Then for a point $t_0 \in I$, and the curve $\beta : I \to E^4$ defined by*

$$\beta(t) = \frac{\kappa_2}{\kappa_1}\sin\left(\int_{t_0}^{t}\kappa_3 du\right)T + \sin\left(\int_{t_0}^{t}\kappa_3 du\right)B_1 + \cos\left(\int_{t_0}^{t}\kappa_3 du\right)B_2$$

there exists a subinterval J of I containing the point t_0 such that the curve $\beta : J \to E^4$ is a Chen curve.

Proof: Using equations in (2.1), we compute

$$\beta'(t) = \left(\frac{\kappa_2}{\kappa_1}\sin\left(\int_{t_0}^{t}\kappa_3 du\right)\right)' T. \tag{3.3}$$

Note that

$$\beta^{'}(t_0) = \left(\frac{\kappa_2\kappa_3}{\kappa_1}\right)(t_0)T(t_0) \neq \overrightarrow{0},$$

and thus by continuity of $\beta^{'}(t)$ on I, there is a subinterval J of I containing t_0 such that $\beta^{'}(t) \neq \overrightarrow{0}$ for each $t \in J$. Hence, $\beta : J \to \mathbb{E}^4$ is a regular curve with speed v_β and tangent vector field T_β given by

$$v_\beta = \left| \left(\frac{\kappa_2}{\kappa_1} \sin \left(\int_{t_0}^{t} \kappa_3 du \right) \right)^{'} \right|, \quad T_\beta = \pm T. \tag{3.4}$$

Differentiating second equation in (3.4) and using equation (2.1), we get

$$\kappa_1^\beta v_\beta N_\beta = \pm \kappa_1 N, \tag{3.5}$$

where κ_1^β is the first curvature and N_β is the principal normal of the regular curve $\beta(t)$. Hence, we have

$$\kappa_1^\beta = \frac{\kappa_1}{v_\beta}, \quad N_\beta = \pm N. \tag{3.6}$$

Similarly, differentiating second equation in (3.6) and using equations (2.1), (3.4), (3.6), we find the second curvature κ_2^β and the first binormal vector field $B_{\beta 1}$ as

$$\kappa_2^\beta = \frac{\kappa_2}{v_\beta}, \quad B_{\beta 1} = \pm B_1. \tag{3.7}$$

Finally, differentiating the second equation in (3.7), we find the third curvature κ_3^β and the second binormal vector field $B_{\beta 2}$ given by

$$\kappa_3^\beta = \frac{\kappa_3}{v_\beta}, \quad B_{\beta 2} = \pm B_2. \tag{3.8}$$

Since all the curvatures $\kappa_i^\beta = \frac{\kappa_i}{v_\beta} \neq 0$, $i = 1, 2, 3$, the curve $\beta(t)$ is a Frenet curve and it follows from equation (3.5) and definition of $\beta(t)$ that

$$\langle \beta(t), N_\beta \rangle = 0.$$

Hence, $\beta(t)$ is a Chen curve. □

Remark: Let s be the arc length parameter of the Chen curve $\beta(t)$ of Theorem 3.1. Then using equation (3.4), we find

$$s = \frac{\kappa_2}{\kappa_1} \sin \left(\int_{t_0}^{t} \kappa_3 du \right) + c$$

for a constant c, and thus, the distance function $f(s) = \|\beta(s)\|$ is given by

$$f(s) = \sqrt{\left(\frac{\kappa_2}{\kappa_1} \sin \left(\int_{t_0}^{t} \kappa_3 du \right) \right)^2 + 1} = \sqrt{(s-c)^2 + 1} = \sqrt{s^2 + c_1 s + c_2},$$

where c_1, c_2 are constants, which is the required representation of the distance function of a Chen curve (cf. [9]).

Example: Theorem 3.1, provides an equal number of Chen curves as those of Frenet curves. For example consider the unit speed curve $\alpha : I \to \mathbb{E}^4$ defined by

$$\alpha(t) = \left(\cos\sqrt{\frac{2}{3}}t, \sin\sqrt{\frac{2}{3}}t, \cos\sqrt{\frac{1}{3}}t, \sin\sqrt{\frac{1}{3}}t \right),$$

The curvatures of $\alpha(t)$ are given by $\kappa_1 = \frac{\sqrt{5}}{3}$, $\kappa_2 = \frac{1}{3}\sqrt{\frac{2}{5}}$ and $\kappa_3 = \sqrt{\frac{2}{5}}$ and Frenet frame vector fields are given by

$$T = \left(-\sqrt{\frac{2}{3}}\sin\sqrt{\frac{2}{3}}t, \sqrt{\frac{2}{3}}\cos\sqrt{\frac{2}{3}}t, -\sqrt{\frac{1}{3}}\sin\sqrt{\frac{1}{3}}t, \sqrt{\frac{1}{3}}\cos\sqrt{\frac{1}{3}}t\right),$$

$$N = -\frac{1}{\sqrt{5}}\left(2\cos\sqrt{\frac{2}{3}}t, 2\sin\sqrt{\frac{2}{3}}t, \cos\sqrt{\frac{1}{3}}t, \sin\sqrt{\frac{1}{3}}t\right),$$

$$B_1 = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}\sin\sqrt{\frac{2}{3}}t, -\sqrt{\frac{2}{3}}\cos\sqrt{\frac{2}{3}}t, -\frac{2}{\sqrt{3}}\sin\sqrt{\frac{1}{3}}t, \frac{2}{\sqrt{3}}\cos\sqrt{\frac{1}{3}}t\right),$$

$$B_2 = \frac{1}{\sqrt{5}}\left(\cos\sqrt{\frac{2}{3}}t, \sin\sqrt{\frac{2}{3}}t, -2\cos\sqrt{\frac{1}{3}}t, -2\sin\sqrt{\frac{1}{3}}t\right).$$

Thus, the corresponding Chen curve is given by

$$\beta(t) = \frac{\sqrt{2}}{5}\sin\left(\sqrt{\frac{2}{5}}t\right)T + \sin\left(\sqrt{\frac{2}{5}}t\right)B_1 + \cos\left(\sqrt{\frac{2}{5}}t\right)B_2$$

and its curvatures are given by

$$\kappa_1^\beta = \frac{25}{6}\sec\left(\sqrt{\frac{2}{5}}t\right), \kappa_2^\beta = \frac{5}{3\sqrt{2}}\sec\left(\sqrt{\frac{2}{5}}t\right) \text{ and } \kappa_3^\beta = \frac{5}{\sqrt{2}}\sec\left(\sqrt{\frac{2}{5}}t\right).$$

4. A differential equation and its applications

Let $\alpha : I \to \mathbb{E}^4$ be a unit speed Frenet curve of class C^6 and $f(t) = \|\alpha(t)\|$ be the distance function. The distance function is assumed to be differentiable (If the curve passes through origin, we could use an isometry of $\mathbb{E}^4$ to make the distance function differentiable). In this section, first we find a differential equation satisfied by the distance function of any Frenet curve and then as an application of this differential equation we derive characterizations of sphere curves and Chen curves in the Euclidean space $\mathbb{E}^4$.

Note that using equations in (2.1), we have the following

$$\langle \alpha(t), T\rangle' = 1 + \kappa_1 \langle \alpha(t), N\rangle, \quad \langle \alpha(t), N\rangle' = -\kappa_1 \langle \alpha(t), T\rangle + \kappa_2 \langle \alpha(t), B_1\rangle, \tag{4.1}$$

$$\langle \alpha(t), B_1\rangle' = -\kappa_2 \langle \alpha(t), N\rangle + \kappa_3 \langle \alpha(t), B_2\rangle, \quad \langle \alpha(t), B_2\rangle' = -\kappa_3 \langle \alpha(t), B_1\rangle. \tag{4.2}$$

Theorem 4.1. *Let $\alpha : I \to E^4$ be a unit speed Frenet curve of class C^5 with Frenet-Serret apparatus $\{\kappa_1, \kappa_2, \kappa_3, T, N, B_1, B_2\}$, $\rho_i = \frac{1}{\kappa_i}$, $i = 1, 2, 3$. Then the function $h(t) = f(t)f'(t)$, where $f(t) = \|\alpha(t)\|$ is the distance function of α, satisfies the differential equation*

$$\begin{aligned}
&\rho_1\rho_2\rho_3 h^{(iv)} + \left[(\rho_1\rho_2\rho_3)' + \rho_3(\rho_1\rho_2)' + \rho_1'\rho_2\rho_3\right] h''' \\
&+ \left[\left(\rho_3\left((\rho_1\rho_2)' + \left(\rho_1'\rho_2\right)\right)\right)' + \rho_3\left(\left(\rho_1'\rho_2\right)' + \frac{\rho_1}{\rho_2} + \frac{\rho_2}{\rho_1}\right) + \frac{\rho_1\rho_2}{\rho_3}\right] h'' \\
&+ \left[\left(\rho_3\left(\left(\rho_1'\rho_2\right)' + \frac{\rho_1}{\rho_2} + \frac{\rho_2}{\rho_1}\right)\right)' + \rho_3\left(\frac{\rho_2}{\rho_1}\right)' + \frac{\rho_2}{\rho_3}\rho_1'\right] h' \\
&+ \left[\left(\rho_3\left(\frac{\rho_2}{\rho_1}\right)'\right)' + \frac{\rho_2}{\rho_1\rho_3}\right] h - \left[\left(\rho_3\left(\left(\rho_1'\rho_2\right)' + \frac{\rho_1}{\rho_2}\right)\right)' + \frac{\rho_2}{\rho_3}\rho_1'\right] = 0.
\end{aligned}$$

Proof: Differentiating $f^2(t) = \langle \alpha(t), \alpha(t) \rangle$ and using equations (4.1) and (4.2), we get

$$\begin{aligned} h(t) &= \langle \alpha(t), T \rangle \langle \alpha(t), T \rangle' + \langle \alpha(t), N \rangle \langle \alpha(t), N \rangle' + \langle \alpha(t), B_1 \rangle \langle \alpha(t), B_1 \rangle' \\ &\quad + \langle \alpha(t), B_2 \rangle \langle \alpha(t), B_2 \rangle' \\ &= \langle \alpha(t), T \rangle . \end{aligned} \tag{4.1}$$

Differentiating equation (4.3) and using equation (4.1), we have

$$\rho_1 \left(h' - 1 \right) = \langle \alpha(t), N \rangle , \tag{4.4}$$

which on differentiating and using equations (4.1), (4.3), leads to

$$\rho_1 \rho_2 h'' + \rho_1' \rho_2 \left(h' - 1 \right) + \frac{\rho_2}{\rho_1} h = \langle \alpha(t), B_1 \rangle . \tag{4.5}$$

Differentiating above equation and using equations (4.2), (4.4), we get

$$\begin{aligned} \rho_1 \rho_2 \rho_3 h''' + \rho_3 \left((\rho_1 \rho_2)' + \left(\rho_1' \rho_2 \right) \right) h'' + \rho_3 \left(\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} + \frac{\rho_2}{\rho_1} \right) h' & \\ + \rho_3 \left(\frac{\rho_2}{\rho_1} \right)' h - \left(\rho_3 \left(\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right) \right) &= \langle \alpha(t), B_2 \rangle . \end{aligned}$$

Finally, differentiating above equation and using equations (4.2), (4.5) after a straight forward computation, we get the differential equation in the statement of the theorem. □

As an application of Theorem 4.1, we have the following corollaries giving characterizations of the spherical curves and Chen curves in the Euclidean space $\mathbb{E}^4$.

Corollary 4.1.[13] *A unit speed Frenet curve $\alpha : I \to \mathbb{E}^4$ is a spherical curve if and only if*

$$\rho_1^2 + \left(\rho_1' \rho_2 \right)^2 + \left(\rho_3 \left[\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right] \right)^2 = r^2,$$

where r is a positive constant.

Proof. Suppose $\alpha(t)$ is a spherical curve that lies on a sphere of radius r. Without loss of generality, we can assume that the center of the sphere is at the origin (for otherwise, we could use an isometry of the Euclidean space $\mathbb{E}^4$ to achieve this objective). Thus, the distance function of $\alpha(t)$ satisfies $f(t) = r$ and consequently, we have $h(t) = 0$ in the differential equation of Theorem 4.1, which gives

$$\left(\rho_3 \left(\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right) \right)' + \frac{\rho_2}{\rho_3} \rho_1' = 0. \tag{4.6}$$

Multiplying equation (4.6) by

$$2 \left(\rho_3 \left[\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right] \right),$$

we get

$$2\rho_3 \left[\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right] \left(\rho_3 \left[\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right] \right)' + 2 \left(\rho_1' \rho_2 \right) \left(\rho_1' \rho_2 \right)' + 2\rho_1 \rho_1' = 0,$$

which on integration gives the required result.

Conversely, suppose $\alpha(t)$ is a unit speed Frenet curve that satisfies the given condition, which is equivalent to equation (4.6). We use equations in (2.1) and (4.6) to compute

$$\left(\alpha(t) + \rho_1 N + \left(\rho_1' \rho_2 \right) B_1 + \rho_3 \left[\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right] B_2 \right)' = \vec{0}.$$

Hence, we get a constant vector $\overrightarrow{m}$ that satisfies $\langle \alpha(t) - \overrightarrow{m}, \alpha(t) - \overrightarrow{m} \rangle = r^2$, that is, $\alpha(t)$ is a spherical curve. $\square$

Corollary 4.2 (Theorem 3.1[9]). *A unit speed Frenet curve $\alpha : I \to \mathbb{E}^4$ is a Chen curve if and only if*

$$\left(\rho_3 \left(\frac{\rho_2}{\rho_1} \right)' (t+c) + \frac{\rho_2 \rho_3}{\rho_1} \right)' + \frac{\rho_2}{\rho_1 \rho_3}(t+c) = 0,$$

where c is a constant.

Proof. Suppose $\alpha(t)$ is a Chen curve. Then the distance function $f(t) = \|\alpha(t)\|$ is given by equation (2.2) (cf. [9])

$$f(t) = \sqrt{t^2 + c_1 t + c_2},$$

where c_1, c_2 are constants. Then it follows that $h(t) = f(t)f'(t) = t + c$, where $2c = c_1$. Using $h(t) = t + c$ and $h'(t) = 1$ in the differential equation of Theorem 4.1, we get

$$\begin{aligned} &\left[\left(\rho_3 \left(\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} + \frac{\rho_2}{\rho_1} \right) \right)' + \rho_3 \left(\frac{\rho_2}{\rho_1} \right)' + \frac{\rho_2}{\rho_3} \rho_1' \right] \\ &+ \left[\left(\rho_3 \left(\frac{\rho_2}{\rho_1} \right)' \right)' + \frac{\rho_2}{\rho_1 \rho_3} \right] (t+c) - \left[\left(\rho_3 \left(\left(\rho_1' \rho_2 \right)' + \frac{\rho_1}{\rho_2} \right) \right)' + \frac{\rho_2}{\rho_3} \rho_1' \right] \;=\; 0, \end{aligned}$$

which gives the condition in the statement. The converse follows from [9]. $\square$

Corollary 4.3 ([5]). *A unit speed Frenet curve $\alpha : I \to \mathbb{E}^4$ satisfies*

$$\langle \alpha(t), N \rangle^2 + \langle \alpha(t), B_1 \rangle^2 + \langle \alpha(t), B_2 \rangle^2 = c^2,$$

for a constant c, if and only if either it is a spherical curve or a Chen curve.

Proof. Suppose $\alpha(t)$ is a unit speed Frenet curve satisfying the given condition. Differentiating the condition in the statement and using equations in (2.1), we get

$$\kappa_1 \langle \alpha(t), T \rangle \langle \alpha(t), N \rangle = 0. \tag{4.7}$$

Also, in view of given condition, the distance function is given by

$$f(t) = \sqrt{\langle \alpha(t), T \rangle^2 + c^2}, \tag{4.8}$$

which on differentiating leads to

$$h(t) = \langle \alpha(t), T \rangle + \kappa_1 \langle \alpha(t), T \rangle \langle \alpha(t), N \rangle .$$

Using equations (4.7) and (4.8) in the above equation, we conclude that

$$f^2 = h^2 + c^2,$$

which on differentiation gives $h = hh'$, that is either $h = 0$ or $h = 1$ holds. Hence, Corollaries 4.1 and 4.2 imply that either $\alpha(t)$ is a sphere curve or a Chen curve. The converse is trivial. $\square$

5. Curves with constant curvature ratios

Helices in the Euclidean space $\mathbb{E}^3$ are very important curves as they appear in physical applications as well as in medical sciences (specially in the structure of DNA). From the view of differential geometry, a helix is a geometric curve with nonvanishing constant curvature $\varkappa$ and non-vanishing constant torsion τ [11]. The helix is also known as circular helix or W-curve which is a special case of the general helix. The main feature of general helix is that the tangent makes a constant angle with a fixed straight line which is called the axis of the general helix. A classical result stated by *Lancret* in 1802 and first proved by de Saint Venant in 1845 says that: A necessary and sufficient condition that a curve be a general helix is that the ratio $\frac{\varkappa}{\tau}$ is constant along the curve.

As generalization of a helix in $\mathbb{E}^3$ to higher dimensional spaces, a curve with constant curvature ratios or a ccr-curve, is introduced and these curves are studied in [13] and [15]. A Frenet curve $\alpha : I \to \mathbb{E}^n$ is said to be ccr-curve if the curvature ratios $\frac{\kappa_{i+1}}{\kappa_i}$ are constants, $i = 1, .., n-1$.

A natural question arises as to whether the property that the tangent vector field makes a constant angle with a constant vector in $\mathbb{E}^3$ that of a helix holds for ccr-curves? In this section, we study this question for the Frenet curves in the Euclidean space $\mathbb{E}^4$ and show that this property of helix in $\mathbb{E}^3$ does not go through for ccr-curves in the Euclidean space $\mathbb{E}^4$.

Proposition 5.1. *The tangent vector field of a unit speed Frenet curve $\alpha : I \to \mathbb{E}^4$ makes a constant angle with a constant unit vector in $\mathbb{E}^4$ if and only if*

$$\frac{\kappa_1}{\kappa_2} = a \sin\left(\int_0^t \kappa_3 du\right),$$

where $a \neq 0$ is a constant.

Proof. Suppose $\overrightarrow{u}$ is a constant unit vector in $\mathbb{E}^4$ such that $\langle \overrightarrow{u}, T\rangle = c$ for a constant c. Then using equation (2.1), we get $\langle \overrightarrow{u}, N\rangle = 0$, which on differentiating and using equation (2.1), gives

$$\langle \overrightarrow{u}, B_1\rangle = c\left(\frac{\kappa_1}{\kappa_2}\right).$$

Differentiating above equation gives

$$\langle \overrightarrow{u}, B_2\rangle = \frac{c}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'$$

and consequently, the unit vector $\overrightarrow{u}$ has the following representation

$$\overrightarrow{u} = cT + c\left(\frac{\kappa_1}{\kappa_2}\right)B_1 + \frac{c}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)' B_2. \tag{5.1}$$

Differentiation of above equation, in view of equation (2.1) leads to

$$c\left(\left(\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'\right)' + \left(\frac{\kappa_1}{\kappa_2}\right)\kappa_3\right)B_2 = 0,$$

which implies either $c = 0$ or

$$\left(\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'\right)' + \left(\frac{\kappa_1}{\kappa_2}\right)\kappa_3 = 0. \tag{5.2}$$

However, using $c = 0$ in equation (5.1) gives $\overrightarrow{u} = \overrightarrow{0}$, a contradiction. Hence, equation (5.2) holds, which on multiplication by $2\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'$ leads to

$$\left(\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'\right)^2 + \left(\frac{\kappa_1}{\kappa_2}\right)^2 = a^2,$$

where constant $a \neq 0$ is a constant (by virtue of the fact that $\alpha(t)$ is a Frenet curve). The above equation could be rearranged as

$$\frac{\left(\frac{\kappa_1}{\kappa_2}\right)'}{\sqrt{a^2 - \left(\frac{\kappa_1}{\kappa_2}\right)^2}} = \kappa_3,$$

(we have considered only positive sign of the radical, as the negative sign could be adjusted with the constant in the final result giving the same result). Integrating above equation, we get the result.

Conversely, assume that the unit speed Frenet curve $\alpha(t)$ in the Euclidean space $\mathbb{E}^4$ satisfies

$$\frac{\kappa_1}{\kappa_2} = a \sin\left(\int_0^t \kappa_3 du\right), \tag{5.3}$$

where a is a nonzero constant. Define a unit vector $\overrightarrow{u}$ in $\mathbb{E}^4$ by

$$\overrightarrow{u} = \frac{1}{\sqrt{1+a^2}}T + \frac{a}{\sqrt{1+a^2}} \sin\left(\int_0^t \kappa_3 du\right) B_1 + \frac{a}{\sqrt{1+a^2}} \cos\left(\int_0^t \kappa_3 du\right) B_2.$$

Differentiation of above equation, in view of equations (2.1) and (5.3), gives

$$\begin{aligned}
\overrightarrow{u}' &= \frac{\kappa_1}{\sqrt{1+a^2}}N - \frac{a\kappa_2}{\sqrt{1+a^2}} \sin\left(\int_0^t \kappa_3 du\right) N \\
&= \frac{\kappa_2}{\sqrt{1+a^2}} \left(\frac{\kappa_1}{\kappa_2} - a \sin\left(\int_0^t \kappa_3 du\right)\right) N = \overrightarrow{0},
\end{aligned}$$

that is, $\overrightarrow{u}$ is a constant unit vector that satisfies $\langle \overrightarrow{u}, T\rangle = \frac{1}{\sqrt{1+a^2}}$. Hence the tangent vector field T of the Frenet curve $\alpha(t)$ makes a constant angle with the constant unit vector $\overrightarrow{u}$. □

As a consequence of above Proposition, we have the following:

Corollary 5.1. *There does not exist a Frenet ccr-curve in the Euclidean space $\mathbb{E}^4$ whose tangent vector field makes a constant angle with a constant unit vector field.*

Proof. Suppose $\alpha : I \to \mathbb{E}^4$ is a Frenet ccr-curve such that $\langle \overrightarrow{u}, T\rangle = c$, where c is a constant. Then differentiating equation (5.3) would give

$$\cos\left(\int_0^t \kappa_3 du\right) = 0,$$

as $\kappa_3 \neq 0$ for a Frenet curve. The above equation implies

$$\int_0^t \kappa_3 du = \text{ a constant},$$

which on differentiation leads to $\kappa_3 = 0$, a contradiction. □

We also have the following Proposition, whose proof is parallel to Proposition 5.1.

Proposition 5.2. *The second binormal vector field B_2 of a unit speed Frenet curve $\alpha : I \to \mathbb{E}^4$ makes a constant angle with a constant unit vector in $\mathbb{E}^4$ if and only if*

$$\frac{\kappa_3}{\kappa_2} = a \sin\left(\int_0^t \kappa_1 du\right),$$

where $a \neq 0$ is a constant.

As a consequence of above Proposition, we have the following:

Corollary 5.2. *There does not exist a Frenet ccr-curve in the Euclidean space $\mathbb{E}^4$ whose second binormal vector field makes a constant angle with a constant unit vector field.*

Corollary 5.3. *There does not exist a Frenet ccr-curve in the Euclidean space $\mathbb{E}^4$ whose principal normal vector field makes a constant angle with a constant unit vector field.*

Proof. Suppose $\alpha(t)$ be a Frenet ccr-curve in $\mathbb{E}^4$ whose principal normal vector field N makes a constant angle with a constant unit vector $\overrightarrow{u}$, that is $\langle \overrightarrow{u}, N\rangle = c$ for a constant c. Since, for a ccr-curve we have $\frac{\kappa_2}{\kappa_1} = c_1$, $\frac{\kappa_3}{\kappa_2} = c_2$, and $\frac{\kappa_3}{\kappa_1} = c_1c_2$, where c_1, c_2 are nonzero constants. Differentiating $\langle \overrightarrow{u}, N\rangle = c$, we get

$$\langle \overrightarrow{u}, T\rangle = c_1 \langle \overrightarrow{u}, B_1\rangle, \tag{5.4}$$

which again on differentiation leads to

$$\langle u, B_2\rangle = \frac{c}{c_2}\left(1 + \frac{1}{c_1^2}\right). \tag{5.5}$$

Differentiating above equation, we get $\langle \overrightarrow{u}, B_1\rangle = 0$ and consequently, by equation (5.4) that $\langle \overrightarrow{u}, T\rangle = 0$. Hence, the unit vector field $\overrightarrow{u}$ admits the following expression

$$\overrightarrow{u} = cN + \frac{c}{c_2}\left(1 + \frac{1}{c_1^2}\right)B_2, \tag{5.6}$$

and differentiation of this equation leads to

$$\overrightarrow{0} = c\left(-\kappa_1 T + \kappa_2 B_1 - \frac{\kappa_3}{c_2}\left(1 + \frac{1}{c_1^2}\right)B_1\right) = c\left(-\kappa_1 T - \frac{\kappa_2}{c_1^2}B_1\right).$$

Thus $c = 0$, which in view of equation (5.6), gives $\overrightarrow{u} = \overrightarrow{0}$, a contradiction and this proves the Corollary. □

We also have the following Corollary, whose proof is parallel to Corollary 5.3.

Corollary 5.4. *There does not exist a Frenet ccr-curve in the Euclidean space $\mathbb{E}^4$ whose first binormal vector field makes a constant angle with a constant unit vector field.*

Combining corollaries 5.1-5.4, we have the following:

Theorem 5.1. *There does not exist a Frenet ccr-curve in the Euclidean space $\mathbb{E}^4$ whose any Frenet frame vector field makes a constant angle with a constant unit vector field.*

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