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Certain Constant Angle Surfaces Constructed on Curves in Minkowski 3-Space

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ABSTRACT

A constant angle surface in Minkowski space is a spacelike or timelike surface whose unit normal vector field makes a constant (hyperbolic) angle with a fixed vector. In this paper, we deal with certain special spacelike and timelike ruled surfaces in $\mathbb{R}^3_1$ under the general theorem of characterization of constant angle surfaces. We study the normal, binormal, rectifiying developable, Darboux developable and conical surfaces from the point of view the constant angle property in $\mathbb{R}^3_1$. Moreover, we give some related examples with their figures.

Keywords: *Constant angle surfaces; ruled surface; helix; slant helix; Minkowski Space.*

AMS Subject Classification (2010): *Primary: 53B25*

1. Introduction

In recent years much work has been done to understand the geometry of surfaces whose unit normal vector field forms a constant angle with a fixed field of directions of the ambient space. These surfaces are called helix surfaces or constant angle surfaces and they have been studied in all the 3-dimensional geometries. This kind of surfaces are models to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids [2]. Constant angle surfaces were studied in product spaces $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{S}^2$ and $\mathbb{H}^2$ represent the unit 2-sphere in $\mathbb{R}^2$ and $\mathbb{R}^2_1$, respectively [3, 4]. The angle was considered between the unit normal of the surface M and the tangent direction to $\mathbb{R}$. Munteanu and Nistor obtained a classification of all surfaces in Euclidean 3-space for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to $\mathbb{R}$ [14]. Moreover in [15] it is also classified certain special ruled surfaces in $\mathbb{R}^3$ under the general theorem of characterization of constant angle surfaces. A classification is given of special developable surfaces and some conical surfaces from the point of view the constant angle property in [17]. Also some characterizations are given for a curve lying on a surface for which the unit normal makes a constant angle with a fixed direction [17].

Lopez and Munteanu studied constant angle surfaces in Minkowski 3-space. They investigated spacelike surfaces with the constant timelike direction [12]. Atalay et al. by choosing the constant direction spacelike, they obtained different parameterization for the spacelike surfaces. It is shown that the minimal spacelike constant angle surfaces are planes [1]. Also, the classifications are given for the timelike surfaces whose the normals make a constant angle with a constant direction and it is shown that the minimal timelike constant angle surfaces are planes by Guler et al in [6].

In this paper, we deal with certain special spacelike and timelike ruled surfaces in $\mathbb{R}^3_1$ under the general theorem of characterization of constant angle surfaces. We study the normal, binormal, rectifiying developable, Darboux developable and conical surfaces from the point of view the constant angle property in $\mathbb{R}^3_1$. Finally, some examples of these surfaces are given with their figures by using the Maple Programme.

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2. Preliminary

In this section, we give the basic notations and some results in the general theory of curves and surfaces in Minkowski 3-space. For more details, we refer to [13][16]. Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The Lorentzian scalar product of x and y is defined by

$$< x, y >_L = x_1y_1 + x_2y_2 - x_3y_3.$$

The vector space on $\mathbb{R}^3$ equipped with the Lorentzian scalar product is called Minkowski 3- space denoted by $\mathbb{E}^3_1$. For a vector $x \in \mathbb{E}^3_1$, the sign of $< x, x >_L$ determines the type of x. The vector $x \in \mathbb{E}^3_1$ is a called a spacelike vector if $< x, x >_L > 0$ or $x = 0$, null vector if $< x, x >_L = 0$ for $x \neq 0$ and a timelike vector if $< x, x >_L < 0$. For $x \in \mathbb{E}^3_1$, the norm of x is defined by $\| x \|_L = \sqrt{|< x, x >_L|}$ and x is called a unit vector if $\| x \|_L = 1$. For any $x = (x_1, x_2, x_3)\,, y = (y_1, y_2, y_3) \in \mathbb{E}^3_1$, Lorentzian cross product is defined by

$$x \times_L y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2)\,.$$

For vectors x, y in $\mathbb{E}^3_1$ are said to be orthogonal if $< x, y >_L = 0$.

An arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}^3_1$ is spacelike, timelike or null if all of its velocity vector $\alpha'(s)$ are respectively spacelike, timelike or null for each $s \in I \subset \mathbb{R}$. The curve $\alpha = \alpha(s)$ is called a unit speed curve if velocity vector α' is unit, i.e $\|\alpha'(s)\|_L = 1$. Throughout this paper we shall assume all curves are parameterized by its arc lenght. Let α be a non-null curve and $\{t(s), n(s), b(s)\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \varepsilon_b\kappa & 0 & \tau \\ 0 & \varepsilon_t\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.1}$$

where $\langle x, x\rangle_L = \varepsilon_x$ and κ, τ are curvature function and torsion function respectively [7].

The Darboux vector is given $w = -\tau t + \kappa b$ and $w = \tau t + \kappa b$ for spacelike curve and timelike curve respectively.

A surface in the Minkowski space $\mathbb{E}^3_1$ is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [16].

A natural extension from curves to the theory of surfaces constructed on curves can be made as follows. Given a curve parameterized by arc length, we can think of constructing ruled surfaces involving and the tangent, normal, binormal or Darboux lines to the curve. As a consequence, we have well known types of surfaces of this kind, namely

- tangent developable surface: $r(s, v) = \alpha(s) + vt(s)$
- normal surface: $r(s, v) = \alpha(s) + vn(s)$
- binormal surface: $r(s, v) = \alpha(s) + vb(s)$
- rectifying developable surface: $r(s, v) = \alpha(s) + vw(s)$
- Darboux developable surface: $r(s, v) = b(s) + vt(s)$
- tangential Darboux developable surface: $r(s, v) = w(s) + vn(s)$

(see for details [5, 8]).

Consider a (connected) surface M and a smooth immersion $x : M \longrightarrow \mathbb{E}^3_1$, x is called a spacelike immersion if the induced metric on M via x is a Riemannian metric. This is equivalent to saying that any unit normal vector field x of M is timelike at each point. In particular, $x : M \longrightarrow \mathbb{E}^3_1$ is a spacelike immersion, then the surface M is orientable [12].

Definition 2.1. Let $x : M \longrightarrow \mathbb{E}^3_1$ be a spacelike immersion and let N be a unit normal vector field on M. M is called a constant angle spacelike surface if there is a fixed timelike vector k such that N makes a constant hyperbolic angle with k, namely $\langle N, k\rangle_L = -\cosh\theta$ [12].

This definition is extended for constant angle spacelike surface and fixed spacelike direction k, and for constant angle timelike surface whose the normal makes a constant angle with a constant direction k in [1] and [6] respectively as follows:

Definition 2.2. Let $x : M \longrightarrow \mathbb{E}^3_1$ be a spacelike immersion and let N be a unit normal vector field on M. M is called a constant angle spacelike surface if there is a fixed spacelike vector k such that N makes a constant hyperbolic angle with k, namely $\langle N, k\rangle_L = \sinh\theta$ [1].

Definition 2.3. Let M be a timelike surface and $N = (n_1, n_2, n_3)$ be a unit normal vector field on M. M is called a constant angle timelike surface if there is a fixed spacelike vector k such that N makes a constant angle with k. There are two cases for angle θ:
a) if $|n_1| > 1$, $\langle N, k\rangle_L = \cosh\theta$ b) if $|n_1| \leq 1$, $\langle N, k\rangle_L = \cos\theta$ [6].

Definition 2.4. Let M be a timelike surface and N be a unit normal vector field on M. M is called a constant angle timelike surface if there is a fixed timelike vector k such that N makes a constant hyperbolic angle with k, namely $\langle N, k\rangle_L = \sinh\theta$ [6].

In this paper, we assume that if the surface is timelike and $N = (n_1, n_2, n_3)$ is a unit normal vector field on M, with $|n_1| \leq 1$.

3. Developable Constant Angle Surfaces

In this section we consider three developable surfaces associated to a space curve. Developable surfaces are ruled surfaces. A ruled surface M in $\mathbb{E}^3_1$ is (locally) the map $r : I \times \mathbb{R} \longrightarrow \mathbb{R}^3_1$ defined by $r(s, v) = \alpha(s) + v\beta(s)$, where $\alpha : I \longrightarrow \mathbb{E}^3_1$, $\beta : I \longrightarrow \mathbb{E}^3_1 \setminus \{0\}$ are smooth mappings and I is an open interval. α is called the base curve and β is called the director curve. In particular, the ruled surface M is said to be cylindrical if the director curve β is constant and non-cylindrical otherwise. First of all, we consider that the base curve α is spacelike or timelike. In this case, the director curve β can be naturally chosen so that it is orthogonal to α. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows: If the base curve α is spacelike or timelike, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is spacelike, it is said to be of type M^1_+ or M^2_+ if β' is non-null or lightlike, respectively. When β is timelike, β' must be spacelike by causal character. In this case, M is said to be of type M^3_+. On the other hand, for the ruled surface of type M_-, it is also said to be of type M^1_- or M^2_- if β' is non-null or lightlike, respectively. Note that in the case of type M_- the director curve β is always spacelike. The ruled surface of type M^1_+ or M^2_+ (resp. M^3_+, M^1_- or M^2_-) is clearly spacelike (resp. timelike). But, if the base curve α is a lightlike curve and the vector field β along α is a lightlike vector field, then the ruled surface M is called a null scroll [9].

A tangent developable surface (resp. cylinder, cone) is a constant angle surface if and only if the generating curve is a helix (resp. a straight line, a circle) [12]. Given a regular curve $\alpha : I \longrightarrow \mathbb{E}^3_1$ the tangent surface M is generated by α as the surface parameterized by $r(s, v) = \alpha(s) + v\alpha'(s)$, $(s, v) \in I \times \mathbb{R}$.

Let us recall the concept of a helix in Minkowski space. A spacelike (or timelike) curve $\alpha = \alpha(s)$ parameterized by the arc-length is called a helix if there exists a vector $v \in \mathbb{E}^3_1$ such that the function $\langle\alpha'(s), v\rangle_L$ is constant. This is equivalent to saying that the function $\frac{\tau}{\kappa}$ is constant.

The characterization of tangent developable spacelike constant angle surfaces in $\mathbb{E}^3_1$ was given in [12]:
"*The tangent developable spacelike constant angle surfaces are generated by helices with* $\tau^2 < \kappa^2$". Moreover the direction k with which M makes a constant hyperbolic angle θ can be taken such that

$$k = \frac{1}{\sqrt{\kappa^2 - \tau^2}} \left(\tau(s)t(s) + \kappa(s)b(s)\right),$$

and the angle θ is determined by the relation

$$\cosh\theta = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}.$$

Also, the concept of constant angle surfaces extended for tangent developable timelike surfaces in [12].

Now, we deal with the other four types of spacelike and timelike surfaces constructed on a spatial curve in $\mathbb{E}^3_1$, normal, binormal, rectifiying developable, Darboux developable surfaces.

Theorem 3.1. *The normal constant angle spacelike surfaces are pieces of Lorentzian planes.*

Proof. Let us consider normal spacelike surface M oriented, immersed in $\mathbb{E}^3_1$ given by

$$r(s, v) = \alpha(s) + vn(s),$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve parameterized by arc length consisting of the edge of regression of M, n is the unit normal to α. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v, and applying the Frenet formulas given in (2.1) we get

$$r_s(s, v) = (1 + \varepsilon_b\kappa v)\, t + \tau v b \text{ and } r_v(s, v) = n.$$

Since M is a spacelike surface $r_s(s,v)$and $r_v(s,v)$ must be spacelike vectors. So, in the Frenet trihedron t, n are spacelike vectors and b is a timelike vector, $\varepsilon_b = -1$. Thus we can write $t(s) \times_L n(s) = -b(s)$ and $n(s) \times_L b(s) = t(s)$. The normal to the spacelike surface is given by

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm \frac{\tau v t + (1 - \kappa v)\, b}{\sqrt{\Delta}}, \text{ where } \Delta = -(\tau v)^2 + (1 - \kappa v)^2 . \tag{3.1}$$

On the other hand, since $\{t, n, N\}$ is an orthonormal frame field along α for spacelike ruled surface M, multiplying Eq.(3.1) by t, we obtain

$$\tau = 0.$$

Since being a plane curve it's binormal coincides with the normal of the spacelike plane. Thinking now the normal spacelike surface like a ruled surface for which the rulings are the normal lines to the generating spacelike plane curve α, we get that the normal constant angle spacelike surface is a portion of spacelike plane. □

Theorem 3.2. *The normal constant angle timelike surfaces are pieces of Lorentzian planes.*

Proof. Let us consider normal timelike surface M oriented, immersed in $\mathbb{E}^3_1$ given by

$$r(s,v) = \alpha(s) + v n(s),$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial timelike curve parameterized by arc length consisting of the edge of regression of M, n is the unit normal to α. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v, and applying the Frenet formulas given in (2.1) we get

$$r_s(s,v) = (1 + \kappa v)\, t + \tau v b \text{ and } r_v(s,v) = n,$$

and the normal to the surface is given by

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{\tau v t + (1 + \kappa v)\, b}{\sqrt{\Delta}}, \text{ where } \Delta = -(\tau v)^2 + (1 + \kappa v)^2 . \tag{3.2}$$

Since $\{t, n, N\}$ is an orthonormal frame field along α for timelike ruled surface M, multiplying Eq.(3.2) by t, we obtain

$$\tau = 0.$$

Since being a plane curve it's binormal coincides with the normal of the timelike plane. Thinking now the normal timelike surface like a ruled surface for which the rulings are the normal lines to the generating timelike plane curve α, we get that the normal constant angle timelike surface is a portion of timelike plane. □

Remark 3.1. In Theorem 3.2, in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve we can obtain

$$N = \mp \frac{\tau v t + (-1 - \varepsilon_b \kappa v)\, b}{\sqrt{\Delta}}, \text{where } \Delta = (\tau v)^2 + (1 + \varepsilon_b \kappa v)^2 .$$

Then it is not difficult to show that this condition is equivalent to saying that $\tau = 0$ and M is affine plane.

Theorem 3.3. *The binormal constant angle spacelike surfaces are generated by the spacelike plane curves.*

Proof. Let us consider first the parameterization of a binormal surface

$$r(s,v) = \alpha(s) + v b(s),$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve parameterized by arc length consisting of the edge of regression of M, b is the unit binormal to α. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v, and applying the Frenet formulas given in (2.1) we get

$$r_s(s,v) = t + \tau v n \text{ and } r_v(s,v) = b.$$

Since M is a spacelike surface $r_s(s,v)$and $r_v(s,v)$ must be spacelike vectors. So, in the Frenet trihedron t, b are spacelike vectors and n is a timelike vector. Thus we can write $t(s) \times_L n(s) = -b(s)$ and $n(s) \times_L b(s) = -t(s)$.

The normal to the spacelike surface is

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm \frac{\tau v t + n}{\sqrt{\Delta}}, \text{where } \Delta = -(\tau v)^2 + 1. \tag{3.3}$$

We are interested in those normal surfaces for which the normal N makes a constant hyperbolic angle with the fixed timelike direction k, namely $\langle N, k\rangle_L = -\cosh\theta$. Substituting (3.3) in this expression we get a vanishing polynomial expression of second order in v. So, all the coefficients must be identically zero, that is the following relations are satisfied:

$$\tau^2 \left(\langle t, k\rangle_L^2 + \cosh^2\theta\right) = 0, \tag{3.4}$$

$$\tau \langle t, k\rangle_L \langle n, k\rangle_L = 0, \tag{3.5}$$

$$\langle n, k\rangle_L^2 - \cosh^2\theta = 0.$$

From (3.4) and (3.5) we get

$$\tau = 0.$$

It follows that α is a spacelike plane curve. □

Theorem 3.4. *The binormal constant angle timelike surfaces are generated by the plane curves.*

Proof. Let us consider first the parameterization of a binormal surface

$$r(s, v) = \alpha(s) + v b(s), \tag{3.6}$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ is a spatial timelike curve parameterized by arc length consisting of the edge of regression of M, b is the unit binormal to α. Let us determine the normal to the surface.

By taking the partial derivatives of (3.6) with respect to s and v, and applying the Frenet formulas given in (2.1), we get

$$r_s(s, v) = t - \tau v n \text{ and } r_v(s, v) = b.$$

Computing the normal to the above surface, one gets

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{\tau v t - n}{\sqrt{\Delta}}, \text{where } \Delta = -(\tau v)^2 + 1. \tag{3.7}$$

We are interested in those normal surfaces for which the normal N makes a constant angle with the fixed spacelike direction k, namely $\langle N, k\rangle_L = \cos\theta$. Substituting (3.7) in this expression we get a vanishing polynomial expression of second order in v. So, all the coefficients must be identically zero, that is the following relations are satisfied:

$$\tau^2 \left(\langle t, k\rangle_L^2 + \cos^2\theta\right) = 0, \tag{3.8}$$

$$\tau \langle t, k\rangle_L \langle n, k\rangle_L = 0, \tag{3.9}$$

$$\langle n, k\rangle_L^2 = \cos^2\theta.$$

From (3.8) and (3.9) we get

$$\tau = 0.$$

It follows that α is a spacelike plane curve. □

Remark 3.2. In Theorem 3.4, in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ is a spatial spacelike curve

(i) If we consider the Frenet frame $\{t(s), n(s), b(s)\}$ for the spacelike curve $\alpha(s)$, in this trihedron $t(s), n(s)$ are spacelike vectors and $b(s)$ is a timelike vector, we get

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{\tau v t - n}{\sqrt{\Delta}}, \text{where } \Delta = (\tau v)^2 + 1.$$

(ii) If we now consider the Frenet frame $\{t(s), n(s), b(s)\}$ for the spacelike curve $\alpha(s)$, in this trihedron $t(s), b(s)$ are spacelike vectors and $n(s)$ is a timelike vector, we get

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm \frac{\tau v t + n}{\sqrt{\Delta}}, \text{where } \Delta = (\tau v)^2 - 1.$$

In two case, it is not difficult to show that this condition is equivalent to saying that $\tau = 0$ and α is a spacelike plane curve. But, since $\tau = 0$, spatial spacelike curve's normal coincides with the normal of the timelike surfaces. Thus, the Frenet frame $\{t(s), n(s), b(s)\}$ for the spacelike curve $\alpha(s)$, in this trihedron $t(s), n(s)$ must be spacelike vectors and $b(s)$ must be a timelike vector.

We present example of constant angle surface that is binormal surface.

Example 3.1. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$, $\alpha(s) = r\left(0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)\right)$ is a spatial spacelike curve. Then

$$t(s) = \left(0, \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right)\right),$$

$t' = \frac{1}{r}\left(0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)\right)$. Then $\kappa = \frac{1}{r}$. Moreover,

$$n(s) = \left(0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)\right),$$

$$b(s) = (1, 0, 0).$$

Here $\tau = 0$. Let us consider binormal constant angle surface M oriented, immersed in $\mathbb{E}^3_1$, parameterized by

$$r(s, v) = \alpha(s) + vb(s),$$

$$r(s, v) = \left(v, r\sinh\left(\frac{s}{r}\right), r\cosh\left(\frac{s}{r}\right)\right).$$

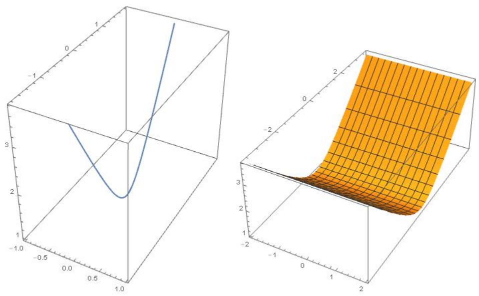

Figure 1

A picture of the curve α and the corresponding binormal constant angle surface appears in Figure 1, where $r = 1$, $s \in [-2, 2]$, $v \in [-2, 2]$.

Theorem 3.5. *The rectifying developable constant angle spacelike surfaces are generated by spacelike slant helices.*

Proof. Let us consider a rectifying developable spacelike surface M oriented, immersed in $\mathbb{E}^3_1$ given by

$$r(s, v) = \alpha(s) + vw(s), \tag{3.10}$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve parameterized by arc length consisting of the edge of regression of M and $w = -\tau t + \kappa b$ is the Darboux vector to α. The surface M is smooth everywhere, except in points of the curve. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v, and use (2.1), we obtain

$$r_s(s, v) = (1 - v\tau')\, t + \kappa' vb \text{ and } r_v(s, v) = -\tau t + \kappa b. \tag{3.11}$$

The normal to the surface is given by

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm n. \tag{3.12}$$

Choosing an orientation of the spacelike surface we take as the normal to the surface equal to the normal of the generating spacelike curve α. In the case of constant angle surfaces it follows that the normal n of the curve makes a constant angle with the fixed timelike direction k, namely

$$\langle n, k\rangle_L = \langle N, k\rangle_L = -\cosh\theta. \tag{3.13}$$

It follows that α is a spacelike slant helix. $\square$

Theorem 3.6. *The rectifying developable constant angle timelike surfaces are generated by slant helices.*

Proof. Let us consider a rectifying developable spacelike surface M oriented, immersed in $\mathbb{E}_1^3$ given by

$$r(s,v) = \alpha(s) + vw(s), \tag{3.14}$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ is a spatial timelike curve parameterized by arc length consisting of the edge of regression of M and $w = \tau t + \kappa b$ is the Darboux vector to α. The surface M is smooth everywhere, except in points of the curve. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v, and use (2.1), we obtain

$$r_s(s,v) = (1 + v\tau')\, t + \kappa' v b \text{ and } r_v(s,v) = \tau t + \kappa b. \tag{3.15}$$

The normal to the surface is given by

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm n. \tag{3.16}$$

Choosing an orientation of the timelike surface we take as the normal to the surface equal to the normal of the generating timelike curve α. In the case of constant angle surfaces it follows that the normal n of the timelike curve makes a constant angle with the fixed spacelike direction k, namely

$$\langle n, k \rangle_L = \langle N, k \rangle_L = \cos\theta. \tag{3.17}$$

It follows that α is a timelike slant helix. □

Remark 3.3. In Theorem 3.6, in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ is a spatial spacelike curve we can obtain

$$N = \mp n.$$

Then saying that easily α is a spacelike slant helix.

Theorem 3.7. *The Darboux developable constant angle spacelike surfaces are generated by binormal curves of spacelike helices.*

Proof. Let us consider Darboux developable surface M oriented, immersed in $\mathbb{E}_1^3$ given by

$$r(s,v) = b(s) + vt(s),$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ is a spatial spacelike curve parameterized by arc length consisting of the edge of regression of M, t and b are the unit tangent and binormal to α recpectively. The surface M is smooth everywhere, except in points of the curve. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v, and use (2.1), we obtain

$$r_s(s,v) = (\tau + \kappa v)\, n \text{ and } r_v(s,v) = t.$$

The normal to the surface is given by

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp b.$$

Choosing an orientation of the surface we take as the normal to the surface equal to the binormal of the generating curve α. In the case of constant angle surfaces it follows that the binormal b of the spacelike curve makes a constant angle with the fixed timelike direction k, namely

$$\langle b, k \rangle_L = \langle N, k \rangle_L = -\cosh\theta.$$

It follows that α is a spacelike helix. □

Theorem 3.8. *The Darboux developable constant angle timelike surfaces are generated by binormal curves of helices.*

Proof. Let us consider Darboux developable surface M oriented, immersed in $\mathbb{E}^3_1$ given by

$$r(s,v) = b(s) + vt(s), \ \ (s,v) \in I \times \mathbb{R}$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike or timelike curve parameterized by arc length consisting of the edge of regression of M, t and b are the unit tangent and binormal to α respectively. The surface M is smooth everywhere, except in points of the curve. Let us determine the normal to the timelike surface. To do this, we compute the partial derivatives of r with respect to s and v.

$$r_s(s,v) = (\varepsilon_t \tau + \kappa v)\, n \text{ and } r_v(s,v) = t.$$

The normal to the timelike surface is given by
(i) in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp b,$$

(ii) in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial timelike curve

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm b.$$

Choosing an orientation of the surface we take as the normal to the timelike surface equal to the binormal of the generating curve α. In the case of constant angle surfaces it follows that the binormal b of the curve makes a constant angle with the fixed spacelike direction k, namely

$$\langle b, k\rangle_L = \langle N, k\rangle_L = \cos\theta.$$

It follows that α is a helix. □

We present example of constant angle surface that is Darboux developable constant angle surface.

Example 3.2. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$, $\alpha(s) = (\cosh\theta \sin s, -\cosh\theta\cos s, (\sinh\theta)\, s)$ is a spatial spacelike curve. Then

$$t(s) = (\cosh\theta\cos s, \cosh\theta \sin s, \sinh\theta).$$

An easy computation leads to

$$n(s) = (-\sin s, \cos s, 0),$$

$$b(s) = (-\sinh\theta\cos s, -\sinh\theta\sin s, -\cosh\theta).$$

Here $\kappa(s) = \cosh\theta, \tau(s) = -\sinh\theta$ and α is a helix where both the curvature and torsion functions are constant. Let us consider Darboux developable constant angle surface M oriented, immersed in $\mathbb{E}^3_1$, parameterized by

$$r(s,v) = b(s) + vt(s),$$

$$r(s,v) = (-\sinh\theta\cos s, -\sinh\theta\sin s, -\cosh\theta) + v(\cosh\theta\cos s, \cosh\theta\sin s, \sinh\theta).$$

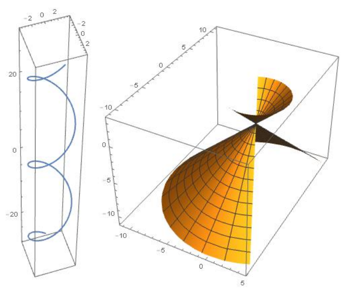

Figure 2.

A picture of the curve α and the corresponding Darboux developable constant angle surface appears in Figure 2, where $\theta = 2$, $s \in [-2,2]$, $v \in [-2,2]$.

4. Conical Constant Angle Surfaces

A cone is a ruled surface that can be parameterized by $r(s,v) = v\alpha(s)$, where α is a regular curve. The vertex of the cone is the origin and the surface is regular wherever $t\,(\alpha(s) \times \alpha'(s)) \neq 0$. A ruled surface is called a cylinder if it can be parameterized by $r(s,v) = \alpha(s) + uv$, where α is a regular curve and u is a fixed vector. The regularity of the cylinder is given by the fact that $\alpha'(s) \times u \neq 0$. The characterization of constant angle spacelike cylinders in $\mathbb{E}^3_1$ was given in [12] *"the only constant angle spacelike cylinders are planes"*.

Now let us consider the case of some conical surfaces regarded from the point of view of constant angle surfaces.

Theorem 4.1. *A tangent conical constant angle spacelike surfaces are generated by tangent curves of spacelike helices.*

Proof. A tangent conical surface with the vertex in the origin is given by

$$r(s,v) = vt(s),$$

where we consider now s, v standard parameters and t is the unit tangent to spacelike curve α. Computing the normal to the above surface, one gets

$$r_s(s,v) = \kappa v n \text{ and } r_v(s,v) = t,$$

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp b.$$

Choosing an orientation of the spacelike surface we take as the normal to the surface equal to the binormal of the generating spacelike curve α. In the case of constant angle surfaces it follows that the binormal b of the curve makes a constant angle with the fixed timelike direction k, namely

$$\langle b, k\rangle_L = \langle N, k\rangle_L = -\cosh\theta.$$

It follows that α is a spacelike helix. □

The concept of tangent conical constant angle surfaces can be extended for timelike surfaces easily and similar results to the previous theorem are obtained.

Definition 4.1. Let α be a curve in $\mathbb{E}^3_1$ with $\frac{\tau}{\kappa} \neq 0$ everywhere. A curve $\alpha(s)$ is said to be a Darboux helix if there is some constant unit vector k such that $< w, k >_L$ is constant along the curve α where $w(s)$ is a unit Darboux vector of α at s. The direction of the vector k is axis of the Darboux helix [19].

Theorem 4.2. *A normal conical constant angle spacelike surfaces are generated by normal curves of Darboux helices.*

Proof. A normal conical surface with the vertex in the origin is given by

$$r(s,v) = vn(s), \tag{4.1}$$

where we consider now s, v standard parameters and n is the unit normal to curve α. The normal to the spacelike surface is given by:

(i) in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve:

By taking the partial derivatives of (4.1) with respect to s and v, and applying the Frenet formulas given in (2.1), we get

$$r_s(s,v) = -\kappa v t + \tau v b \text{ and } r_v(s,v) = n.$$

Since the surface is spacelike, unit normal to curve α must be a spacelike vector and $\kappa^2 - \tau^2 > 0$. Computing the normal to the above surface, one gets

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{1}{\sqrt{\kappa^2 - \tau^2}}(-\tau t + \kappa b) = \mp \frac{w}{\|w\|_L}.$$

(ii) in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial timelike curve:

By taking the partial derivatives of (4.1) with respect to s and v, and applying the Frenet formulas given in (2.1), we get

$$r_s(s,v) = \kappa v t + \tau v b \text{ and } r_v(s,v) = n.$$

Since the surface is spacelike, unit normal to curve α must be a spacelike vector and $\tau^2 - \kappa^2 > 0$. Computing the normal to the above surface, one gets

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{1}{\sqrt{\tau^2 - \kappa^2}} (\tau t + \kappa b) = \mp \frac{w}{\|w\|_L}.$$

Choosing an orientation of the spacelike surface we take as the normal to the surface equal to the Darboux vector of the generating curve α. In the case of constant angle surfaces it follows that Darboux vector w of the curve makes a constant angle with the fixed timelike direction k, namely

$$\langle w, k \rangle_L = \langle N, k \rangle_L = -\cosh\theta. \tag{4.2}$$

It follows that α is a Darboux helix. □

Theorem 4.3. *A normal conical constant angle timelike surfaces are generated by normal curves of Darboux helices.*

Proof. A normal conical surface with the vertex in the origin is given by

$$r(s, v) = vn(s), \tag{4.3}$$

where we consider now s, v standard parameters and n is the unit normal to curve α. The normal to the timelike surface is given by:

(i) in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial timelike curve:

By taking the partial derivatives of (4.3) with respect to s and v, and applying the Frenet formulas given in (2.1), we get

$$r_s(s, v) = \kappa vt + \tau vb \text{ and } r_v(s, v) = n.$$

Computing the normal to the above surface, one gets with $\kappa^2 - \tau^2 > 0$

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{1}{\sqrt{-\tau^2 + \kappa^2}} (\tau t + \kappa b) = \mp \frac{w}{\|w\|_L}.$$

(ii) in the case that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3_1$ is a spatial spacelike curve:

(ii-a) If we consider the Frenet frame $\{t(s), n(s), b(s)\}$ for the spacelike curve $\alpha(s)$, in this trihedron $t(s), n(s)$ are spacelike vectors and $b(s)$ is a timelike vector. Thus we can write $t(s) \times_L n(s) = -b(s)$ and $n(s) \times_L b(s) = t(s)$. By taking the partial derivatives of (4.3) with respect to s and v, and applying the Frenet formulas given in (2.1), we get

$$r_s(s, v) = -\kappa vt + \tau vb \text{ and } r_v(s, v) = n.$$

Computing the normal to the above surface, one gets with $\tau^2 - \kappa^2 > 0$

$$N = \mp \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \mp \frac{1}{\sqrt{\tau^2 - \kappa^2}} (-\tau t + \kappa b) = \mp \frac{w}{\|w\|_L}.$$

(ii-b) If we now consider the Frenet frame $\{t(s), n(s), b(s)\}$ for the spacelike curve $\alpha(s)$, in this trihedron $t(s), b(s)$ are spacelike vectors and $n(s)$ is a timelike vector. By taking the partial derivatives of (4.3) with respect to s and v, and applying the Frenet formulas given in (2.1), we get

$$r_s(s, v) = \kappa vt + \tau vb \text{ and } r_v(s, v) = n.$$

Computing the normal to the above surface, one gets

$$N = \pm \frac{r_s \times_L r_v}{\|r_s \times_L r_v\|_L} = \pm \frac{1}{\sqrt{\tau^2 + \kappa^2}} (-\tau t + \kappa b) = \pm \frac{w}{\|w\|_L}.$$

In all case, choosing an orientation of the timelike surface we take as the normal to the surface equal to the Darboux vector of the generating curve α. In the case of constant angle surfaces it follows that Darboux vector w of the curve makes a constant angle with the fixed spacelike direction k, namely

$$\langle w, k \rangle_L = \langle N, k \rangle_L = \cos\theta. \tag{4.4}$$

It follows that α is a Darboux helix. □

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