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Concircular Vectors Field in (k, μ)-Contact Metric Manifolds

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ABSTRACT

The aim of the present paper is to study (k, μ)-contact manifolds admitting a non-null concircular vector field and concurrent vector field. We prove that in both the cases the manifold becomes a Sasakian manifold under certain restriction on k, μ.

Keywords: *(k, μ) contact manifolds; Concircular vector field; Concurrent vector field.*
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1. Introduction

In 1995 Blair, Koufogiorgos and Papantoniou [1] introduced the notion of (k, μ)- contact metric manifolds M of dimension $(2n + 1)$, where k and μ are real constants and a full classification of such manifolds was given by E. Boeckx [4]. Actually this class of space was obtained through D-homothetic deformation [14] to a contact metric manifold whose curvature satisfying $R(X, Y)\xi = 0$. There exist contact metric manifolds for which $R(X, Y)\xi = 0$. For instance the tangent sphere bundle of flat Riemannian manifold admits such structure. Further it is well known that [1] the tangent sphere bundle T_1M of a Riemannian manifold of constant curvature c is a (k, μ)-contact metric space where $k = c(2 - c)$ and $\mu = -2c$. Thus in one hand there exists examples of (k, μ)-contact manifolds in all dimensions and on the other this class is invariant under D-homothetic deformation. It is evident that the class of (k, μ)-contact manifolds contains the class of Sasakian manifolds, in which $k = 1$. The study of torseforming vector field, concircular vector field has a long history starting in 1925 by the work of Shirokov [13] and Yano [18]. These vector fields have been used in many areas of differential geometry, for example in conformal mappings and transformations, geodesics and holomorphically projective mappings and transformation, and others. Recently De and Pathok [7], Bagewadi et. al. [5] study these vector fields in Kenmotsu manifolds and Tran-Sasakian manifolds. Motivated by the above studies we plan is to study concircular vector fields and concurrent vector fields in (k, μ)-contact metric manifolds.

Definition 1.1. A vector field V on a Riemannian manifold is said to be concircular vector field [18] if it satisfies an equation of the form

$$\nabla_X V = \rho X, \tag{1.1}$$

for all smooth vector fields X, where ρ is a scalar.
If ρ is constant, then V is called a concurrent vector field.

The paper is organized as follows: Section 2 is eqquiped with some prerequisites about (k, μ)- contact metric manifolds. In section 3, we study (k, μ)-contact manifold admitting a non-null concircular vector field and concurrent vector field.

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2. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold. If there exits a triplet (ϕ, ξ, η) of a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on M^{2n+1} which satisfies the relation [2, 3]

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0,$$

then we say the triplet (ϕ, ξ, η) is an almost contact structure and the manifold is an almost contact manifold. If an almost contact manifold M^{2n+1} with an almost contact structure (ϕ, ξ, η) admits a Riemannian metric g such that [2]

$$g(X, Y) = g(\phi X, \phi Y) - \eta(X)\eta(Y),$$

then we say that M^{2n+1} is an almost contact metric structure (ϕ, ξ, η, g) and such a metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1, n)$. The fundamental 2-form of M^{2n+1} is defined by

$$\Phi(X, Y) = g(X, \phi Y).$$

An almost contact metric structure becomes a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y),$$

for all vector fields X, Y where

$$d\eta(X, Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X, Y])].$$

Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ define a (1.1) tensor field h by $h = L_\xi \phi$, where L denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Also we have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation hold:

$$\nabla_X \xi = -\phi X - \phi h X.$$

An almost contact structure is said to be normal if and only if the tensor $N_\phi - 2d\eta \otimes \xi$ vanishes identically, where N_ϕ is the Nijenhuis tensor of $\phi : N_\phi(X, Y) = [\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. A normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold M is Sasakian manifold if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for all vectors field X, Y, where ∇ is the Levi-Civita connection of the Riemannian metric ; while a contact metric manifold M is Sasakian [2] if and only if the curvature tensor R satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

As a generalization of $R(X, Y)\xi = 0$ and the Sasakian case: D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [1] consider the (k, μ)-nulitty condition on a contact metric manifold and gave several resons for studying it. The (k, μ)-nuliity distribution $N(k, \mu)$ ([1],[3]) of a contact metric manifold is defined by

$$\begin{aligned} N(k, \mu) : p \to N_p(k, \mu) \quad &= \quad [W \in T_p M | R(X, Y)W = \\ &\quad\ (kI + \mu h)(g(Y, W)X - g(X, W)Y)], \end{aligned}$$

for all vector fields $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. Thus we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y)] + \mu[\eta(Y)hX - \eta(X)hY].$$

On (k, μ)-contact metric manifold, $k \leq 1$. If $k = 1$, the structure is Sasakian and if $k < 1$, the (k, μ)-nullity condition completely determines the curvature of M^{2n+1} [1]. Again a (k, μ) contact manifold reduces to an $N(k)$-contact manifold if and only if $\mu = 0$.

Moreover, in a (k, μ) -contact manifold the following relation holds :

$$h^2 = (k-1)\phi^2, k \leq 1.$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX]. \tag{2.1}$$

(k, μ)-contact manifolds have been studied by several authors such as Papantoniou [12], De et al. ([6],[8],[9],[10],[11]), Yildiz et al. ([15],[16],[17]) and many others.

3. Concircular vector field

We suppose that a (k, μ)-contact manifold metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ admits a non-null concircular vector field. Then the relation (1.1) holds. Differentiating (1.1) covariently we get

$$\nabla_Y \nabla_X V = \rho \nabla_Y X + (Y\rho)X. \tag{3.1}$$

From (3.1) it follows that

$$\nabla_Y \nabla_X V - \nabla_X \nabla_Y V - \nabla_{[X,Y]} = (X\rho)Y - (Y\rho)X. \tag{3.2}$$

By the help of Ricci identity we obtain from (3.2)

$$R(X, Y)V = (X\rho)Y - (Y\rho)X.$$

$$\bar{R}(X, Y, V, W) = (X\rho)g(Y, Z) - (Y\rho)g(X, Z),$$

where $\bar{R}(X, Y, V, W) = g(R(X, Y)V, W)$.
Putting $W = \xi$ in the above equation yields

$$\eta(R(X, Y)V) = (X\rho)\eta(Y) - (Y\rho)\eta(X). \tag{3.3}$$

Making use of (2.1), we obtain

$$\begin{aligned} \eta(R(X, Y)V) &= k[\eta(X)g(Y, V) - \eta(Y)g(X, V)] \\ &\quad +\mu[g(hY, V)\eta(X) - g(hX, V)\eta(Y)]. \end{aligned} \tag{3.4}$$

By virtue of (3.3) and (3.4) we have

$$\begin{aligned} (X\rho)\eta(Y) - (Y\rho)\eta(X) &= k[\eta(X)g(Y, V) - \eta(Y)g(X, V)] \\ &\quad +\mu[g(hY, V)\eta(X) - g(hX, V)\eta(Y)]. \end{aligned} \tag{3.5}$$

Putting $Y = \xi$ and multiplying by $\eta(V)$ in (3.5) we get

$$[\xi\rho + k\eta(X)][g(V, X) - \eta(X)\eta(V)] = -\mu[g(hV, X)\eta(V)]. \tag{3.6}$$

Interchanging X and V in (3.6) implies

$$[\xi\rho + k\eta(V)][g(V, X) - \eta(X)\eta(V)] = -\mu[g(hV, X)\eta(X)]. \tag{3.7}$$

Subtracting (3.7) from (3.6) yields

$$\begin{aligned} k[\eta(V) - \eta(X)][g(V, X) - \eta(V)\eta(X)] &= \mu[g(hV, X)\eta(X) \\ &\quad -g(hV, X)\eta(V)]. \end{aligned} \tag{3.8}$$

Putting $X = \phi X$ in (3.8) we have

$$kg(V, \phi X) = -\mu g(h\phi X, V), \tag{3.9}$$

since $\eta(V) \neq 0$, which implies

$$k\phi V = \mu h\phi V. \tag{3.10}$$

Substituting $X = hX$ in (3.9) and using the fact $h^2 = (k-1)\phi^2$ we obtain

$$kg(V, \phi hX) = -(k-1)g(\phi X, V). \tag{3.11}$$

Multiplying (3.10) by k and (3.9) by $(k-1)$ and then adding we have

$$g(V, \phi hX) = 0, \tag{3.12}$$

provided $k^2 - \mu(k-1) \neq 0$. Substituting $X = hX$ in the above equation implies

$$g(V, \phi h^2 X) = 0. \tag{3.13}$$

Again using the fact $h^2 = (k-1)\phi^2$ in (3.13) yields

$$k = 1.$$

Thus we can state the following:

Theorem 3.1. *If a (k,μ)-contact manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ admitting a non-null concircular vector field, then the manifold is a Sasakian manifold, provided $[k^2-\mu(k-1)]\neq 0$.*

Next we assume that the vector field V is concurrent, that is, ρ is constant, so from (3.5) it follows that

$$\begin{aligned} & k[\eta(X)g(Y,V)-\eta(Y)g(X,V)] \\ & +\mu[g(hY,V)\eta(X)-g(hX,V)\eta(Y)]=0. \end{aligned} \tag{3.14}$$

Putting $X=hX$ in (3.14) and using the fact $h^2=(k-1)\phi^2$ we obtain

$$kg(hX,V)-\mu(k-1)g(X,V)+\mu(k-1)\eta(X)\eta(V)=0. \tag{3.15}$$

Substituting $X=hX$ in (3.15) implies

$$-k(k-1)g(X,V)+k(k-1)\eta(X)\eta(V)-\mu(k-1)g(hX,V)=0. \tag{3.16}$$

Subtracting μ multiple of (3.16) from k multiple of (3.15) we have

$$[k^2+\mu^2(k-1)]g(hX,V)=0. \tag{3.17}$$

Putting $X=hX$ and making use of $h^2=(k-1)\phi^2$ in (3.17) implies

$$k=1,$$

provided $[k^2+\mu^2(k-1)]\neq 0$.

Thus we can state the following:

Theorem 3.2. *If a (k,μ)-contact manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ admits a non-null concurrent vector field, then the manifold is a Sasakian manifold, provided $[k^2+\mu^2(k-1)]\neq 0$.*

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