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Upper Bound Inequalities for $\delta-$Casorati Curvatures of Submanifolds in Generalized Sasakian Space Forms Admitting a Semi-Symmetric Metric Connection

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ABSTRACT

In this paper, we establish two sharp inequalities, which involve the generalized normalized $\delta-$Casorati curvatures and the generalized normalized scalar curvature of any submanifold in generalized Sasakian space forms with semi-symmetric metric connection by using T Oprea's technique. Afterwards, we examine that the equality holds if and only if the submanifold is invariantly quasi-umbilical in both inequalities. We also develop these inequalities for invariant, anti-invariant, CR, slant, semi-slant, hemi-slant and bi-slant submanifolds in the same ambient space form with SSMC.

Keywords: *$\delta-$Casorati curvature; bi-slant submanifold; generalized Sasakian space forms; semi-symmetric metric connection; quasi-umbilical submanifold*
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1. Introduction

In 1993, B.-Y Chen [5] initiated the theory of $\delta-$invariants. B.-Y. Chen established a sharp inequality for a submanifold into the real space form using the scalar curvature and the sectional curvature, both being intrinsic invariants, and squared mean curvature, the main extrinsic invariant. That is, he established in [4] simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold in real space forms with any codimension. Now it has become one of the most interesting research topics in differential geometry of submanifolds. Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form. The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. It was preferred by Casorati over the traditional Gauss curvature. Several geometers in [7, 8, 13, 27, 28] found geometrical meaning and the importance of the Casorati curvature. Therefore, it attracts the geometers to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. Decu, Haesen and Verstraelen introduced the normalized $\delta-$Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ and established inequalities involving $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ for submanifolds in real space forms [7]. Moreover, the same authors proved in [8] an inequality in which the scalar curvature is estimated from above by the normalized Casorati curvatures, while Ghisoiu obtained in [10] some inequalities for the Casorati curvatures of slant submanifolds in complex space forms. Recently, Lee et al. in [16] obtained optimal inequalities for submanifolds in real space forms, endowed with a semi-symmetric metric connection. Many authors obtained the optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces [14, 15, 17, 22, 23, 25, 32].

The idea of a semi-symmetric linear connection in a differentiable manifold was introduced by Friedmann and Schouten in [9]. Later, Hayden [11] introduced the idea of a metric connection with torsion in a Riemannian

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manifold. Yano [31] studied semi-symmetric metric connection in a Riemannian manifold. Many other geometers have used this idea of connection in different ambient spaces such as real space forms, complex space forms, Sasakian space forms and so on (see [18, 19, 21]). On the other hand, Blair, Carriazo and Alegre [1] introduced the notion of a generalized Sasakian space form and proved some of its basic properties. In [22], Aliya and Shahid obtained some optimal inequalities involving the normalized Casorati curvatures and the normalized scalar curvature of bi-slant submanifold in generalized Sasakian space form and also studied these inequalities in different kinds of submanifolds and ambient space forms. So, in this paper we wish to obtain these optimal Casorati inequalities on any submanifold of a generalized Sasakian space form with semi-symmetric metric connection (SSMC). This paper can be considered as the next version of [22].

The paper is structured as follows: *Section 2* is devoted to preliminaries. *Section 3* deals with the study of Casorati curvatures for any submanifold of $(n+1)-$dimension. In *Section 4*, we establish two sharp inequalities that relate the normalized scalar curvature with generalized normalized δ-Casorati curvature for any submanifold in a generalized Sasakian space form with semi-symmetric metric connection (SSMC) with some immediate consequences. In *Section 5*, we give some applications as consequences of our derived inequalities in *Section 4*.

2. Preliminaries

A (2m + 1)-dimensional differentiable manifold $\overline{\mathcal{M}}$ is said to have an almost contact structure (ϕ, ξ, η, g) if there exists on $\overline{\mathcal{M}}$ a tensor field ϕ of type (1, 1), a vector field ξ, a 1-form η and a Riemannian metric g such that [30]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta \; o \; \phi = 0, \quad \eta(X) = g(X, \xi) \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) + g(X, \phi Y) = 0 \tag{2.2}$$

Here X, Y, Z denote arbitrary vector fields on $\overline{\mathcal{M}}$. The fundamental 2-form Φ on $\overline{\mathcal{M}}$ is defined by

$$\Phi(X, Y) = g(\phi X, Y)$$

Alegre et al. [1] introduced and studied the generalized Sasakian space forms. An almost contact metric manifold $(\overline{\mathcal{M}}, \phi, \xi, \eta, g)$ is said to be a *generalized Sasakian space form* if there exist differentiable functions f_1, f_2, f_3 such that the curvature tensor R of $\overline{\mathcal{M}}$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z \;=\; & f_1\big[g(Y, Z)X - g(X, Z)Y\big] + f_2\big[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ & +2g(X, \phi Y)\phi Z\big] + f_3\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & -g(Y, Z)\eta(X)\xi\big] \end{aligned} \tag{2.3}$$

for all vector fields $X, Y, Z \in T\overline{\mathcal{M}}$.

The generalized Sasakian space form generalizes the concept of Sasakian space form, Kenmotsu space form and cosymplectic space form.

(i) A Sasakian space form is the generalized Sasakian space form with $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

(ii) A Kenmotsu space form is the generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$.

(iii) A cosymplectic space form is the generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

Definition 2.1. A linear connection ∇^* on an $n-$dimensional Riemannian manifold $\overline{\mathcal{M}}$ with Riemannian metric g is called a semi-symmetric connection if the torsion tensor T of the connection ∇^* satisfies [31]

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a $1-$form associated with the vector field ξ on $\overline{\mathcal{M}}$ defined by $\eta(X) = g(X, \xi)$ and ∇^* is called a semi-symmetric metric connection if it satisfies $\nabla^* g = 0$.

Remark 2.1. A semi-symmetric metric connection ∇^* on $\overline{\mathcal{M}}$ is given by [31]

$$\nabla^*_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi,$$

where ∇ is the Levi-Civita connection of $\overline{\mathcal{M}}$.

Let $\overline{R}$ and $\overline{R}^*$ be curvature tensors of ∇ and ∇^* of a Riemannian manifold $\overline{\mathcal{M}}$, respectively. Then we have the following relation [31]:

$$\begin{aligned}\overline{R}^*(X,Y)Z &= \overline{R}(X,Y)Z - \gamma(Y,Z)X + \gamma(X,Z)Y \\ &\quad -g(Y,Z)FX + g(X,Z)FY \end{aligned} \tag{2.4}$$

for all vector fields $X, Y, Z \in T\overline{\mathcal{M}}$, where γ is the $(0,2)-$tensor field defined by

$$\gamma(X,Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X,Y)$$

and

$$g(FX,Y) = \gamma(X,Y).$$

In the following we consider $\overline{\mathcal{M}}$ as a generalized Sasakian space form $\overline{\mathcal{M}}(f_1,f_2,f_3)$ of dimension $(2m+1)$ with a semi-symmetric metric connection and let $\mathcal{M}$ be an $(n+1)-$dimensional submanifold of $\overline{\mathcal{M}}(f_1,f_2,f_3)$. Let $T\mathcal{M}$ and $T^\perp\mathcal{M}$ denote the Lie algebra of vector fields and set of all normal vector fields on $\mathcal{M}$, respectively. The operator of covariant differentiation with respect to the Levi-Civita connection in $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by ∇ and $\overline{\nabla}$, respectively. Let $\overline{R}$ and R be the curvature tensor of $\overline{\mathcal{M}}(f_1,f_2,f_3)$ and $\mathcal{M}$, respectively. The Gauss equation is given by [30]

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) \\ &\quad +g(h(X,Z),h(Y,W)) \end{aligned} \tag{2.5}$$

for all vector fields $X, Y, Z \in T\overline{\mathcal{M}}$.

The curvature tensor $\overline{R}^*$ of a generalized Sasakian space form with a semi-symmetric metric connection is given by

$$\begin{aligned}\overline{R}^*(X,Y)Z &= f_1\big[g(Y,Z)X - g(X,Z)Y\big] + f_2\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad +2g(X,\phi Y)\phi Z\big] + f_3\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &\quad -g(Y,Z)\eta(X)\xi\big] - \gamma(Y,Z)X + \gamma(X,Z)Y - g(Y,Z)FX \\ &\quad +g(X,Z)FY, \end{aligned} \tag{2.6}$$

where we have used equations (2.3), (2.4).

For any vector field $X \in T\mathcal{M}$, we put [30]

$$\phi X = PX + QX, \tag{2.7}$$

where PX and QX denote the tangential and normal components of ϕX, respectively. Then P is an endomorphism of $T\mathcal{M}$, and Q is the normal bundle valued 1-form on $T\mathcal{M}$.

In the same way, for any vector field $V \in T^\perp\mathcal{M}$, we put [30]

$$\phi V = BV + CV, \tag{2.8}$$

where BV and CV denote tangential and normal components of ϕV, respectively.

It is easy to see that F and B are skew-symmetric and they are related by

$$g(QX,V) = -g(X,BV) \tag{2.9}$$

for any vector fields $X \in T\mathcal{M}$ and $V \in T^\perp\mathcal{M}$.

The structural vector field ξ can be decomposed as

$$\xi = \xi_1 + \xi_2, \tag{2.10}$$

where ξ_1 and ξ_2 are the tangential and the normal components of ξ, respectively.

A submanifold $\mathcal{M}$ of an almost contact metric manifold $\overline{\mathcal{M}}$ is said to be *invariant* if $Q \equiv 0$, that is, $\phi X \in T\mathcal{M}$, and *anti-invariant* if $P \equiv 0$, that is, $\phi X \in T^\perp\mathcal{M}$, for any vector field $X \in T\mathcal{M}$.

There are some other important classes of submanifolds which are determined by the behavior of tangent bundle of the submanifold under the action of an almost contact metric structure ϕ of $\overline{\mathcal{M}}$:

(i) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called a *contact CR-submanifold* [29] of $\overline{\mathcal{M}}$ if there exists a differentiable distribution D on $\mathcal{M}$ whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant.

(ii) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called a *slant submanifold* [3] of $\overline{\mathcal{M}}$ if, the angle between ϕX and $T_x\mathcal{M}$ is constant for all $X \in T\mathcal{M} - \{\xi_x\}$ and $x \in \mathcal{M}$.

(iii) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *semi-slant submanifold* [2] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions D and D_θ such that D is invariant and D_θ is proper slant.

(iv) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *hemi-slant submanifold* (or *pseudo-slant*) [12] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions $D^{\perp}$ and D_θ such that $D^{\perp}$ is anti-invariant and D_θ is proper slant.

Bi-slant submanifolds were first defined by A. Cariazo et al. in [2] as a generalization of CR and semi-slant submanifolds. Such submanifolds generalize complex, totally real, slant and hemi-slant submanifolds as well. Here we define a bi-slant submanifold of an almost contact metric manifold as follows:

Definition 2.2. A submanifold $\mathcal{M}$ of an almost contact metric manifold $\overline{\mathcal{M}}$ is said to be a *bi-slant submanifold* if there exists a pair of orthogonal distributions D_{θ_1} and D_{θ_2} of $\mathcal{M}$ such that

(i) $T\mathcal{M} = D_{\theta_1} \oplus D_{\theta_2} \oplus \{\xi\}$;

(ii) $\phi D_{\theta_1} \perp D_{\theta_2}$ and $\phi D_{\theta_2} \perp D_{\theta_1}$;

(iii) Each distribution D_{θ_i} is slant with the slant angle θ_i for $i = 1, 2$.

A bi-slant submanifold of an almost contact metric manifold $\overline{\mathcal{M}}$ is called *proper* if the slant distributions D_{θ_1} and D_{θ_2} are of the slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Remark 2.2. If we assume

(i) $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then $\mathcal{M}$ is a *CR-submanifold*.

(ii) $\theta_1 = 0$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then $\mathcal{M}$ is a *semi-slant submanifold*.

(iii) $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then $\mathcal{M}$ is a *hemi-slant submanifold*.

Suppose that $\mathcal{M}$ is a bi-slant submanifold of dimension $n + 1 = 2n_1 + 2n_2 + 1$ in $\overline{\mathcal{M}}(f_1, f_2, f_3)$. Let us assume the orthonormal basis of $\mathcal{M}$ as follows [6, 26]:

$$E_1, E_2 = sec\theta_1 Pe_1, \ldots, E_{2n_1-1}, E_{2n_1} = sec\theta_1 Pe_{2n_1-1}, E_{2n_1+1}, E_{2n_1+2} =$$
$$sec\theta_2 Pe_{2n_1+1}, \ldots, E_{2n_1+2n_2-1}, E_{2n_1+2n_2} = sec\theta_2 Pe_{2n_1+2n_2-1}, E_{2n_1+2n_2+1} = \xi.$$

Also,

$$g^2(\phi E_{i+1}, E_i) = \begin{cases} cos^2\theta_1 & \text{for} \quad i = 1, \ldots, 2n_1 - 1 \\ cos^2\theta_2 & \text{for} \quad i = 2n_1 + 1, \ldots, 2n_1 + 2n_2 - 1 \end{cases} \tag{2.11}$$

Hence, we have

$$\sum_{i,j=1}^{n+1} g^2(\phi E_j, E_i) = 2\{n_1 cos^2\theta_1 + n_2 cos^2\theta_2\}.$$

3. Casorati Curvatures

In this section, we study the Casorati curvature of any submanifold $\mathcal{M}$ of dimension $(n+1)$ in a $(2m+1)-$dimensional generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ with SSMC. Consider a local orthonormal tangent frame $\{E_1, \ldots, E_{n+1}\}$ of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and a local orthonormal normal frame $\{E_{n+2}, \ldots, E_{2m+1}\}$ of the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$ in $\overline{\mathcal{M}}(f_1, f_2, f_3)$. At any $p \in \mathcal{M}$, the scalar curvature τ at that point is given by

$$\tau = \sum_{1 \leq i < j \leq n+1} R(E_i, E_j, E_j, E_i)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n+1)}.$$

The mean curvature vector denoted by $\mathcal{H}$ of $\mathcal{M}$ is given by

$$\mathcal{H} = \sum_{i=1}^{n+1} \frac{1}{n+1} \sigma(E_i, E_i).$$

Conveniently, let us put

$$h_{ij}^r = g(h(E_i, E_j), E_r)$$

for $i, j = \{1, \ldots, n+1\}$ and $r = \{n+2, \ldots, 2m+1\}$. Then the squared norm of mean curvature vector of $\mathcal{M}$ is defined as

$$||\mathcal{H}||^2 = \frac{1}{(n+1)^2} \sum_{r=n+2}^{2m+1} \left\{ \sum_{i=1}^{n+1} h_{ii}^r \right\}^2.$$

and the squared norm of second fundamental form h is denoted by

$$\mathcal{C} = \frac{1}{n+1} ||h||^2, \tag{3.1}$$

where

$$||h||^2 = \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n+1} \left(h_{ij}^r\right)^2.$$

It is known as the *Casorati curvature* $\mathcal{C}$ of $\mathcal{M}$.

If we suppose that $\mathcal{L}$ is an s-dimensional subspace of $T\mathcal{M}$, $s \geq 2$, and $\{E_1, \ldots, E_s\}$ is an orthonormal basis of $\mathcal{L}$, then the scalar curvature of the $s-$plane section $\mathcal{L}$ is given by

$$\tau(\mathcal{L}) = \sum_{1 \leq i < j \leq s} R(E_i, E_j, E_j, E_i)$$

and the Casorati curvature of the subspace $\mathcal{L}$ is as follows

$$\mathcal{C}(\mathcal{L}) = \frac{1}{s} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{s} \left(h_{ij}^r\right)^2.$$

The normalized δ-Casorati curvatures $\delta_c(n)$ and $\widehat{\delta}_c(n)$ are defined as

$$[\delta_c(n)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+2}{2(n+1)} inf\{\mathcal{C}(\mathcal{L}) | \mathcal{L} : \text{a hyperplane of } T_p\mathcal{M}\}$$

and

$$[\widehat{\delta}_c(n)]_p = 2\mathcal{C}_p - \frac{2n+1}{2(n+1)} sup\{\mathcal{C}(\mathcal{L}) | \mathcal{L} : \text{a hyperplane of } T_p\mathcal{M}\}.$$

Now we define the generalized normalized δ-Casorati curvatures $\delta_C(t; n-1)$ and $\widehat{\delta}_C(t; n-1)$ as follows:

1. For $0 < t < n^2 - n$

$$[\delta_C(t;n)]_x = t\mathcal{C}_p + b(t)\ inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xM\}$$

2. For $t > n^2 - n$

$$[\widehat{\delta}_C(t;n)]_x = t\mathcal{C}_p - b(t)\ sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xM\}$$

where

$$b(t) = \frac{1}{t(n+1)}(n)(n+1+t)((n+1)^2 - n - 1 - t), \quad t \neq n(n-1).$$

Throughout this paper, we use the above notations.

A point $p \in \mathcal{M}$ is said to be an *invariantly quasi-umbilical point* if there exist $2m - n$ orthogonal unit normal vector $\{E_{n+2}, \dots, E_{2m+1}\}$ such that the shape operator with respect to all directions E_r have an eigenvalue of multiplicity n and that for each E_r the distinguished eigendirection is the same. The submanifold $\mathcal{M}$ is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point.

4. T Oprea's Optimization Method

Here we construct some optimal inequalities consisting of the normalized scalar curvature and the normalized $\delta-$Casorati curvatures for bi-slant submanifolds $\mathcal{M}$ in a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ with SSMC.

The following lemmas play a key role in the proof of our theorem:

Lemma 4.1. *[25] Let $\vartheta = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \dots + x_n = k\}$ be a hyperplane of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ a quadratic form given by*

$$f(x_1, x_2, \dots, x_n) = \alpha \sum_{i=1}^{n-1} (x_i)^2 + \beta(x_n)^2 - 2 \sum_{1 \le i < j \le n} x_i x_j, \ \alpha > 0, \beta > 0.$$

Then, by the constrained extremum problem, f has a global solution as follows,

$$x_1 = x_2 = \dots = x_{n-1} = \frac{k}{\alpha + 1},$$
$$x_n = \frac{k}{\beta + 1} = \frac{k(n-1)}{(\alpha + 1)\beta} = (\alpha - n + 2)\frac{k}{\alpha + 1},$$

provided that

$$\beta = \frac{n-1}{\alpha - n + 2}.$$

Lemma 4.2. *[20] Let N be a Riemannian submanifold of Riemannian manifold (M, G), where g is the metric induced on N by G and $f : N \to \mathbb{R}$ be a differentiable function. If $x_0 \in N$ is the solution of the constrained extremum problem* $\min_{x \in N} f(x)$*, then*

(i) $(gradf)(x_0) \in T_{x_0}^{\perp} N$;
(ii) the bilinear form $A : T_{x_0}N \times T_{x_0}N \to \mathbb{R}$;
$A(X, Y) = Hess_f(X, Y) + G(h(X, Y), (gradf)(x_0))$

is positive semidefinite, where h is the second fundamental form of N in M.

Now we prove our main result:

Theorem 4.1. *Let $\mathcal{M}$ be an $(n+1)-$dimensional submanifold in a $(2m+1)-$dimensional generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ with SSMC. Then the generalized normalized δ-Casorati curvatures $\delta_c(t;n)$ and $\widehat{\delta}_c(t;n)$ satisfy*

(i)

$$\rho \leq \delta_c(t;n) + f_1 - \frac{2f_3}{n+1}||\xi_1||^2 + \frac{6f_2}{n(n+1)}(||P||^2) - \frac{2}{n}trace(\gamma) \tag{4.1}$$

for any $t \in \mathbb{R}$ with $0 < t < n(n-1)$, and

(ii)

$$\rho \leq \widehat{\delta}_c(t;n) + f_1 - \frac{2f_3}{n+1}||\xi_1||^2 + \frac{6f_2}{n(n+1)}(||P||^2) - \frac{2}{n}trace(\gamma) \tag{4.2}$$

for any $t \in \mathbb{R}$ with $t > n(n-1)$,

respectively. Moreover, the equality case holds in (i) and (ii) if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold.

Proof. Let $\{E_1, \ldots, E_{n+1}\}$ and $\{E_{n+2}, \ldots, E_{2m+1}\}$ be the orthonormal basis of $T\mathcal{M}$ and and $T^{\perp}\mathcal{M}$, respectively, at any point $p \in M$. Putting $X = W = E_i$, $Y = Z = E_j$ into (2.6) and considering $i \neq j$, then we have

$$\begin{aligned}\sum_{i,j=1}^{n+1} R(E_i, E_j, E_j, E_i) &= \sum_{i,j=1}^{n+1} \Big\{ f_1\{g(E_j, E_j)g(E_i, E_i) - g(E_i, E_j)g(E_j, E_i)\} \\ &+ f_2\{g(E_i, \phi E_j)g(\phi E_j, E_i) - g(\phi E_i, E_i)g(E_j, \phi E_j) \\ &+ 2g(E_i, \phi E_j)g(E_i, \phi E_j)\} + f_3\{\eta(E_i)\eta(E_j)g(E_i, E_i) \\ &- \eta(E_j)\eta(E_j)g(E_i, E_i) + \eta(E_i)\eta(E_j)g(E_i, E_j) \\ &- \eta(E_i)\eta(E_i)g(E_j, E_j)\} \Big\} - \gamma(E_j, E_j)g(E_i, E_i) \\ &+ \gamma(E_i, E_j)g(E_j, E_i) - g(E_j, E_j)g(FE_i, E_i) + g(E_i, E_j)g(FE_j, E_i).\end{aligned}$$

From this and together with Gauss equation, we have

$$\begin{aligned}2\tau(p) &= n(n+1)f_1 + 6f_2(||P||^2) - 2nf_3||\xi_1||^2 \\ &+ (n+1)^2||\mathcal{H}||^2 - (n+1)\mathcal{C} - 2(n+1)trace(\gamma),\end{aligned} \tag{4.3}$$

where we have used (3.1).

We define now the following function, denoted by Λ, which is a quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned}\Lambda &= t\mathcal{C} + b(t)\mathcal{C}(\mathcal{L}) - 2\tau(p) + n(n+1)f_1 \\ &+ 6f_2(||P||^2) - 2nf_3||\xi_1||^2 - 2(n+1)trace(\gamma),\end{aligned} \tag{4.4}$$

where $\mathcal{L}$ is a hyperplane of $T_p\mathcal{M}$. We can assume without loss of generality that $\mathcal{L}$ is spanned by $\{E_1, \ldots, E_n\}$. Then we have

$$\begin{aligned}\Lambda &= \frac{n+1+t}{n+1}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n+1}(h_{ij}^r)^2 + \frac{b(t)}{n}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - 2\tau(p) + n(n+1)f_1 \\ &+ 6f_2(||P||^2) - 2nf_3||\xi_1||^2 - 2(n+1)trace(\gamma).\end{aligned} \tag{4.5}$$

From (4.3) and (4.5), we obtain

$$\begin{aligned}\Lambda &= \frac{n+1+t}{n+1}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n+1}(h_{ij}^r)^2 + \frac{b(t)}{n}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ &- \sum_{r=n+2}^{2m+1}\Big(\sum_{i,j=1}^{n+1}h_{ij}^r\Big)^2.\end{aligned}$$

Now we can easily derive that

$$\begin{aligned}\Lambda &= \sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}\left[a(h_{ii}^r)^2+\frac{2(n+1+t)}{n+1}(h_{in+1}^r)^2\right]\\ &\quad+\sum_{r=n+2}^{2m+1}\Big[2(a+1)\sum_{i<j=1}^{n}(h_{ij}^r)^2-2\sum_{i<j=1}^{n+1}h_{ii}^rh_{jj}^r\\ &\quad+\frac{t}{n+1}(h_{n+1n+1}^r)^2\Big]\\ &\geq \sum_{i=1}^{n}a(h_{ii}^r)^2-2\sum_{1\leq i\neq j\leq n+1}h_{ii}^rh_{jj}^r+\frac{t}{n+1}(h_{n+1n+1}^r)^2, \qquad (4.6)\end{aligned}$$

where

$$a=\left(\frac{t}{n+1}+\frac{b(t)}{n}\right).$$

For $r=n+2,\dots,2m+1$, let us take the quadratic form $\phi_r:\mathbb{R}^{n+1}\to\mathbb{R}$, defined by

$$\begin{aligned}\phi_r(h_{11}^r,\dots,h_{n+1n+1}^r) &= \sum_{i=1}^{n}\left[a(h_{ii}^r)^2\right]-2\sum_{i<j=1}^{n+1}h_{ii}^rh_{jj}^r\\ &\quad+\frac{t}{n+1}(h_{n+1n+1}^r)^2 \qquad (4.7)\end{aligned}$$

and the constrained extremum problem $\min\phi_r$ subject to the component of trace $\mathcal{H}$,

$$\theta: h_{11}^r+\cdots+h_{n+1n+1}^r=k^r,$$

where k^r is a real constant.

Comparing (4.7) with the quadratic function in Lemma 4.1, we find that

$$\alpha=a=\left(\frac{t}{n+1}+\frac{b(t)}{n}\right),\ \beta=\frac{t}{n+1}.$$

Therefore, we can find the critical point $(h_{11}^r,\dots,h_{n+1n+1}^r)$:

$$\begin{aligned}h_{11}^r=h_{22}^r=\cdots=h_{nn}^r&=\frac{k^r}{a+1},\\ h_{n+1n+1}^r&=\frac{(n+1)k^r}{n+1+t}. \qquad (4.8)\end{aligned}$$

Now here we use Lemma 4.2 and for this, we fix an arbitrary point $x_0\in\theta$. The bilinear form

$$A: T_{x_0}\theta\times T_{x_0}\theta\to\mathbb{R}$$

is defined by

$$A(X,Y)=Hess_{\phi_r}(X,Y)+\left\langle\hbar(X,Y),(grad\phi_r)(x_0)\right\rangle,$$

where $\hbar$ is the second fundamental form of θ in $\mathbb{R}^{n+1}$ and $\langle,\rangle$ is the standard inner product on $\mathbb{R}^{n+1}$. So, we have the following:

$$\begin{aligned}A(X,X) &= (X_1,\dots,X_n,X_{n+1})\begin{pmatrix}2a&-2&-2&\dots&-2&-2\\-2&2a&-2&\dots&-2&-2\\-2&-2&2a&\dots&-2&-2\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\-2&-2&-2&\dots&2a&-2\\-2&-2&-2&\dots&-2&\frac{2t}{n+1}\end{pmatrix}\begin{pmatrix}X_1\\X_2\\ \vdots\\ \vdots\\X_n\\X_{n+1}\end{pmatrix}\\ &\geq 0.\end{aligned}$$

Thus, the point $(h^r_{11},\ldots,h^r_{n+1n+1})$ (see (4.8)) is a global minimum point. From relation (4.6) and (4.8), we get $\Lambda \geq 0$ and hence we have

$$\begin{aligned}2\tau(p) &\leq tC + b(t)C(\mathcal{L}) + n(n+1)f_1\\ &\quad +6f_2(||P||^2) - 2nf_3||\xi_1||^2 - 2(n+1)trace(\gamma),\end{aligned}$$

whereby, we obtain

$$\begin{aligned}\rho &\leq \frac{t}{n(n+1)}\mathcal{C} + \frac{b(t)}{n(n+1)}\mathcal{C}(\mathcal{L}) + f_1 - \frac{2f_3}{n+1}||\xi_1||^2\\ &\quad +\frac{6f_2}{n(n+1)}(||P||^2) - \frac{2}{n}trace(\gamma).\end{aligned}$$

By the definition of $\delta_C(n)$, we can obtain our desired inequality (4.1). Moreover, the equality sign holds if and only if

$$h^r_{ij} = 0,\ \forall\, i,j \in \{1,\ldots,n+1\},\ i \neq j,\ r \in \{n+2,\ldots,2m+1\} \tag{4.9}$$

and

$$h_{n+1n+1} = 2h^r_{11} = \cdots = 2h^r_{nn},\ \forall\, r \in \{n+2,\ldots,2m+1\}. \tag{4.10}$$

From (4.9) and (4.10), we conclude that the equality sign holds in the inequality (4.1) if and only if the submanifold $\mathcal{M}$ is invariantly quasi-umbilical submanifold.

In the same manner, we can establish the inequality (4.2) as a second part of the theorem. □

Following is an immediate consequence of Theorem 4.1:

Corollary 4.1. *Let M be an $(n+1)-$dimensional submanifold $\mathcal{M}$ in a $(2m+1)-$dimensional generalized Sasakian space form $\overline{\mathcal{M}}(f_1,f_2,f_3)$ with SSMC. Then the normalized δ-Casorati curvature $\delta_c(n)$ and $\widehat{\delta}_c(n)$ satisfy*

(i)

$$\rho \leq \delta_c(n) + f_1 + \frac{6f_2}{n(n+1)}(||P||^2) - \frac{2f_3}{n+1}||\xi_1||^2 - \frac{2}{n}trace(\gamma) \tag{4.11}$$

and

(ii)

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{6f_2}{n(n+1)}(||P||^2) - \frac{2f_3}{n+1}||\xi_1||^2 - \frac{2}{n}trace(\gamma), \tag{4.12}$$

respectively. Moreover, the equality sign holds in (i) and (ii) if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold.

5. Applications of Theorem 4.1

In this section, we see the developed optimal Casorati inequalities for bi-slant, hemi-slant, semi-slant, slant, CR, anti-invariant and invariant submanifolds in the same ambient space, that is, a generalized Sasakian space form with SSMC. We assume that dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. This is as follows:

Submanifolds	Two Optimal Casorati Inequalities
bi-slant	(i) $\rho \leq \delta_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}(n_1cos^2\theta_1 + n_2cos^2\theta_2) - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}(n_1cos^2\theta_1 + n_2cos^2\theta_2) - \frac{2}{n}trace(\gamma)$
hemi-slant	(i) $\rho \leq \delta_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}(n_1cos^2\theta_1) - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}(n_1cos^2\theta_1) - \frac{2}{n}trace(\gamma)$
semi-slant	(i) $\rho \leq \delta_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}(n_1 + n_2cos^2\theta_2) - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}(n_1 + n_2cos^2\theta_2) - \frac{2}{n}trace(\gamma)$
slant	(i) $\rho \leq \delta_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}cos^2\theta - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 + \frac{6f_2}{n(n+1)}cos^2\theta - \frac{2}{n}trace(\gamma)$
CR	(i) $\rho \leq \delta_c(t;n) + f_1 + \frac{6f_2}{n(n+1)}n_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 + \frac{6f_2}{n(n+1)}n_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 - \frac{2}{n}trace(\gamma)$
anti-invariant	(i) $\rho \leq \delta_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 - \frac{2}{n}trace(\gamma)$
invariant	(i) $\rho \leq \delta_c(t;n) + f_1 + \frac{6f_2}{n(n+1)} - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 - \frac{2}{n}trace(\gamma)$ (ii) $\rho \leq \widehat{\delta}_c(t;n) + f_1 + \frac{6f_2}{n(n+1)} - \frac{2f_3}{n+1}\|\|\xi_1\|\|^2 - \frac{2}{n}trace(\gamma)$

Remark 5.1. Theorem 4.1 shows that the normalized scalar curvature for any submanifold of dimension $(n+1)$ in a generalized Sasakian space form with SSMC is bounded above by the generalized normalized Casorati curvatures $\delta_c(t;n)$ and $\widehat{\delta}_c(t;n)$.

Remark 5.2. With similar proof of Theorem 4.1, we can show that the normalized scalar curvature is bounded above by the generalized normalized Casorati curvatures $\delta_c(t;n)$ and $\widehat{\delta}_c(t;n)$ when ambient space form is, respectively,

(i) Sasakian space form with SSMC
(ii) Kenmotsu space form with SSMC
(iii) cosymplectic space form with SSMC

Remark 5.3. The present paper can be considered as the next version of [22].

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