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A Natural Morphing Between Two Epicycloids

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ABSTRACT

Let A be a point which moves with constant angular speed on the circle with centre O and radius 1, and B a point wich moves with an angular speed which is q times ($q > 1$) that of A on the circle with centre O and radius r ($r > 1$). We study the envelope of the straight lines AB. Both limit cases $r = 1$ and $r = +\infty$ (with q constant) are epicycloids; the relation between the shapes of these epicycloids does not depend on q.

Keywords: *envelope; straight line; angular speed; epicycloid.*

AMS Subject Classification (2010): *Primary: 53A04 ; Secondary: 58K05.*

1. Introduction.

An epicycloid is the trajectory of a point on a circle which rolls without slipping around another circle; the ratio of the radii of these circles determines the shape of the epicycloid. Examples of epicycloids are the cardioid and the nephroid. My attention was drawn to epicycloids by a recent article [4] where it is shown by direct calculations how the cardioid can be obtained as the envelope of the family of chords joining a point to its 'double' on a circle. Now, this result is not new: it is exercise 5.7(2) in [1, p.185], where a generalization is hinted at. In fact, the general case is already in Gomes Teixeira's three volume work [2, volume II, §561 p.166]. It can be worded as follows.

On the circle α with centre $O := (0, 0)$ and radius 1, let A and B be two points which, starting from $(1, 0)$, move with constant speed. If the angular velocity of B is q times ($q > 1$) that of A, the chord AB generates a family of straight lines whose envelope is an epicycloid (see Figure 1 for the case $q = 4$).

Here we investigate what occurs if B moves on the circle β with centre O and radius r ($r > 1$) starting from $(r, 0)$, its angular velocity being always q times that of A. We calculate explicitly the parametric representation of the curve which is the envelope of the straight lines AB; we find the critical points of this curve and we study its position relatively to α. The curve is not an epicycloid, but, as r tends to 1, we evidently get the epicycloid from Gomes Teixeira's cited result; moreover, as r tends to $+\infty$, we also get an epicycloid. We show that the shapes of these epicycloids are related in a way which does not depend on q.

2. Envelope and curve.

We may suppose that A and B start at $t = 0$ from the points $(1, 0)$ and $(r, 0)$ respectively; we can then choose the following parametrizations:

$$A(t) = (\cos t, \sin t), \quad B(t) = (r \cos qt, r \sin qt).$$

The straight line through A and B is given by the equation

$$\frac{x - \cos t}{r \cos qt - \cos t} = \frac{y - \sin t}{r \sin qt - \sin t}.$$

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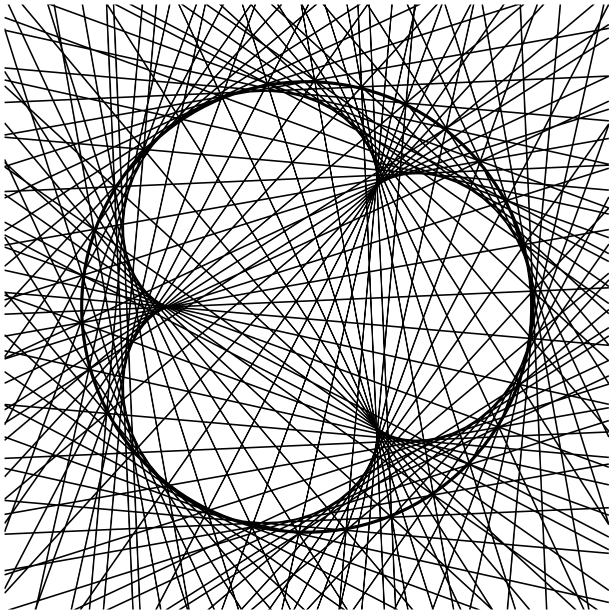

Figure 1. The circle α and some of the chords AB for $q = 4$.

Hence the set of these lines is the zero set of the function

$$\begin{aligned} F(t,x,y) &:= (x-\cos t)(r\sin qt - \sin t) - (y - \sin t)(r\cos qt - \cos t) \\ &= x(r\sin qt - \sin t) - y(r\cos qt - \cos t) - r\cos t \sin qt + r\sin t\cos qt \\ &= x(r\sin qt - \sin t) - y(r\cos qt - \cos t) - r\sin(q-1)t. \end{aligned}$$

We deduce:

$$\frac{\partial F}{\partial t}(t,x,y) = x(rq\cos qt - \cos t) + y(rq\sin qt - \sin t) - r(q-1)\cos(q-1)t.$$

The envelope of the straight lines AB is the set of points (x, y) such that there exists t with $F(t, x, y) = 0$ and $(\partial F/\partial t)(t, x, y) = 0$ [1, Definition 5.3 p.102]. So we must solve the system

$$\begin{cases} x(r\sin qt - \sin t) - y(r\cos qt - \cos t) = r\sin(q-1)t \\ x(rq\cos qt - \cos t) + y(rq\sin qt - \sin t) = r(q-1)\cos(q-1)t. \end{cases}$$

This is a linear system in the unknown x and y of the form $ax - by = u$ and $cx + dy = v$, whose solution is

$$x = \frac{bv + du}{ad + bc}, \quad y = \frac{av - cu}{ad + bc}$$

if its determinant $ad + bc$ is non zero. So we first calculate this determinant:

$$\begin{aligned} ad + bc &= (r\sin qt - \sin t)(rq\sin qt - \sin t) + (r\cos qt - \cos t)(rq\cos qt - \cos t) \\ &= r^2 q\sin^2 qt - r\sin qt\sin t - rq\sin qt\sin t + \sin^2 t \\ &\qquad + r^2 q\cos^2 qt - r\cos qt\cos t - rq\cos qt\cos t + \cos^2 t \\ &= r^2 q + 1 - r(q+1)(\sin qt\sin t + \cos qt\cos t) \\ &= r^2 q + 1 - r(q+1)\cos(q-1)t. \end{aligned}$$

Suppose $ad + bc = 0$; this would mean

$$\cos(q-1)t = \frac{r^2 q + 1}{r(q+1)}. \tag{2.1}$$

We will prove that the right-hand of (2.1) is greater than 1, which implies that (2.1) has no solution and $ad + bc \neq 0$. Let, for $r > 0$, $f(r) := (r^2 q + 1)/(r(q + 1))$. We have $f'(r) = (r^2 q - 1)/(r^2(q + 1))$. Then $f'(r) = 0$

implies $r^2q - 1 = 0$, that is, $r = 1/\sqrt{q} < 1$. Since $\lim_{r\to+\infty} f'(r) = q/(q+1) > 0$, f is increasing on $]1/\sqrt{q}; +\infty[$; in particular, for all $r > 1$, $f(r) > f(1) = 1$, as wanted. Next we calculate the numerator of x:

$$bv + du = (r\cos qt - \cos t)r(q-1)\cos(q-1)t + (rq\sin qt - \sin t)r\sin(q-1)t.$$

We concentrate on the second term of the right hand:

$$\begin{aligned}(rq\sin qt - \sin t)r\sin(q-1)t &= (rq\sin qt - \sin t)r(\cos t\sin qt - \sin t\cos qt)\\ &= r^2q\sin^2 qt\cos t - r^2q\sin qt\sin t\cos qt\\ &\quad - r\sin t\cos t\sin qt + r^2\sin^2 t\cos qt\\ &= r^2q(1-\cos^2 qt)\cos t - r^2q\sin qt\sin t\cos qt\\ &\quad - r\sin t\cos t\sin qt + r(1-\cos^2 t)\cos qt\\ &= r^2q\cos t - r^2q\cos^2 qt\cos t - r^2q\sin qt\sin t\cos t\\ &\quad + r\cos qt - r\cos^2 t\cos qt - r\sin t\cos t\sin qt\\ &= r^2q\cos t - r^2q\cos qt(\cos qt\cos t + \sin qt\sin t)\\ &\quad + r\cos qt - r\cos t(\cos t\cos qt + \sin t\sin qt)\\ &= r^2q\cos t + r\cos qt - r(rq\cos qt + \cos t)\cos(q-1)t.\end{aligned}$$

Therefore

$$\begin{aligned}bv + du &= (r\cos qt - \cos t)r(q-1)\cos(q-1)t\\ &\quad - r(rq\cos qt + \cos t)\cos(q-1)t + r^2q\cos t + r\cos qt\\ &= r^2(q-1)\cos qt\cos(q-1)t - r(q-1)\cos t\cos(q-1)t\\ &\quad - r^2q\cos qt\cos(q-1)t - r\cos t\cos(q-1)t + r^2q\cos t + r\cos qt\\ &= r^2q\cos t - r^2\cos qt\cos(q-1)t + r\cos qt - rq\cos t\cos(q-1)t\\ &= rq\cos t[r - \cos(q-1)t] + r\cos qt[1 - r\cos(q-1)t].\end{aligned}$$

The numerator of y can be calculated along the same lines and we find:

$$av - cu = rq\sin t[r - \cos(q-1)t] + r\sin qt[1 - r\cos(q-1)t].$$

We conclude that the envelope of the straight lines AB is the curve γ_r given by

$$\gamma_r(t) = \begin{pmatrix} \dfrac{rq\cos t[r-\cos(q-1)t] + r\cos qt[1 - r\cos(q-1)t]}{r^2q + 1 - r(q+1)\cos(q-1)t} \\ \\ \dfrac{rq\sin t[r-\cos(q-1)t] + r\sin qt[1 - r\cos(q-1)t]}{r^2q + 1 - r(q+1)\cos(q-1)t} \end{pmatrix} \tag{2.2}$$

for all $t \in \mathbb{R}$.

3. Critical points.

The critical points of γ_r are those points where the derivative of γ_r is zero: $\dot\gamma_r(t) = 0$. Now, if the expression we have found for γ_r is not simple, the one for $\dot\gamma_r$ will certainly be complicated, and to solve $\dot\gamma_r(t) = 0$ from it not less.

However, it is possible to find the critical points of γ_r without messy calculations if we allow ourselves the use of some old-style reasoning. Suppose γ_r has a critical point at t_0. Let $\Delta t > 0$ be so small that we may consider the arcs on the circles α and β between $t_- := t_0 - \Delta t$ and $t_+ := t_0 + \Delta t$ as straight segments. We write $A_- := A(t_-)$, $A_0 := A(t_0)$, $A_+ := A(t_+)$ and similarly for B_-, B_0 , B_+, so that A_0 (respectively B_0) is the middlepoint of the segment A_-A_+ (resp. B_-B_+). Let I be the intersection of the straight lines A_-B_- and A_0B_0, and J the intersection of A_0B_0 and A_+B_+. We have essentially two possibilities: see Figure 2. We may consider I and J as points on the curve γ_r. The equality $\dot\gamma_r(t_0) = 0$ means that $\gamma_r(t)$ does not vary for t near

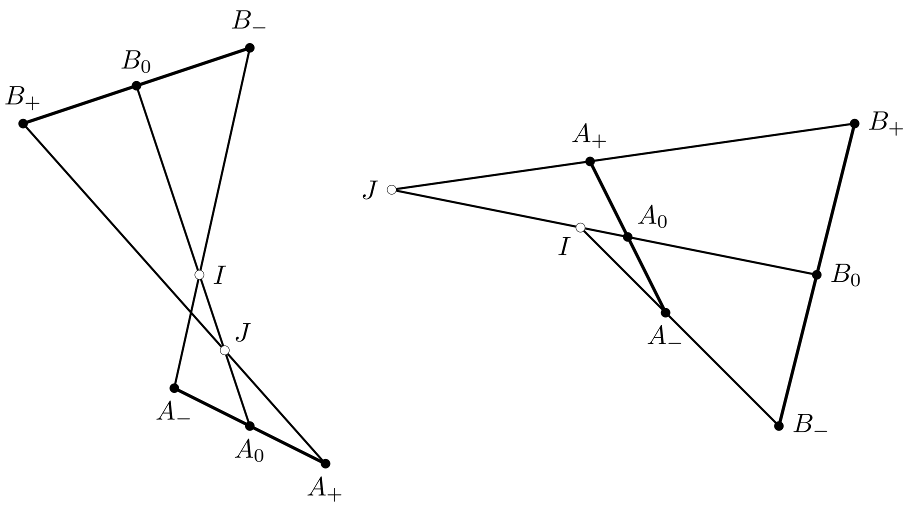

Figure 2. Position of I and J with respect to A_-A_+ and B_-B_+.

t_0; hence $I = J$, which is only possible if the segments A_-A_+ and B_-B_+ are parallel. Now, these segments are on the concentric circles α and β respectively: the tangents at A_0 to α and at B_0 to β can only be parallel when A_0 and B_0 are aligned with the centre O. Going back to the parametrizations of A and B, this gives two possibilities:

(a) $\cos qt_0 = -\cos t_0$ and $\sin qt_0 = -\sin t_0$;

(b) $\cos qt_0 = \cos t_0$ and $\sin qt_0 = \sin t_0$.

The case (a) occurs when $qt_0 = t_0 + (2l+1)\pi$ for some $l \in \mathbb{Z}$, that is,

$$t_0 = \frac{(2l+1)\pi}{q-1}.$$

Then $\cos(q-1)t_0 = \cos(2l+1)\pi = -1$ and

$$x = \frac{rq\cos t_0[r+1] - r\cos t_0[1+r]}{r^2q+1+r(q+1)} = \frac{r(r+1)(q-1)\cos t_0}{(rq+1)(r+1)} = \frac{r(q-1)\cos t_0}{rq+1}.$$

Similarly,

$$y = \frac{r(q-1)\sin t_0}{rq+1}.$$

Hence all these critical points are on the circle with centre O and radius

$$R_a := \frac{r(q-1)}{rq+1}. \tag{3.1}$$

The case (b) occurs when $qt_0 = t_0 + 2l\pi$ for some $l \in \mathbb{Z}$, that is,

$$t_0 = \frac{2l\pi}{q-1}.$$

Then $\cos(q-1)t_0 = \cos 2l\pi = 1$ and

$$x = \frac{rq\cos t_0[r-1] + r\cos t_0[1-r]}{r^2q+1-r(q+1)} = \frac{r(r-1)(q-1)\cos t_0}{(rq-1)(r-1)} = \frac{r(q-1)\cos t_0}{rq-1}.$$

Similarly,

$$y = \frac{r(q-1)\sin t_0}{rq-1}.$$

Hence all these critical points are on the circle with centre O and radius

$$R_b := \frac{r(q-1)}{rq-1}. \tag{3.2}$$

Moreover, $R_a < R_b < 1$.

4. Periodicity.

Suppose q is a rational number: $q = m/n$ with $m, n \in \mathbb{Z}$, $m > n \geq 1$ and $\gcd(m, n) = 1$. We show that γ_r is in this case periodic and therefore its image is a closed curve.

To find the period of γ_r, we must find when the points A and B have regained their initial position at $(1, 0)$ and $(r, 0)$ respectively, that is, when both points have toured a whole number of times around O. As A tours n times, B tours $q \cdot n = \frac{m}{n} \cdot n = m$ times: they are at their initial position. Moreover, since $\gcd(m, n) = 1$, this cannot occur before. Hence γ_r is periodic of period $2n\pi$. (This does not preclude the curve intersecting itself for t strictly between 0 and $2n\pi$; in fact, when $n > 1$ some selfintersection is necessary.) The critical points of type (a) (see section 3) occur at

$$t = \frac{(2l+1)\pi}{m/n-1} = \frac{(2l+1)n\pi}{m-n}.$$

It follows that there are $m - n$ distinct critical points of type (a), corresponding to $l = 0, \ldots, m - n - 1$. The critical points of type (b) (see section 3) occur at

$$t = \frac{2l\pi}{m/n-1} = \frac{2ln\pi}{m-n}.$$

It follows that there are $m - n$ distinct critical points of type (b), corresponding to $l = 0, \ldots, m - n - 1$. In total, γ_r has $2(m - n)$ critical points.

When q is not rational, γ_r is not periodic, so its image is not a closed curve, and it has infinitely many critical points of both types.

5. Within the unit circle.

First, we prove that γ_r cannot go outside the circle β. Let t_0 be such that $\gamma_r(t_0)$ is one of the points on the curve which are farthest from O, and suppose $\gamma_r(t_0)$ is outside β. Then t_0 cannot be a critical point, since all critical points are inside α (see section 3). Hence it is regular and $\dot{\gamma}_r(t_0) \neq 0$, so the tangent to the curve γ_r at $\gamma_r(t_0)$ is well defined. Moreover, $\dot{\gamma}_r(t_0)$ is orthogonal to $\gamma_r(t_0)$: since $\|\gamma_r(t)\|^2$ is maximal at t_0, its derivative at t_0 is zero, i.e. $2\gamma_r(t_0)\dot{\gamma}_r(t_0) = 0$. But the curve γ_r is tangent at $\gamma_r(t_0)$ to the line given by $F(t_0, x, y) = 0$ [1, proposition 5.25 p.114]. Now it follows from the first part of our discussion that the tangent to the curve at $\gamma_r(t_0)$ does not touch the circle β, and hence cannot be a straight line through A and B: contradiction.

The same argument shows that γ_r cannot go outside the circle α: see Figure 3.

Next we show that there are points of γ_r which lie on α. We already know that, if such a point $\gamma_r(t)$ exists, $\dot{\gamma}_r(t) \neq 0$ and $\gamma_r(t)$ it is farthest from the centre O and hence $\dot{\gamma}_r(t)$ is orthogonal to $\gamma_r(t)$. Moreover the tangent to γ_r at $\gamma_r(t)$ is the straight line given by $F(t, x, y) = 0$, in other words: it goes through $A(t)$ and $B(t)$. This implies $\gamma_r(t) = A(t)$, i.e.

$$\begin{cases} \dfrac{rq\cos t[r-\cos(q-1)t] + r\cos qt[1-r\cos(q-1)t]}{r^2q+1-r(q+1)\cos(q-1)t} = \cos t \\[2ex] \dfrac{rq\sin t[r-\cos(q-1)t] + r\sin qt[1-r\cos(q-1)t]}{r^2q+1-r(q+1)\cos(q-1)t} = \sin t. \end{cases} \tag{5.1}$$

The first equation can be written:

$$r^2q\cos t - rq\cos t\cos(q-1)t + r\cos qt[1-r\cos(q-1)t] = r^2q\cos t + \cos t - r(q+1)\cos t\cos(q-1)t$$

that is,

$$r\cos qt[1-r\cos(q-1)t] - \cos t + r\cos t\cos(q-1)t = 0$$

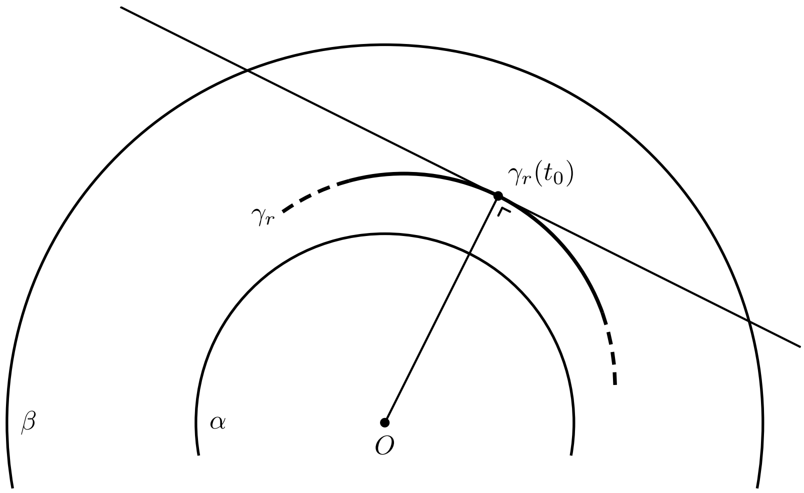

Figure 3. The curve γ_r cannot go outside α.

or

$$[r\cos qt - \cos t]\cdot[1 - r\cos(q-1)t] = 0.$$

The calculations with the second equation of (5.1) are analogue and we get the system

$$\begin{cases}[r\cos qt - \cos t]\cdot[1 - r\cos(q-1)t] = 0\\ [r\sin qt - \sin t]\cdot[1 - r\cos(q-1)t] = 0.\end{cases}$$

If $1 - r\cos(q-1)t \neq 0$, we have

$$\cos qt = \frac{1}{r}\cos t, \quad \sin qt = \frac{1}{r}\sin t$$

and

$$\cos(q-1)t = \cos qt\cos t + \sin qt \sin t = \frac{1}{r}\cos^2 t + \frac{1}{r}\sin^2 t = \frac{1}{r}.$$

We conclude that the curve γ_r touches (and is tangent to) the circle α exactly for every t with $\cos(q-1)t = 1/r$.

(When γ_r is periodic, the existence of points on the curve for which $\|\gamma_r(t)\|^2$ is maximal is clear by compactness. In the aperiodic case, one must reason on a piece of the curve joining two critical points.)

6. Away from the origin.

We prove here that there is no point on the curve whose distance from O is less than R_a. Suppose such a point exists: we may even choose t such that $\|\gamma_r(t)\|^2$ be minimal. Then t cannot be a critical point (see section 3). Hence $\dot\gamma_r(t) \neq 0$, so the tangent to the curve γ_r at $\gamma_r(t)$ is well defined. Moreover, $\dot\gamma_r(t)$ is orthogonal to $\gamma_r(t)$ and the tangent to the curve at $\gamma_r(t)$ is the straight line through $A(t)$ and $B(t)$, by a reasoning already used in section 5. This implies that the scalar product of $\gamma_r(t)$ with the vector joining $A(t)$ to $B(t)$ is zero:

$$0 = \begin{pmatrix}\dfrac{rq\cos t[r-\cos(q-1)t] + r\cos qt[1 - r\cos(q-1)t]}{r^2q + 1 - r(q+1)\cos(q-1)t}\\ \\ \dfrac{rq\sin t[r-\cos(q-1)t] + r\sin qt[1 - r\cos(q-1)t]}{r^2q + 1 - r(q+1)\cos(q-1)t}\end{pmatrix}\cdot\begin{pmatrix} r\cos qt - \cos t\\ r\sin qt - \sin t\end{pmatrix}.$$

Multiplying by the common denominator (which is never zero), we have

$$\begin{aligned}0 = {} & r^2q\cos t\cos qt[r - \cos(q-1)t] - rq\cos^2 t[r-\cos(q-1)t]\\ & + r^2\cos^2 qt[1 - r\cos(q-1)t] - r\cos qt\cos t[1 - r\cos(q-1)t]\\ & + r^2 q\sin t\sin qt[r - \cos(q-1)t] - rq\sin^2 t[r - \cos(q-1)t]\\ & + r^2\sin^2 qt[1 - r\cos(q-1)t] - r\sin qt\sin t[1 - r\cos(q-1)t],\end{aligned}$$

or

$$0 = \{r^2 q \cos t \cos qt + r^2 q \sin t \sin qt - rq\cos^2 t - rq\sin^2 t\}[r - \cos(q-1)t] \\ + \{r^2\cos^2 qt + r^2\sin^2 qt - r\cos qt\cos t - r\sin qt \sin t\}[1 - r\cos(q-1)t].$$

We simplify it to

$$0 = \{r^2 q\cos(q-1)t - rq\}[r - \cos(q-1)t] + \{r^2 - r\cos(q-1)t\}[1 - r\cos(q-1)t],$$

that is,

$$0 = rq\{r\cos(q-1)t - 1\}[r - \cos(q-1)t] + r\{r - \cos(q-1)t\}[1 - r\cos(q-1)t],$$

or

$$0 = (rq - r)[r - \cos(q-1)t]\cdot[r\cos(q-1)t - 1],$$

But $r - \cos(q-1)t = 0$ has no solution since $r > 1$; and the solutions to $r\cos(q-1)t - 1 = 0$ give the points on α already found. The conclusion follows.

7. The limit case $r = 1$.

When r tends to 1, the expression (2.2) for γ_r gives

$$\gamma_1(t) = \begin{pmatrix} \dfrac{q\cos t[1-\cos(q-1)t] + \cos qt[1-\cos(q-1)t]}{q+1-(q+1)\cos(q-1)t} \\ \\ \dfrac{q\sin t[1-\cos(q-1)t] + \sin qt[1-\cos(q-1)t]}{q+1-(q+1)\cos(q-1)t} \end{pmatrix}.$$

The terms $1 - \cos(q-1)t$ (which give a possible zero of the denominator) cancel out to

$$\gamma_1(t) = \begin{pmatrix} \dfrac{q\cos t + \cos qt}{q+1} \\ \\ \dfrac{q\sin t + \sin qt}{q+1} \end{pmatrix}.$$

Comparing with the representation of an epicycloid in [2, volume II, formula (8) p.162], we see that γ_1 is the epicycloid obtained as the trajectory of a point on a circle of radius $\rho := 1/(q+1)$ rolling around a circle of radius $R := (q-1)/(q+1)$. Note that $R + 2\rho = 1$ and that the ratio R/ρ equals $q-1$. We observe further (see (3.1) and (3.2)) that

$$\lim_{r\to 1_+} R_a = R, \quad \lim_{r\to 1_+} R_b = 1.$$

This means that the critical points of type (a) remain but that those of type (b) disappear: they merge with the points of contact of γ_r with α in those particular points where $\cos t_0 = \cos qt_0$ and $\sin t_0 = \sin qt_0$, i.e. $A(t_0) = B(t_0)$ where there is no chord possible! In fact, the envelope is not defined at these points: we closed the gap by continuity (these points correspond to the zeros of the term $1 - \cos(q-1)t$ in our first expression for γ_1).

8. The limit case $r = +\infty$.

When r tends to $+\infty$, the expression (2.2) for γ_r gives

$$\gamma_\infty(t) = \begin{pmatrix} \dfrac{q\cos t - \cos qt\cos(q-1)t}{q} \\ \\ \dfrac{q\sin t - \sin qt\cos(q-1)t}{q} \end{pmatrix}.$$

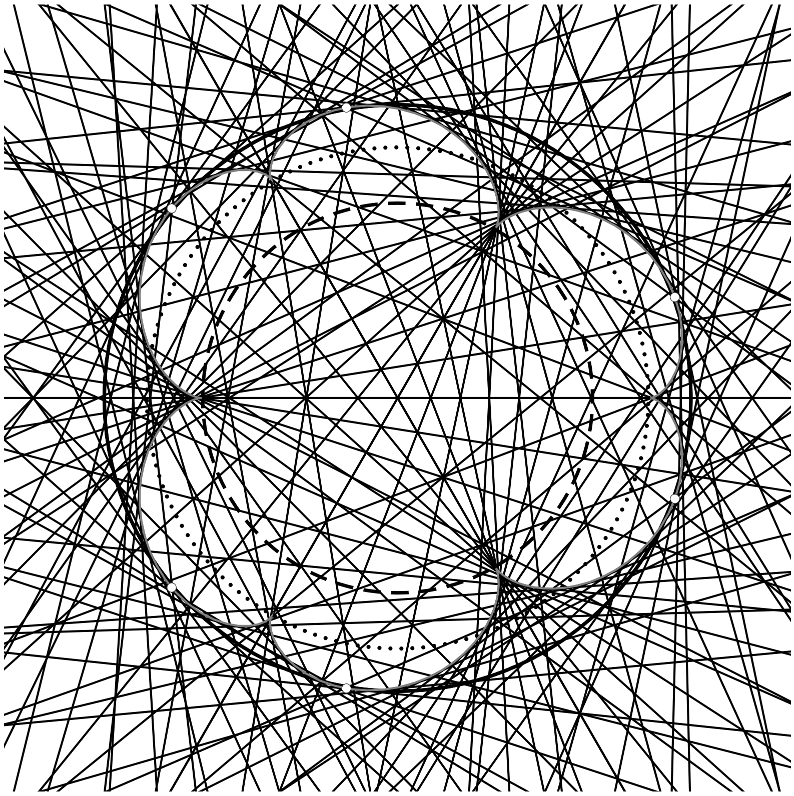

Figure 4. The case $q = 4$ and $r = 2$ in detail.

Now

$$\cos qt \cos(q-1)t = \frac{1}{2}\cos(2q-1)t + \frac{1}{2}\cos t;$$

hence

$$q\cos t - \cos qt \cos(q-1)t = \frac{1}{2}[(2q-1)\cos t - \cos(2q-1)t].$$

An analogue calculation for the second coordinate gives finally

$$\gamma_\infty(t) = \begin{pmatrix} \dfrac{(2q-1)\cos t - \cos(2q-1)t}{2q} \\ \\ \dfrac{(2q-1)\sin t - \sin(2q-1)t}{2q} \end{pmatrix}.$$

Comparing with the representation of an epicycloid in [2, volume II, formula (2) p.156], we see that γ_∞ is the epicycloid obtained as the trajectory of a point on a circle of radius $\rho := 1/2q$ rolling around a circle of radius $R := (q-1)/q$. Note that $R + 2\rho = 1$ and that the ratio R/ρ equals $2(q-1)$, which is the double of the ratio R/ρ in the case $r = 1$. We observe further (see (3.1) and (3.2)) that

$$\lim_{r\to+\infty} R_a = R, \quad \lim_{r\to+\infty} R_b = R.$$

This means that all critical points of γ_r remain, but the distinction between type (a) and type (b) disappears.

That in this case we also have an epiycloid is not a surprise. When r tends to $+\infty$, the straight line AB tends to a line through A which makes an angle qt with the x-axis. We can then use [2, volume II, §560 p.165] or [3, p.140]: the diameter of a circle rolling around another circle envelopes an epicycloid.

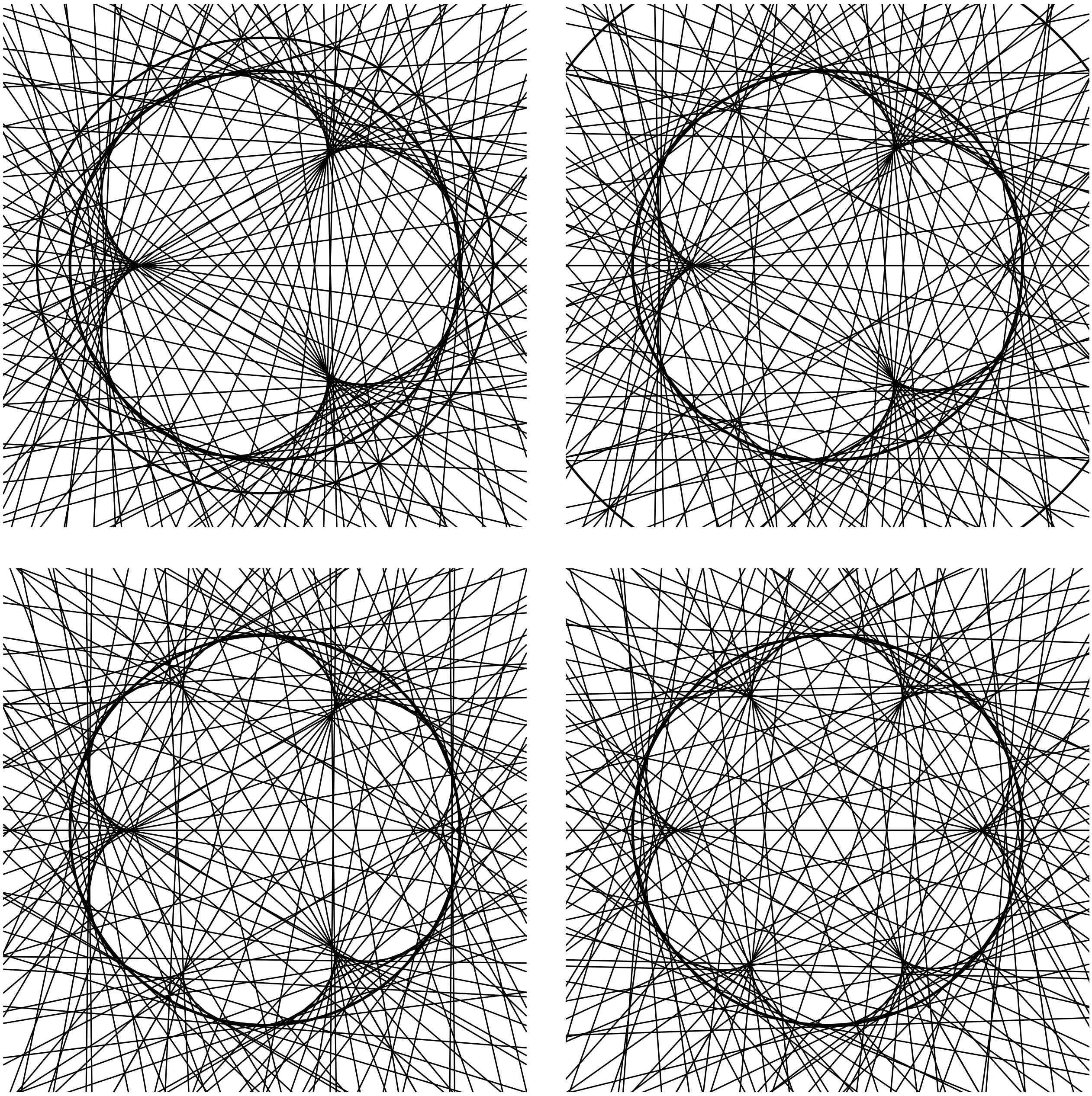

Figure 5. With $q = 4$, top left: $r = 7/6$, top right: $r = 5/3$, bottom left: $r = 3$, bottom right: $r = 30$.

9. Examples.

We take $q = 4$ and $r = 2$. The curve γ_2 has period 2π. The three critical points of type (a) are on the circle of radius $R_a = 2/3$ at angles $\pi/3$, π and $5\pi/3$. The three critical points of type (b) are on the circle of radius $R_b = 6/7$ at angles 0, $2\pi/3$ and $4\pi/3$. The points of contact of γ_2 with α are at the angles t such that $\cos 3t = 1/2$ i.e. $\cos 3t = \cos \pi/3$; we find the six angles $\pi/9$, $5\pi/9$, $7\pi/9$, $11\pi/9$, $13\pi/9$ and $17\pi/9$. All this can be seen in Figure 4.

In Figure 5 we illustrate, also for $q = 4$, the cases $r = 7/6$, $r = 5/3$, $r = 3$ and $r = 30$, without drawing the curve γ_r itself.

Sections 7 and 8 and the preceding example may suggest that the epicycloid for $r = +\infty$ shows two times more critical points than the one for $r = 1$. It is not always so. Take $q = 3/2$; for $r = 1$ as well as for $r = +\infty$ the epicycloids show one critical point.

10. Another natural morphing.

Almost all calculations made in the preceding sections for $q > 1$ are still valid when $q < -1$, that is, when B, starting from $(r, 0)$, moves on β in the sense opposite to A. We only state some salient differences in the results: $1 < R_b < R_a$; if $q = -m/n$ with m, $n \in \mathbb{Z}$, $m > n \geq 1$ and $\gcd(m, n) = 1$, γ_r has $m + n$ critical points of type (a) and $m + n$ critical points of type (b); the curve γ_r is again tangent to the circle α but does not go inside it; finally, the limit cases γ_1 and γ_∞ are hypocycloids.

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